Conditions on N For Writing it as A Sum of Four Cubes

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Abstract

The aim of this paper is to give a conditions on n, so can be written as a sum of four cubes, but this conditions work on another problem that deals with cubic equation and the solution of non-linear system of equation are also required This work is very important in different branches of applied mathematics especially in number theory and cryptography.

الشروط على N لكتابته على شكل مجموع اربع مكعبات

الخلاصة:

هدف البحث هو اعطاء الشروط على n لغرض كتابته على شكل مجموع أربع مكعبات.للأجابة عن هذا السؤال نحتاج العمل على مسألة أخرى التي تتعلق بالمعادلات التكعيبية وكذلك الحل لنظام غير خطي الناتج أثناء حل المسألة. العمل مهم جدا في تطبيقات الرياضيات المختلفة وخاصة نظرية الاعداد والتشفير.

1. Introduction:

There are many application in different branch of science that depend on the solution of some open problems therefore [1] gives some open problems in Mathematics especially in number theory and the solution for these problems could be broke the RSA cryptosystems that used in internet transaction, if we can find the number of ways for writing an integer n as a sum of four cubic [2]. Our work can be obtained by writing the problem as a cubic equation then try to solve this equation using a modified method; we get a system of non linear equation which is the condition for the open problems.

1. Formulation of the method:

Now, we discuss how every integer n can be written as a sum of four cubes and the condition on it

Assume that $n = a^3 + b^3 + c^3 + d^3$

Where a, b, c and d belongs to the set of integer numbers .Using the assumption,

$a^3 + b^3 = (a+b)^3 - 3ab^2 - 3a^2b$

$c^3 + d^3 = (c+d)^3 - 3cd^2 - 3c^2d$

Then,

$n = (a+b)^3 - 3ab^2 - 3a^2b + (c+d)^3 - 3cd^2 - 3c^2d$

$n = (a+b)^3 + (c+d)^3 - 3ab(a+b) - 3cd(c+d)$

$n = (a+b+c+d)^3 - 3[(a+b)(c+d)$
$(a+b+c+d) - 3ab(a+b) - 3cd(c+d)]$

Let $r = a+b+c+d$

Therefore,

$n = r^3 - 3r(a+b)(c+d) - 3(ab^2 + a^2b + cd^2 + c^2d)$

Let $A = -3(a+b)(c+d)$

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B=−3

$$(ab^2 + a^2b + cd^2 + c^2d)$$

Then,

$$n = r^3 + Ar + B$$

$$r^3 + Ar + (B - n) - 0$$

Let D = B – n,

Therefore,

$r^3 + Ar + D = 0$

The values of A and D in the equation above are not zero .So we will solve the above equation.

Consider that for two numbers u and v,

$$(u - v)^3 + 3uv(u - v) = u^3 - v^3$$

And if we substitute for A = 3uv, $-D = u^3 - v^3$ and $r = u - v$ .Since the result is zero, so u−v is a root of our equation .When we find the values of u and v the cubic equation is able to solve. Substitute for v, using $\frac{A}{3u}$ the problem now, is to find the root of the above polynomial equations, [3], [4], and [5].

Note that u can not be zero, because A would also be zero, so neither u nor v is zero, hear

$-D = u^3 - v^3$

To get $u^3 + D - (\frac{A}{3u})^3 = 0$

Multiplying throughout by u^3 to get

$$u^6 + Du^3 - \left(\frac{A}{3}\right)^3 = 0$$

Now, this equation is quadratic in u^3, Therefore,

$$u^3 = \frac{\left(-D \pm \sqrt{D^2 + 4\left(\frac{A}{3}\right)^3}\right)}{2}$$

Or simplified to get,

$$u^3 = \frac{-D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{A}{3}\right)^3}$$

There are three roots of cubic, so we can find v^3 from $-D = u^3 - v^3$,

$v^3 = u^3 - D$

It is clearly that the discriminate Δ, is the bit in the square root

$$\Delta = \left(\frac{D}{2}\right)^2 + \left(\frac{A}{3}\right)^3$$

There are three cases

1. If $\Delta = 0$, then all the roots are real, and at least two equal, [6].
2. If $\Delta > 0$, then $\sqrt{\Delta}$ is a real number, and so one root is real, and the other two are complex
3. If $\Delta < 0$, then $\sqrt{\Delta}$ is imaginary, so all the roots are u and v will be complex number.

By substation Δ in equation (1) and (2) we get,

$$u^3 = \frac{-D}{2} + \sqrt{\Delta}$$

And

$$v^3 = \frac{D}{2} + \sqrt{\Delta}$$

We take only the case one,

If $\Delta = 0$, then from equation (1) and (2) we get,

$u = \sqrt[3]{\frac{-D}{2}}$ and $v = \sqrt[3]{\frac{D}{2}}$

Since $r = u - v$ then,

$r = -2\sqrt[3]{\frac{-D}{2}}$ where

$$r = a + b + c + d \qquad (1)$$

$$A = -3(a + b)(c + d)$$

$$D = -3(ab^2 + a^2b + cd^2 + c^2d) - n$$

And $\Delta = 0$, so

$$a+b+c+d = -2\sqrt[3]{\frac{-3(ab^2+a^2b+cd^2+c^2d)-n}{2}}$$

(3)

$$\Delta = \left(\frac{D}{2}\right)^2 + \left(\frac{A}{3}\right)^3$$

$$\left(\frac{-3(ab^2+a^2b+cd^2+c^2d)-n}{2}\right) + \left(\frac{3(a+b)(c+d)^3}{3}\right) = 0$$

(4)

So

$$(a+b+c+d)^3 = 2^3.\frac{1}{2}[3(ab^2+a^2b+cd^2+c^2d)+n]$$

$$(a+b+c+d)^3 = 4[3(ab^2+a^2b+cd^2+c^2d)+n]$$

Therefore,

$$(a+b+c+d)^3 = 8.\frac{[3(ab^2+a^2b+cd^2+c^2d)+n]}{2}$$

$$\left[\frac{3(ab^2+a^2b+cd^2+c^2d)+n}{2}\right]^2 + [(a+b)(c+d)]^3 = 0$$

Put

$$\frac{[3(ab^2+a^2b+cd^2+c^2d)+n]}{2} = \frac{(a+b+c+d)^3}{8}$$

We get,

$$\frac{(a+b+c+d)^6}{64} + (a+b)^3(c+d)^3 = 0$$

$$(a+b+c+d)^6 + 64(a+b)^3(c+d)^3 = 0$$

Taking the third root we get,

$$(a+b+c+d)^3 + \sqrt[3]{64}(a+b)(c+d) = 0$$

The conditions on writing any integer n as a sum of four cubes are

1) $(a+b+c+d)^3 + \sqrt[3]{64}(a+b)(c+d) = 0$
2) $\left(\frac{3(ab^2+a^2b+c^2d)+n}{2}\right)^2 + [(a+b)(c+d)]^3 = 0$
3) $a^3 + b^3 + c^3 + d^3 = n$

Therefore any integer n can be written as a cubic if it satisfied the above equations.

3. conclusion and recommendation

1. Find the values of a, b, c and d exactly to form any integers n
2. Make use of the above results in different application in number theory.
3. Make use of MATLAB software or Maple software to find these values.

4. References

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