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RESEARCH ARTICLE - MATHEMATICS

Approximate Solution for Solving Integro-Differential Equation via Hybrid Method

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Abstract

The integro-differential equations can be solved effectively using a hybrid technique is shown in this article. Finding approximate solutions to both linear and nonlinear problems using this novel method 1is based on combining the Gupta transform and the homotopy perturbation method (GHPM). To evaluate the effectiveness and potency of the suggested approach, problems are tested. Additionally, Maple software incorporates the outcomes of the hybrid technique.

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Keywords: Homotopy perturbation method; Gupta transform method; Integro-differential equations.

1. Introduction

Numerous numerical, analytical, and semi-analytical methods, such as the natural transform with homotopy perturbation method [1], the least squares approach [2], the spectral homotopy method [3], the Taylor extansion method [4], the Adomian decomposition together with variational iteration method [5], were applied for solving integro-differential equations. The initial approximation and sluggish convergence are two challenges associated with the traditional approach used to solve the integro-differential equations. In order to get around these drawbacks, numerous hybrid strategies have been developed. A variety of research methods have been developed to solve these integro-differential equations, including the Differential Transform Method for Solving Integral and Integro-Differential Equation Systems [6]. Applied Sawi transform for solving Volterra integral and Integro-differential Equations [7]. In [8] the author reported an efficient technique based on a monotone method. Additionally, In [9] a modified homotopy perturbation method used to solve a class of Fredholm-Volterra integro-differential equation and get approximate solutions. Solved second kind mixed Fredholm-volterra integral equations using a novel collocation technique is presented in [10]. To get the solution of integro-differential equations a hybrid methodology utilizing the Laplace transform, Adomian approach, and homotopy analysis approach given in [11]. The authors in [12] employed the orthogonal function operational matrix to solve the integral and integro-differential equations. To solve the two-dimensional integro-differential equations of the second class, a unique differential transform approach was developed [13]. Furthermore, the authors in [14] employed a hybrid technique that combines the Pade' approximation, Laplace transform, and the differential transform approach to improve it. The authors in [15] suggested the homotopy perturbation approach in conjunction with the variational iteration method to solve nonlinear mixed integro-differential equations.. In order to provide an analytical and approximation solution of integral and integro-differential equations, the authors of [16] compared He's method with three other conventional approaches. An applications involving living biological species was solved system of integro-differential equations numerically in [17] using the homotopy perturbation technique. In [18] existence and uniqueness of solutions for nonlinear integro-differential equations using Banach contraction mapping principle. Existence and uniqueness of a solution for third-order integro-differential equations using a fixed point technique in [19].

2. Gupta Transform

The definition of Gupta transform and the important theorems are given in this section.

Definition 2.1: Let $f(t)$ be integrable function defined for $t \geq 0$, Gupta transform is defined as follows: [20]

$$G\{f(t)\} = \mathrm{T}(q) = \frac{1}{q^3} \int_o^\infty e^{-qt} f(t) dt \qquad (2.1)$$

where q can be a real or complex parameter.

Theorem 2.1 [20]
Let $G\{f(t)\} = \mathrm{T}(q)$, then Gupta transform for some fundamental functions is as follows:

1- $G\{1\} = \frac{1}{q^4}$,
2- $G\{t^n\} = \frac{n!}{q^{n+4}}$,
3- $G\{e^{bt}\} = \frac{1}{q^3(q-b)}$,
4- $G\{\sin bt\} = \frac{b}{q^3(q^2+b^2)}$,
5- $G\{\cos bt\} = \frac{1}{q^2(q^2+b^2)}$,
6- $G\{\sinh bt\} = \frac{b}{q^3(q^2-b^2)}$,
7- $G\{\cosh bt\} = \frac{1}{q^2(q^2-b^2)}$,

Theorem 2.2: [20]
Let the inverse Gupta Transform of the function $\mathrm{T}(q)$ denoted by $G^{-1}\{\mathrm{T}(q)\}$ is a function $f(t)$, such that: $G\{f(t)\} = \mathrm{T}(q) \Rightarrow G^{-1}\{\mathrm{T}(q)\} = f(t)$, then inverse Gupta transform of elementary functions is as follows:

- $G^{-1}\left\{\frac{1}{q^4}\right\} = 1$,
- $G^{-1}\left\{\frac{1}{q^n}\right\} = \frac{t^{n-4}}{(n-4)!}$,
- $G^{-1}\left\{\frac{1}{q^3(q-b)}\right\} = e^{bt}$,
- $G^{-1}\left\{\frac{b}{q^3(q^2+b^2)}\right\} = \frac{1}{b}\sin(bt)$,
- $G^{-1}\left\{\frac{b}{q^3(q^2-b^2)}\right\} = \frac{1}{b}\sinh(bt)$,
- $G^{-1}\left\{\frac{b}{q^2(q^2+b^2)}\right\} = \cos(bt)$,
- $G^{-1}\left\{\frac{b}{q^2(q^2-b^2)}\right\} = \cosh(bt)$,

Theorem 2.3: [20] (Derivative) Let $f(t)$ is differentiable and Gupta transform of the function $f(t)$ given by $G\{f(t)\} = \mathrm{T}(q)$, then

I. $G\{f'(t)\} = q\mathrm{T}(q) - \frac{1}{q^3} f(0)$,

II. $G\{f''(t)\} = q^2\mathrm{T}(q) - \frac{1}{q^2} f(0) - \frac{1}{q^3} f'(0)$,

III. $G\{f^{(n)}(t)\} = q^{(n)}\mathrm{T}(q) - \sum_{k=0}^{n-1} \frac{1}{q^{4-n+k}} f^{(k)}(0)$,

3. Formulation and Methodology integro-differential equation

The general form of integro-differential equation can be written as:

$$y^{(n)}(t) = f(t) + \lambda \int_{a(t)}^{b(t)} k(t,\tau) F\big(y(\tau)\big) d\tau, \qquad n \in N \tag{3.1}$$

where $y^{(n)}(t) = \frac{d^n(t)}{dt^n}$, $a(t)$ and $b(t)$ are variables, constants or may be mixed, λ be a constant parameter.

In order to illustrate the fundamental concept of the hybrid GHPM approach, the integro-differential equation is given as follows:

$$y^{(n)}(t) = f(t) + \lambda \int_a^b k(t,\tau) F(y(\tau)) d\tau, \qquad n \in N \tag{3.2}$$

With initial conditions

$$y(0) = s_0, y'(0) = s_1, \dots, y^{n-1}(0) = s_{n-1}. \tag{3.3}$$

Take Gupta transform to both sides of the equation (3.2) and using (3.3) we get:

$$q^n G\{y(t)\} = \frac{s_0}{q^{4-n}} + \frac{s_1}{q^{5-n}} + \cdots + \frac{s_{n-1}}{q^3} + G\{f(t)\} + \lambda G\left[\int_a^b k(t,\tau) F(y(\tau)) d\tau\right]$$

$$T(q) = \frac{s_0}{q^4} + \frac{s_1}{q^5} + \cdots + \frac{s_{n-1}}{q^{n+3}} + \frac{G\{f(t)\}}{q^n} + \frac{1}{q^n} \lambda G\left[\int_a^b k(t,\tau) F\big(y(\tau)\big) d\tau\right] \tag{3.4}$$

Where $T(q) = G\{y(\tau)\}$

Applying Gupta inverse to equation (3.4) yields the following result:

$$y(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)!} s_{n-1} + G^{-1}\left\{\frac{1}{q} G\{f(t)\} + \frac{1}{q^n} \lambda \mathrm{G}\left[\int_a^b k(t,\tau) F\big(y(\tau)\big) d\tau\right]\right\} \tag{3.5}$$

Applying the homotopy perturbation method on the equation above, we obtain:

$$\sum_{n=0}^{\infty} p^n y_n(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)!} s_{n-1} + G^{-1} \frac{1}{q^n} G\{f(t)\} + p\left(G^{-1} \frac{1}{q^n} \lambda\, G\left[\int_a^b k(t,\tau) F\big(y(\tau)\big) d\tau\right]\right) \tag{3.6}$$

Now we have two cases depends on $F\big(y(t)\big)$ as follows:

<u>Case 1:</u>

If $F\big(y(t)\big)$ is linear function then Eq. (3.6) becomes as:

$$\sum_{n=0}^{\infty} p^n\, y_n(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)} s_{n-1} + G^{-1}\frac{1}{q^n} G\{f(t)\}$$

$$+p\ \left(G^{-1}\frac{1}{q^n}\lambda G\left[\int_a^b k(t,\tau)\sum_{n=0}^{\infty} p^n y_n(t)\, d\tau\right]\right) \qquad (3.7)$$

By contrasting the coefficients of similar powers of p, we obtain

$p^0 = y_0(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)} z_{n-1} + G^{-1}\frac{1}{q^n} G\{f(t)\}$

$p^1 = y_1(t) = G^{-1}\frac{1}{q^n}\lambda\, G\left[\int_a^b k(t,\tau)\, y_0(t)\, d\tau\right]$

$p^i = y_j(t) = G^{-1}\frac{1}{q^n}\lambda\, G\left[\int_a^b k(t,\tau)\, y_{i-1}(t)\, d\tau\right]$, $i = 2,3,4 \ldots$

<u>Case 2:</u>

If $F(y(t))$ is nonlinear function then Eq. (3.6) becomes as:

$$\sum_{n=0}^{\infty} p^n y_n(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)} s_{n-1} + G^{-1}\frac{1}{q^n} G\{f(t)\}$$

$$+p\ \left(G^{-1}\frac{1}{q^n}\lambda G\left[\int_a^b k(t,\tau)\sum_{n=0}^{\infty} p^n A_n(\tau)\, d\tau\right]\right)$$

Where $A_n(\tau)$ is the Adomain polynomial which defined as

$p^0 = y_0(t) = s_0 + ts_1 + \cdots + \frac{t^{n-1}}{(n-1)} s_{n-1} + G^{-1}\frac{1}{q^n} G\{f(t)\}$

$p^1 = y_1(t) = G^{-1}\frac{1}{q^n}\lambda\, G\left[\int_a^b k(t,\tau)\, A_0(t) d\tau\right]$

$p^i = y_i(t) = G^{-1}\frac{1}{q^n}\lambda\, G\left[\int_a^b k(t,\tau)\, A_{i-1}(t) d\tau\right], \quad i = 2,3,4, \ldots$

For both cases

$y(t) = \lim_{p\to 1}\left[\sum_{n=0}^{\infty} p^n y_n\right] = \sum_{n=0}^{\infty} y_n.$

which is the solution of (3.2).

4. Test Problems

To demonstrate the competence of the current GHPM approach, four problems is solved in this section.

Problem 1.

$y'(t) = 1 - \frac{1}{3}t + \int_0^1 t\tau y(\tau) d\tau \qquad y(0) = 0.$

Exact solution is $y(t) = t$.

Solution: Take Gupta transform on both sides

$$G\{y'(t)\} = G\{1\} - \frac{1}{3}\, G\{t\} + G\left\{\int_0^1 t\tau y(\tau)d\tau\right\}$$

$$qT(q) - \frac{1}{q^3}y(0) = \frac{1}{q^4} - \frac{1}{3q^5} + G\left\{\int_0^1 t\tau y(\tau)d\tau\right\}$$

$$T(q) = \frac{1}{q^5} - \frac{1}{3q^6} + \frac{1}{q}G\left\{\int_0^1 t\tau y(\tau)d\tau\right\}$$

On both sides apply inverse Gupta transform, which gives

$$y(t) = t - \frac{t^2}{6} + G^{-1}\left[\frac{1}{q}\left\{G\int_0^1 t\tau y(\tau)\right\}\right]$$

$$\sum_{n=0}^{\infty} p^n y_n(t) = t - \frac{t^2}{6} + pG^{-1}\left[\frac{1}{q}\; G\left\{t\int_0^1 \sum_{n=0}^{\infty} p^n y_n(\tau)d\tau\right\}\right]$$

$$p^0 = t - \frac{t^2}{6}$$

$$p^1 = y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{t\int_0^1 \tau\left(\tau - \frac{\tau^2}{6}\right)d\tau\right\}\right]$$

$$=G^{-1}\left[\frac{1}{q}\; G\left\{t\int_0^1 \tau\left(\tau - \frac{\tau^2}{6}\right)d\tau\right\}\right]$$

$$=G^{-1}\left[\frac{1}{q}\; G\left\{t\,\frac{7}{24}\right\}\right]$$

$$=\frac{7}{24}\; G^{-1}\left\{\frac{1}{q^6}\right\} = \frac{7}{48}t^2$$

$$p^2 = y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t\tau y(\tau)d\tau\right\}\right]$$

$$= G^{-1}\left[\frac{1}{q}G\left\{t\int_0^1 \tau\left(\frac{7\tau^2}{48}\right)d\tau\right\}\right]$$

$$=G^{-1}\left[\frac{1}{q}G\left\{\frac{7t}{192}\right\}\right]$$

$$= \frac{7}{192}G^{-1}\left\{\frac{1}{q^6}\right\} = \frac{7t^2}{384}$$

$$p^3 = y_3(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t\tau y_2(\tau)d\tau\right\}\right]$$

$$y_3(t) = G^{-1}\left[\frac{1}{q}G\left\{t\int_0^1 \tau\left(\frac{7\tau^2}{384}\right)d\tau\right\}\right]$$

$$y_3(t) = \frac{7}{1536} G^{-1}\left[\frac{1}{q} G\{t\}\right] = \frac{7}{1536} G^{-1}\left\{\frac{1}{q^6}\right\} = \frac{7}{3072} t^2$$

And can continue to find y_4, y_5 and so on

$$p^4 = y_4(t) = \frac{7t^2}{24576}$$

$$p^5 = y_5(t) = \frac{7t^2}{196608}$$

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = t - \frac{t^2}{6} + \frac{7}{48}t^2 + \frac{7}{384}t^2 + \frac{7}{3072}t^2 + \frac{7}{24576}t^2 + \frac{7}{196608}t^2 + \cdots$$

Table 1. Numerical Results and Comparisons for Problem 1

x	Exact	GHPM	SHPM	AHPM	ERRG	ERRS	ERRA
0	0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.1	9.9999949e-02	9.9994320e-02	9.9999712e-02	5.0862630e-08	5.6803385e-06	2.8808910e-07
0.2	0.2	1.9999980e-01	1.9997728e-01	1.9999885e-01	2.0345052e-07	2.2721354e-05	1.1523564e-06
0.3	0.3	2.9999954e-01	2.9994888e-01	2.9999741e-01	4.5776367e-07	5.1123047e-05	2.5928019e-06
0.4	0.4	3.9999919e-01	3.9990911e-01	3.9999539e-01	8.1380208e-07	9.0885417e-05	4.6094257e-06
0.5	0.5	4.9999873e-01	4.9985799e-01	4.9999280e-01	1.2715658e-06	1.4200846e-04	7.2022276e-06
0.6	0.6	5.9999817e-01	5.9979551e-01	5.9998963e-01	1.8310547e-06	2.0449219e-04	1.0371208e-05
0.7	0.7	6.9999751e-01	6.9972166e-01	6.9998588e-01	2.4922689e-06	2.7833659e-04	1.4116366e-05
0.8	0.8	7.9999674e-01	7.9963646e-01	7.9998156e-01	3.2552083e-06	3.6354167e-04	1.8437703e-05
0.9	0.9	8.9999588e-01	8.9953989e-01	8.9997666e-01	4.1198730e-06	4.6010742e-04	2.3335218e-05
1	1	9.9999491e-01	9.9943197e-01	9.9997119e-01	5.0862630e-06	5.6803385e-04	2.8808910e-05

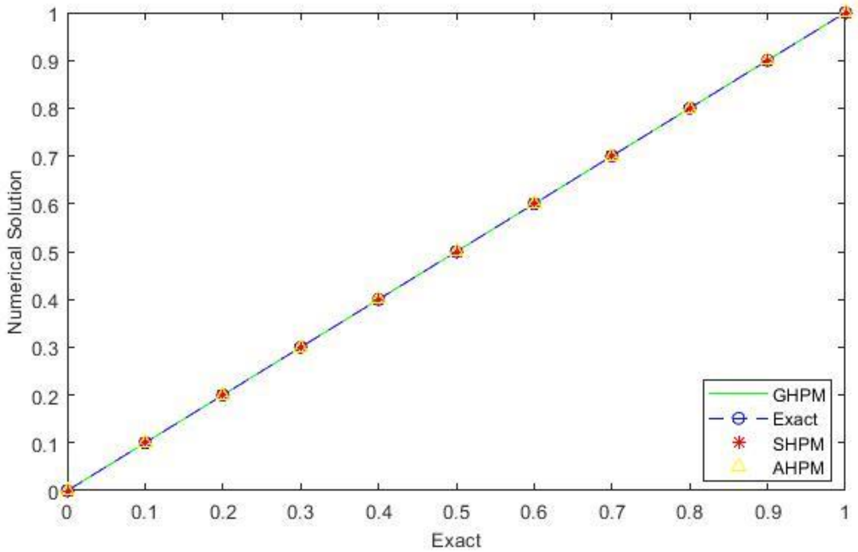

Fig. 1. Numerical Solution for Problem 1

Problem 2.

$y'(t) = 1 - \frac{1}{3}t^3 + \int_0^1 t^3\, y^2(\tau)d\tau$ $\quad ,y(0) = 0$. Exact solution is $y(t) = t$.

Solution: After utilizing Gupta transform, we get

$$G\{y'(t)\} = \mathrm{G}\{1\} - \frac{1}{3}G\{t^3\} + \mathrm{G}\left\{\int_0^1 t^3 y^2(\tau)d\tau\right\}$$
$$qT(q) - \frac{1}{q^3}\mathrm{y}(0) = \frac{1}{q^4} - \frac{3!}{3q^7} + G\left\{\int_0^1 t^3 y^2(\tau)d\tau\right\}$$
$$T(q) = \frac{1}{q^5} - \frac{3!}{3q^8} + \frac{1}{q}G\left\{\int_0^1 t^3 y^2(\tau)d\tau\right\}$$

Using inverse Gupta transform we get

$$y(t) = t - \frac{t^4}{12} + G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t^3 y^2(\tau)d\tau\right\}\right]$$
$$p^0 = t - \frac{t^4}{12}$$
$$A_0 = \frac{1}{0!}.\frac{d^0}{d\lambda^0}[(\textstyle\sum_{k=0}^{0}\lambda^k y_k)^2]_{\lambda=0} = y_0^2$$
$$p^1 = y_1(\mathrm{t}) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t^3 y_0^2(\tau)d\tau\right\}\right]$$
$$y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{t^3\int_0^1\left(\tau - \frac{\tau^2}{12}\right)d\tau\right\}\right]$$
$$y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{t^3\int_0^1(\tau^2 - \frac{\tau^5}{6} + \frac{\tau^8}{144})d\tau\right\}\right]$$
$$= y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{t^3\left(\frac{\tau^3}{3} - \frac{\tau^6}{36} + \frac{\tau^9}{1296}\right)\begin{matrix}1\\0\end{matrix}\right\}\right] = \frac{397}{1296}G^{-1}\left\{\frac{3!}{q^8}\right\} = \frac{397}{5184}t^4$$
$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}[(\textstyle\sum_{k=0}^{n}\lambda^k y_k)^2]_{\lambda=0} = \frac{d}{d\lambda}[(y_0 + \lambda y_0)^2]_{\lambda=0}$$
$$A_1 = \frac{d}{d\lambda}[y_0 + 2\lambda y_0 y_1 + \lambda^2 y_1^2]_{\lambda=0} = 2y_0 y_1$$
$$p^2 = y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t^3 A_1(\tau)d\tau\right\}\right]$$
$$y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^1 t^3(\tau - \frac{\tau^4}{12})(\frac{397\tau^4}{5184})(\tau)d\tau\right\}\right]$$
$$y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{t^3\int_0^1\frac{794\tau^5}{5184} - \frac{797\tau^8}{92208}d\tau\right\}\right]$$
$$y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\frac{13498}{559872}t^3\right\}\right] = \frac{13498}{559872}G^{-1}\left\{\frac{3!}{q^8}\right\} = \frac{13498}{2239488}t^4$$
$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[(y_0 + \lambda y_1 + \lambda^2 y_2)^2]_{\lambda=0}$$
$$A_2 = y_1^2 + 2y_0 y_2 = \frac{130613}{26873856}t^8 + \frac{6749}{559872}t^5$$
$$p^3 = y_3(t) = G^{-1}\left[\frac{1}{q}G\left\{t^3\int_0^1 A_2(\tau)d\tau\right\}\right] = G^{-1}\left[\frac{1}{q}Gt^3\left\{\frac{616541}{241864704}\right\}\right]$$
$$P^3 = y_3(t) = \frac{616541}{241864704}G^{-1}\left\{\frac{3!}{q^8}\right\} = \frac{616541}{967458816}t^4$$
$$y(t) = \sum_{n=0}^{\infty} y_n(t).$$
$$y(t) = t - \frac{t^4}{12} + \frac{397}{5184}t^4 + \frac{13498}{2239488}t^4 + \frac{616541}{967458816}t^4 \ldots = t - \frac{84163}{967458816}t^4 + \cdots.$$

Table 2. Numerical Results and Comparisons for Problem 2

x	Exact	GHPM	SHPM	AHPM	ERRG	ERRS	ERRA
0	0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

0.1	0.1	9.9999991e-02	9.9999326e-02	9.9999619e-02	8.6993884e-09	6.7356047e-07	3.8124246e-07
0.2	0.2	1.9999986e-01	1.9998922e-01	1.9999390e-01	1.3919021e-07	1.0776968e-05	6.0998794e-06
0.3	0.3	2.9999930e-01	2.9994544e-01	2.9996912e-01	7.0465046e-07	5.4558398e-05	3.0880640e-05
0.4	0.4	3.9999777e-01	3.9982757e-01	3.9990240e-01	2.2270434e-06	1.7243148e-04	9.7598071e-05
0.5	0.5	4.9999456e-01	4.9957902e-01	4.9976172e-01	5.4371177e-06	4.2097530e-04	2.3827654e-04
0.6	0.6	5.9998873e-01	5.9912707e-01	5.9950591e-01	1.1274407e-05	8.7293438e-04	4.9409023e-04
0.7	0.7	6.9997911e-01	6.9838278e-01	6.9908464e-01	2.0887232e-05	1.6172187e-03	9.1536316e-04
0.8	0.8	7.9996437e-01	7.9724110e-01	7.9843843e-01	3.5632695e-05	2.7589037e-03	1.5615691e-03
0.9	0.9	8.9994292e-01	8.9558077e-01	8.9749867e-01	5.7076687e-05	4.4192303e-03	2.5013318e-03
1	1	9.9991301e-01	9.9326440e-01	9.9618758e-01	8.6993884e-05	6.7356047e-03	3.8124246e-03

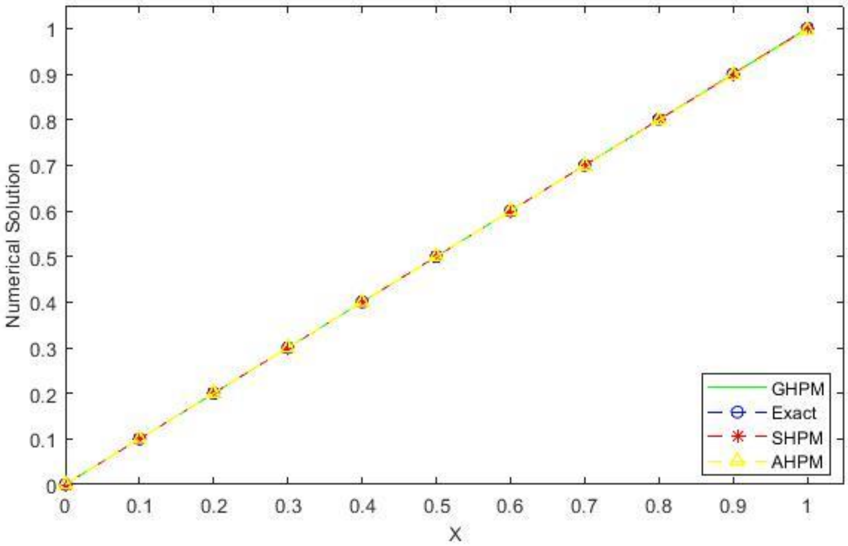

Fig. 2. Numerical Solution for Problem 2

Problem 3.

$y'(t) = -\frac{1}{2} + \int_0^t y'^2(\tau)d\tau \quad y(0) = 0$. Exact solution is $y(t) = -\ln\left(\frac{1}{2}(t+1)\right)$.

Solution:

$G\{y'(t)\} = -\frac{1}{2}G\{1\} + G\left\{\int_0^t y'^2(\tau)\,d\tau\right\}$

$qT(q) - \frac{1}{q^3}y(0) = \frac{-1}{2q^4} + G\left\{\int_0^t y'^2(\tau)\,d\tau\right\}$

$T(q) = \frac{-1}{2q^5} + \frac{1}{q}G\left\{\int_0^t y'^2(\tau)\,d\tau\right\}$

By using inverse of Gupta transform we get

$y(t) = \frac{-t}{2} + G^{-1}\left[\frac{1}{q}G\left\{\int_0^t y'^2(\tau)\,d\tau\right\}\right]$

$p^0 = y_0(t) = \frac{-t}{2}$

$A_0 = y_0'^2$

$p^1 = y_1(x) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t A_0(t)\,dt\right\}\right]$

$y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t (\frac{-\tau}{2})^2 d\tau\right\}\right] = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t \frac{1}{4}d\tau\right\}\right]$

$$y_1(t) = G^{-1}\left[\frac{1}{q}G\left\{\frac{\tau}{4}\Big|\begin{matrix}t\\0\end{matrix}\right\}\right] = \frac{1}{4}G^{-1}\left\{\frac{1}{q}G(t)\right\} = \frac{1}{4}G^{-1}\left\{\frac{1}{q^6}\right\} = \frac{1}{8}t^2$$

$$A_1 = \frac{d}{d\lambda}\left[\left(\sum_{k=0}^{1}\lambda^k y_k\right)'^2\right]_{\lambda=0} = \frac{d}{d\lambda}\left[(y_0+\lambda y_1)'^2\right]_{\lambda=0} = \frac{d}{d\lambda}\left[\left(\frac{-t}{2}+\lambda\frac{t^2}{8}\right)'^2\right]_{\lambda=0}$$

$$A_1 = \frac{d}{d\lambda}\left[\left(\frac{-1}{2}+\frac{2\lambda t}{8}\right)^2\right]_{\lambda=0} = \frac{d}{d\lambda}\left[\frac{1}{4}-\frac{\lambda t}{4}+\frac{\lambda^2t^2}{16}\right]_{\lambda=0} = \frac{-t}{4}$$

$$p^2 = y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t A_1(\tau)\,d\tau\right\}\right]$$

$$y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t \frac{-\tau}{4}\,d\tau\right\}\right]$$

$$y_2(t) = G^{-1}\left[\frac{1}{q}G\left\{\frac{\tau^2}{8}\Big|\begin{matrix}t\\0\end{matrix}\right\}\right]$$

$$y_2(t) = G^{-1}\left[\frac{1}{q}\left\{\frac{t^2}{8}\right\}\right] = \frac{-1}{8}G^{-1}\left\{\frac{2!}{q^7}\right\} = -\frac{1}{24}t^3$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\left(\sum_{k=0}^{2}\lambda^k y_k\right)'^2\right]\Big|_{\lambda=0}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[(y_0+\lambda y_1+\lambda^2 y_2)'^2\right]_{\lambda=0}$$

$$A_2 = \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left[\frac{-t}{2}+\frac{t^2}{8}-\frac{t^3}{24}\right]'^2\right]_{\lambda=0} = \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left[\frac{-1}{2}+\frac{t\lambda}{4}-\frac{t^2\lambda^2}{8}\right]^2\right]_{\lambda=0}$$

$$A_2 = \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left[\frac{1}{4}-\frac{t\lambda}{8}+\frac{t^2\lambda^2}{8}-\frac{t^3\lambda^3}{16}+\frac{t^2\lambda^2}{16}+\frac{t^4\lambda^4}{64}\right]\right]_{\lambda=0}$$

$$A_2 = \left[\frac{1}{2}\frac{d}{d\lambda}\left[\frac{-t}{8}+\frac{t^2\lambda}{4}-\frac{2t^3\lambda^2}{16}+\frac{t^2\lambda}{8}+\frac{4t^4\lambda^3}{64}\right]\right]_{\lambda=0} = \frac{1}{2}\left[\frac{t^2}{8}+\frac{t^2}{8}\right] = \frac{3t^2}{16}$$

$$p^3 = y_3(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t A_2(\tau)\,d\tau\right\}\right] = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t \frac{3\tau^2}{16}\,d\tau\right\}\right]$$

$$y_3(t) = G^{-1}\left[\frac{1}{q}G\left\{\int_0^t \frac{3\tau^3}{48}\Big|\begin{matrix}t\\0\end{matrix}\right\}\right] = \frac{1}{16}G^{-1}\left[\frac{1}{q}G\{t^3\}\right] = \frac{1}{16}G^{-1}\left\{\frac{1}{q}\,\frac{3!}{q^7}\right\} = \frac{t^4}{64}$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\left[\left(\sum_{k=0}^{3}\lambda^k y_k\right)'^2\right]_{\lambda=0}$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\left[(y_0+\lambda y_1+\lambda^2 y_2+\lambda^3 y_3)'^2\right]_{\lambda=0} = \frac{1}{6}\frac{d^3}{d\lambda^3}\left[\left(\frac{-t}{2}+\frac{t^2\lambda}{8}-\frac{t^3\lambda^2}{24}+\frac{t^2\lambda^3}{64}\right)'^2\right]_{\lambda=0}$$

$$A_3 = \frac{1}{6}\left[\frac{-6t^3}{8}\right] = \frac{-t^3}{8}$$

$$p^4 = y_4(t) = G^{-1}\left[\frac{1}{q}G\int_0^t A_3\,d\tau\right] = G^{-1}\left[\frac{1}{q}G\int_0^t \frac{-\tau^3}{8}\,d\tau\right]$$

$$y_4(t) = G^{-1}\left\{\frac{1}{q}G\left[\frac{-\tau^4}{32}\right]\Big|\begin{matrix}t\\0\end{matrix}\right\} = \frac{-t^5}{160}$$

$$A_4 = \frac{5t^4}{64}$$

$$p^5 = y_5(t) = \frac{t^6}{384}$$

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

$$y(t) = \frac{-t}{2}+\frac{t^2}{8}-\frac{t^3}{24}+\frac{t^4}{64}-\frac{t^5}{160}+\frac{t^6}{384}-\cdots$$

Table 3. Numerical Results and Comparisons for Problem 3

x	Exact	GHPM	SHPM	AHPM	ERRG	ERRS	ERRA
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000

0.1	-4.879016e-02	-4.879016e-02	-4.879054e-02	-4.879054e-02	4.675094e-12	3.744838e-07	3.744838e-07
0.2	-9.531018e-02	-9.531018e-02	-9.532215e-02	-9.532215e-02	1.148056e-09	1.196781e-05	1.196781e-05
0.3	-1.397619e-01	-1.397620e-01	-1.398527e-01	-1.398527e-01	2.827216e-08	9.077358e-05	9.077358e-05
0.4	-1.823216e-01	-1.823218e-01	-1.827037e-01	-1.827037e-01	2.717775e-07	3.821384e-04	3.821384e-04
0.5	-2.231436e-01	-2.231451e-01	-2.243088e-01	-2.243088e-01	1.561223e-06	1.165298e-03	1.165298e-03
0.6	-2.623643e-01	-2.623707e-01	-2.652624e-01	-2.652624e-01	6.478390e-06	2.898178e-03	2.898178e-03
0.7	-3.001046e-01	-3.001261e-01	-3.063674e-01	-3.063674e-01	2.148489e-05	6.262834e-03	6.262834e-03
0.8	-3.364722e-01	-3.365327e-01	-3.486842e-01	-3.486842e-01	6.048719e-05	1.221195e-02	1.221195e-02
0.9	-3.715636e-01	-3.717139e-01	-3.935804e-01	-3.935804e-01	1.502961e-04	2.201688e-02	2.201688e-02
1	-4.054651e-01	-4.058036e-01	-4.427827e-01	-4.427827e-01	3.384633e-04	3.731763e-02	3.731763e-02

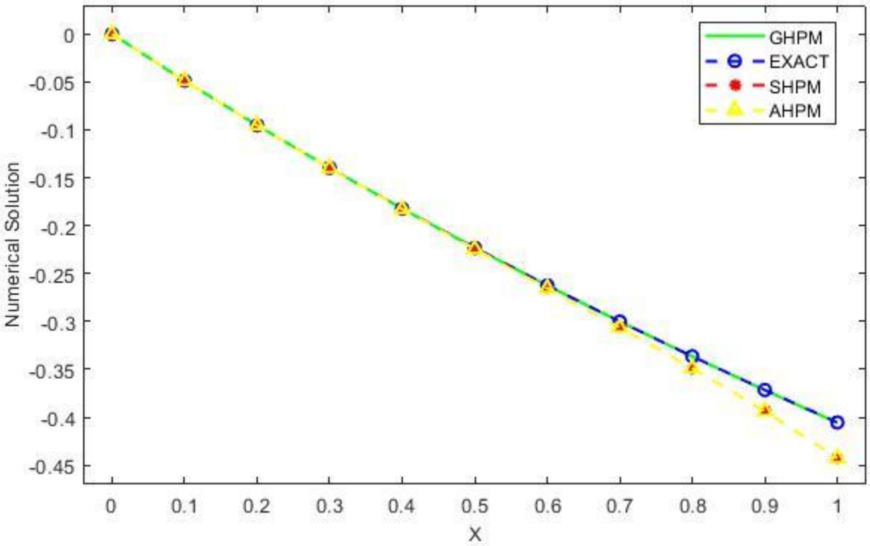

Fig. 3. Numerical Solution for Problem 3

Problem 4.

$y(t) = t + \int_0^t y^2(\tau)d\tau$. Exact solution is $y(t) = \tan t$.

Solution:

$T(q)=\frac{1}{q^5} + q^3 G\{1\}G\{y^2(\tau)\}$

$\mathrm{T(q)} = \frac{1}{q^5} + q^3\left(\frac{1}{q^4}\right)G\{y^2(\tau)\}$

Take the inverse of Gupta transform

$y(t) = t + G^{-1}\left\{\frac{1}{q}G\{y^2(\tau)\}\right\}$

$y_0 = t \quad , \quad A_0 = y_0^2 = t^2$

$y_1 = G^{-1}\left\{\frac{1}{q}G(t^2)\right\} = G^{-1}\left\{\frac{1}{q}.\frac{2!}{q^6}\right\} = \frac{t^3}{3} = y_1$

$A_1 = 2y_0y_1 = \frac{2t^3}{3}$

$$y_2 = G^{-1}\left\{\frac{1}{q}G(\frac{2}{3}t^4)\right\} = \frac{2}{3}G^{-1}\left\{\frac{1}{q}.\frac{4!}{q^8}\right\} = \frac{2}{15}t^5$$

$$A_2 = y_1^2 + 2y_0y_1 = \frac{t^6}{9} + \frac{4t^6}{15}$$

$$y_3 = G^{-1}\left\{\frac{1}{q}G\left[\frac{t^6}{9} + \frac{4t^6}{15}\right]\right\} = G^{-1}\left\{\frac{1}{q}\left[\frac{6!}{9q^{10}} + \frac{4(6!)}{15q^{10}}\right]\right\}$$

$$y_3 = G^{-1}\left\{\frac{80}{q^{11}} + \frac{192}{q^{11}}\right\} = G^{-1}\left\{\frac{272}{q^{11}}\right\} = \frac{272}{5040}t^7$$

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

$$y(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{272t^7}{5040} + \cdots$$

Table 4. Numerical Results and Comparisons for Problem 4

x	Exact	GHPM	SHPM	AHPM	ERRG	ERRS	ERRA
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1.003347e-01	1.003347e-01	1.003373e-01	1.003380e-01	2.195846e-11	2.666645e-06	3.333311e-06
0.2	2.027100e-01	2.027100e-01	2.027954e-01	2.028167e-01	1.138169e-08	8.532195e-05	1.066553e-04
0.3	3.093362e-01	3.093358e-01	3.099838e-01	3.101458e-01	4.467525e-07	6.475532e-04	8.095532e-04
0.4	4.227932e-01	4.227871e-01	4.255178e-01	4.262004e-01	6.130484e-06	2.724536e-03	3.407203e-03
0.5	5.463025e-01	5.462550e-01	5.545883e-01	5.566716e-01	4.752953e-05	8.285804e-03	1.036914e-02
0.6	6.841368e-01	6.838788e-01	7.046148e-01	7.097988e-01	2.580426e-04	2.047796e-02	2.566196e-02
0.7	8.422884e-01	8.411872e-01	8.860059e-01	8.972105e-01	1.101196e-03	4.371747e-02	5.492214e-02
0.8	1.029639e+00	1.025675e+00	1.113057e+00	1.134902e+00	3.963261e-03	8.341807e-02	1.052634e-01
0.9	1.260158e+00	1.247545e+00	1.405009e+00	1.444375e+00	1.261337e-02	1.448506e-01	1.842166e-01
1	1.557408e+00	1.520635e+00	1.787302e+00	1.853968e+00	3.677280e-02	2.298939e-01	2.965605e-01

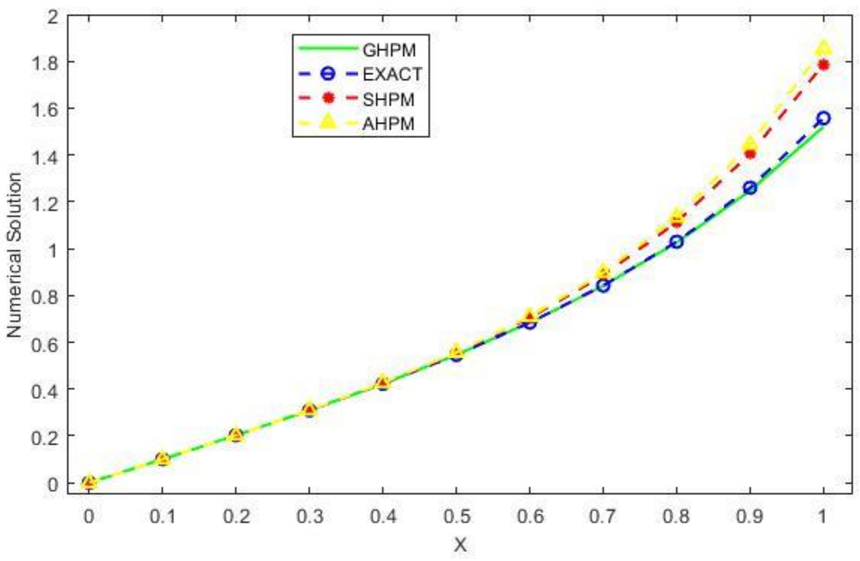

Fig. 4. Numerical Solution for Problem 4

In Tables 1-4, we have compared the between the exact solution, GHPM, Aboodh homotopy perturbation method (AHPM) [21], Sawi homotopy perturbation method (SHPM) [22], and absolute error for each method (ERRG, ERRS, ERRA). The numerical results of GHPM, APHM, and SHPM against exact solution are depicted in Figs 1-4, which reveal that the numerical solutions are very near to the exact solution. From Table 1-4 and Figs. 1-4, we observed that hybrid GHPM technique is more effective to solve integro-differential equations.

5. Conclusion

Integro-differential equations are handled effectively by combining the Homotopy Perturbation approach with the Gupta transform. Through numerical comparison, it is evident that the Cupta transform HPM produces extremely accurate approximations. We have contrasted the outcomes of the Sawi transform HPM, Aboodh transform HPM, and Gupta transform HPM with the precise solution. We deduce that the Gupta transform HPM is a proficient method for handling integro-differential equations.

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