B-Spline Functions for Solving n^{th} Order Linear Delay Integro-Differential Equations of Convolution Type

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Abstract

The paper is devoted to solve n^{th} order linear delay integro-differential equations of convolution type (DIDE's-CT) using Galerkin's method with B-spline functions. A new algorithm with the aid of Matlab language is derived to treat three types (retarded, neutral and mixed) of n^{th} order linear DIDE's-CT using Galerkin's method with the aid of B-spline functions and Bool rule for calculating the required integrals for the proposed method where the procedure can be used comparatively greater computational efficiency. Comparison between approximate and exact results has been given in test examples for solving three types of linear DIDE's-CT of different orders for conciliated the accuracy of the approximate results. Finally, the results are arranged in tabulated form and suitable graphing is given for every example.

Key words_: n^{th} Order Linear Delay Integro-Differential Equation of Convolution type, B-spline Functions, Galerkin method and Bool method.

الدوال الثلمة التوصيلية لحل المعادلات التكاملية- التفاضلية التباطؤية الألتفافيــة الخطية من الرتبــة n

الخلاصة

البحث مكرس لحل المعادلات التكاملية التفاضلية التباطؤية الألتفافية الخطية من الرتبة n باستخدام طريقة كالكن مع الدوال الثلمة التوصيلية. حيث تم اشتقاق خوارزمية جديـــدة تمـــت برمجتها بلغة (Matlab) لمعالجة ثلاثة أنواع من المعادلات التكاملية- التفاضـــلية التباطؤيـــة الألتفافية الخطية و المتضمنة (التراجعية ، المتعادلة والمختلطة) من الرتبة n باستخدام طريقة كالكن مع الدوال الثلمة التوصيلية مع استخدام قاعدة بول لحساب التكاملات للطريقة المقترحـــة حيث من الممكن ملاحظة كفاءة الطريقة و دقة الحسابات فيها. كما تمت مقارنة النتائج التقريبية و الحقيقية لثلاثة أنواع من الرتب المختلفة لهذه المعادلات من خلال بعض الأمثلة و الرسوم وقد تم الحصول على نتائج دقيقة.

1. Introduction

One of the most important and applicable subjects of applied mathematics, and in developing modern mathematics is the integral equations. The names of many modern mathematicians notably, Volterra, Fredholm, Cauchy and others are associated with this topic [1].

The name integral equation was introduced by Bois-Reymond in 1888 [2]. However, in 1959 Volterra's book "Theory of Functional and of Integral and Integro-Differential Equations" appeared [1].

The integral and integro-differential equations formulation of physical problems are more elegant and compact than the differential equation formulation, since the

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boundary conditions can be satisfied and embedded in the integral or integro-differential equation. Also the form of the solution to an integro-differential equation is often more stable for today's extremely fast machine computation. Delay integro-differential equation of convolution type has been developed over twenty years ago where one of its types widely is used in control systems and digital communication systems as, lag-lead compensation and spread spectrum designs [1,3].

In this work, B-spline functions were employed with Galerkin method to solve n^{th} order linear (DIDE's-CT) where they are standard representation of smooth geometry in numerical calculations and the required integrals in this method are calculated using Bool rule as well as Gauss elimination method has been used to solve the resulting equations.

To facilitate the presentation of the material that followed, a brief review of some background on the linear DIDE's-CT and their types are given in the following section.

2. Delay Integro-Differential Equation of Convolution Type (DIDE-CT):

The integro-differential equation is an equation involving one (or more) unknown function $y(t)$ together with both differential and integral operations on y. It means that it is an equation containing derivative of the unknown function $y(t)$, which appears outside the integral sign [1,4].

The delay integro-differential equation is a delay differential equation in which the unknown function $y(t)$ can appear under an integral sign [5]. The main difference between delay differential equation and ordinary differential equation is the kind of initial condition that should be used in delay differential equation differs from ordinary differential equation, so that one should specify in delay differential equations an initial functions on some intervals say $[t_0 - t, t_0]$ and then try to find the solution for all $t \geq t_0$ [6,7].

The general form of n^{th} order linear delay integro-differential equation is given by [6,7]:

$$\sum_{i=0}^{n} p_i(t)\frac{d^i y(t)}{dt^i} + \sum_{i=1}^{n} q_i(t)\frac{d^i y(t-t_i)}{dt^i} + \sum_{i=0}^{n} r_i(t)y(t-t_i) = g(t) + l\int_a^{b(t)} k(t,x)\, y(x-t)dx \quad t \in [a, b(t)] \qquad \ldots (1)$$

with initial functions:

$$\left.\begin{aligned} y(t) &= f(t) \\ y'(t) &= f'(t) \\ &\vdots \\ y^{(n-1)}(t) &= f^{(n-1)}(t) \end{aligned}\right\} \quad for \quad t \leq t_0$$

where $g(t), p_i(t), q_i(t), r_i(t), k(t,x)$ are known functions of t, $y(t)$ is the unknown function, l is a scalar parameter (in this work l =1), a and $b(t)$ are the limits of the integral where a is a constant and $b(t)$ either is given constant or function of t and $t, t_0, t_1, \ldots, t_n$ are fixed positive numbers. The integral term of eq.(1) can be classified into different kinds according to the limits of integral and the kernel. If the limit $b(t)$ in eq.(1) is constant $(b(t) = b)$ then equation

(1) is called a delay Fredholm integro-differential equation while if $b(t)=t$ in eq.(1), then eq.(1) is called a delay Volterra integro-differential equation [8,9]. If the kernel $k(t,x)$ in eq.(1) depends only on the difference $t-x$, such a kernel is called a difference kernel and eq.(1) with this kind of kernel is called a delay integro-differential equation of convolution type (DIDE-CT) [4,8]. So, the general form of nth order linear DIDE-CT is given by:

$$\sum_{i=0}^{n} p_i(t)\frac{d^i y(t)}{dt^i}+\sum_{i=1}^{n} q_i(t)\frac{d^i y(t-t_i)}{dt^i}+\sum_{i=0}^{n} r_i(t)y(t-t_i)=g(t)+l\int_a^{b(t)} k(t-x)\,y(x-t)\,dx \qquad t\in[a,b(t)] \qquad \ldots(2)$$

with initial functions:

$$\left.\begin{aligned} y(t)&=f(t)\\ y'(t)&=f'(t)\\ &\vdots\\ y^{(n-1)}(t)&=f^{(n-1)}(t)\end{aligned}\right\} \quad for \quad t\le t_0 .$$

The DIDE-CT is an important equation in many applications. Convolution can be found in various places in applied mathematics since it plays an important role in heat conduction, wave motion, time series analysis, control systems and digital communication systems [5,6].

DIDE's-CT are classified into three types [10, 11]:-

First type:- Equation (2) is called Retarded type if the derivatives of unknown function appear without difference argument (i.e. the delay comes in y only) and the delay appears in the integrand unknown function (i.e. $t \neq 0$).

Second type: Equation (2) is called a neutral type if the highest-order derivative of unknown function appears both with and without difference argument and the delay does not appear in the integrand function (i.e. $t = 0$).

Third type:- All other DIDE's-CT in eq.(2) are called mixed types, which are combination of the previous two types.

3. B-Spline Functions

The nth order B-splines as appropriately scaled nth divided difference of truncated power function; these functions have several mathematical definitions [4].

B-splines were introduced around 1940's in the context of approximation theory [4]. Schoenberg [12] introduced the B-spline in 1949 and B-splines have been applied to geometric modeling since 1970's [4]. According Schoenberg, B-spline means spline basis and the letter B in B-spline stands for basis [4].

Given $t_0, t_1, \ldots, t_m$ knots with $t_0 < t_1 < \ldots < t_m$. Then, the composed of basis B-spline of degree n is: $B(t)=\sum_{i=0}^{m+1} p_i B_{i,n}(t)$ where the pi , i=0,1,…,m+1 are called control points or de Boor points and $t\in(-\infty,\infty)$.

The B-spline of degree n can be defined using the Cox-de Boor recursion formula as [4,12]:

$$\left.\begin{aligned} &B_{k,o}(t)=\begin{cases}1 & if\ t_k\le t<t_{k+1}\\ 0 & otherwise\end{cases}\\ &B_{k,n}(t)=\frac{t-t_k}{t_{k+n}-t_k}B_{k,n-1}(t)+\frac{t_{k+n+1}-t}{t_{k+n+1}-t_{k+1}}B_{k+1,n-1}(t)\\ &\qquad n\ge 1,\ k\ge 0\end{aligned}\right\} \qquad \ldots(3)$$

When the knots are equidistant, the B-spline is said to be uniform otherwise, it is non-uniform [13].

The B-spline can be defined in another way which is [12,14]:

$$B_{k,n}(t)=\binom{n}{k}t^k(1-t)^{n-k} \quad k\ge 0,\, n\ge 0$$

$$,\ t\in(-\infty,\infty) \qquad \dots (4)$$

where $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$.

There are (n+1) n^{th} degree B-spline polynomials for mathematical convenience, we usually set $B_{k,n}(t)=0 \;\; if \;\; k<0 \;\; or \;\; k>n$.

3.1 Some Types of B-Spline Functions [4,12,13]:

3.1.1 The Constant B-spline $B_{k,0}(t)$:

The constant B-spline or B-spline of order 0 is the simplest spline. It is defined on only one knot span and is not even continues on the knots.

$$B_{k,o}(t)=\begin{cases}1 & if\ t_k\le t<t_{k+1}\\ 0 & otherwise\end{cases}$$

3.1.2 The Linear B-spline $B_{k,1}(t)$:

The linear B-spline or the first order of B-spline is defined on two consecutive knot spans and is continues on the knots.

$$B_{k,1}(t)=\begin{cases}\dfrac{t-t_k}{t_{k+1}-t_k} & if\ t_k\le t<t_{k+1}\\[2ex] \dfrac{t_{k+2}-t}{t_{k+2}-t_{k+1}} & if\ t_{k+1}\le t<t_{k+2}\\[2ex] 0 & t\ge t_{k+2}\quad or\quad t<t_k\end{cases}$$

or $B_{0,1}(t)=1-t \quad , \quad B_{1,1}(t)=t$

3.1.3 Quadratic B-spline $B_{k,2}(t)$:

Quadratic B-spline (or the 2[nd] order of B-spline) with uniform knot-vector is a commonly used form of B-spline which is:

$$B_{k,2}(t)=\begin{cases}\dfrac{(t-t_k)^2}{(t_{k+2}-t_k)(t_{k+1}-t_k)} & if\ t_k<t\le t_{k+1}\\[2ex] \dfrac{(t-t_k)(t_{k+2}-t)}{(t_{k+2}-t_k)(t_{k+2}-t_{k+1})}+\dfrac{(t_{k+3}-t)(t-t_{k+1})}{(t_{k+3}-t_{k+1})(t_{k+2}-t_{k+1})} & if\ t_{k+1}<t<t_{k+2}\\[2ex] \dfrac{(t_{k+3}-t)^2}{(t_{k+3}-t_{k+1})(t_{k+3}-t_{k+2})} & if\ t_{k+2}\le t<t_{k+3}\\[2ex] 0 & t\ge t_{k+3}\quad or\quad t\le t_k\end{cases}$$

or

$B_{0,2}(t)=(1-t)^2 \quad , \quad B_{1,2}(t)=2t(1-t)$.

$and \quad B_{2,2}(t)=t^2$

3.1.4 Cubic B-spline $B_{k,3}(t)$:

Cubic B-spline (or the 3[rd] order of B-spline) with uniform knot-vector is the most commonly used form of B-spline which is:

$B_{0,3}(t)=(1-t)^3 \quad , \quad B_{1,3}(t)=3t(1-t)^2,$

$B_{2,3}(t)=3t^2(1-t) \quad and \quad B_{3,3}(t)=t^3$

3.2 Some Properties of B-Spline Functions [12,14]:

3.2.1 The Integration property:

For $k=0,1,\dots,n \;\; and \;\; n\ge 0$:

$$\int_0^1 B_{k,n}(t)=\frac{1}{n+1}$$

3.2.2 The Differentiation property:

The i[th] derivative of B-spline polynomials $B_{k,n}(t)$ is given by:

$$\frac{d^i B_{k,n}(t)}{dt^i} = \frac{n!}{(n-i)!} \sum_{r=0}^{i} (-1)^r \binom{i}{r} B_{k+r-i,n-i}(t)$$

3.2.3 The Product property:

For $n, m \geq 0, \quad i = 0,1,\ldots,n$

$and \quad j = 0,1,\ldots,m$:

$$B_{i,n}(t) * B_{j,m}(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j,n+m}(t) \cdot$$

4. Bool Method:

Bool method is one of basic formula of quadrature approximation methods for integration. It approximates the function on the interval $[t_0, t_4]$ by a curve that possesses through five points. When it is applied over the interval [a,b], the composite Bool rule is obtained as [1,3] :

$$\int_a^b f(t)dt = \frac{2H}{45} \begin{bmatrix} 7f_0 + 32f_1 + 12f_2 + \\ 32f_3 + 14f_4 + 32f_5 + \\ 12f_6 + 32f_7 + \mathbf{L} + \\ 14f_{N-4} + 32f_{N-3} + \\ 12f_{N-2} + 32f_{N-1} + f_N \end{bmatrix} \quad \ldots (5)$$

where a, b are the limit of the integral, $H = \frac{(b-a)}{N}$, N is the number of intervals $([t_0,t_1],[t_1,t_2],\mathbf{K},[t_{N-1},t_N])$ which is the multiple of (4), $f_i = f(t_i)$ $t_0 = a$, $t_N = b$ and $t_i = a + iH$ are called the integration nodes which are lying in the interval [a,b] where $i = 0,1,\mathbf{K},N$.

5. The Solution of n[th] Order Linear DIDE-CT Using Galerkin's Method with B-Spline Functions and Bool Rule:

Galerkin method [15] is one of the most efficient methods used to solve differential and integro-differential equations without time lag. In this section, Galerkin Method with the aid of B-Spline functions and Bool rule are candidates to find the approximated solutions for three types of n[th] order DIDE's-CT as follows:

Recall eq.(2), to solve it the unknown function y(t) is approximated by a set of B-spline functions as:

$$y(t) \cong y_M(t) = \sum_{a=0}^{M} c_a B_{a,M}(t) \quad \ldots (6)$$

where $M > 0 \quad and \; c_0, c_1, \mathbf{K}, c_M$ are (M+1) unknown coefficients.

By substituting eq.(6) into eq.(2) one gets the following formula:

$$\begin{aligned} &\sum_{i=0}^{n} p_i(t) \frac{d^i}{dt^i} \sum_{a=0}^{M} c_a B_{a,M}(t) + \\ &\sum_{i=1}^{n} q_i(t) \frac{d^i}{dt^i} \sum_{a=0}^{M} c_a B_{a,M}(t - t_i) + \\ &\sum_{i=0}^{n} r_i(t) \sum_{a=0}^{M} c_a B_{a,M}(t - t_i) = \\ &g(t) + \int_a^{b(t)} k(t-x) \sum_{a=0}^{M} c_a B_{a,M}(x - t)dx \end{aligned} \quad \ldots (7)$$

Hence, by using B-spline's property (3.2.2) for eq.(7) yields:

$$\sum_{i=0}^{n} p_i(t)\sum_{a=0}^{M} c_a\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+$$
$$\sum_{i=1}^{n} q_i(t)\sum_{a=0}^{M} c_a\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)$$
$$+\sum_{i=0}^{n} r_i(t)\sum_{a=0}^{M} c_a B_{a,M}(t-t_i)-$$
$$\int_a^{b(t)} k(t-x)\sum_{a=0}^{M} c_a B_{a,M}(x-t)dx=g(t)$$

then,

$$\sum_{a=0}^{M} c_a\left[\begin{array}{l}\sum_{i=0}^{n} p_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+\\ \sum_{i=1}^{n} q_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)+\\ \sum_{i=0}^{n} r_i(t)B_{a,M}(t-t_i)-\int_a^{b(t)}k(t-x)B_{a,M}(x-t)dx\end{array}\right]$$
$$=g(t)$$

… (8)

In Galerkin method the unknown coefficients $c_0, c_1, \mathbf{K}, c_M$ in eq.(8) are chosen to minimize the residual equation E_M (t) by setting its weighted integral equal to zero, i.e.

$$\int_D w_j E_M(t)dt=0 \qquad j=0,1,\ldots,M$$

... (9)

where D is a prescribed domain and w_j are weighting functions which are:

$$w_j(t)=\frac{\partial y_M(t)}{\partial c_j}=$$
$$\frac{\partial\sum_{a=0}^{M} c_a B_{a,M}(t)}{\partial c_j}=B_{j,M}(t)$$

$j=0,1,\ldots,M$

… (10)

By substituting eq.(10) into eq.(9) yields:

$$\int_D B_{j,M}(t)E_M(t)dt=0 \qquad j=0,1,\ldots,M$$

… (11)

The residual equation E_M (t) of DIDE-CT is defined by:

$$E_M(t)=$$
$$\sum_{a=0}^{M} c_a\left[\begin{array}{l}\sum_{i=0}^{n} p_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+\\ \sum_{i=1}^{n} q_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)+\\ \sum_{i=0}^{n} r_i(t)B_{a,M}(t-t_i)-\int_a^{b(t)}k(t-x)B_{a,M}(x-t)dx\end{array}\right]-g(t)$$

… (12)

By substituting eq.(12) into eq.(11) we get:

$$\int_D B_{j,M}(t)\sum_{a=0}^{M} c_a\left[\begin{array}{l}\sum_{i=0}^{n} p_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+\\ \sum_{i=1}^{n} q_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)+\\ \sum_{i=0}^{n} r_i(t)B_{a,M}(t-t_i)-\int_a^{b(t)}k(t-x)B_{a,M}(x-t)dx\end{array}\right]dt$$
$$=\int_D B_{j,M}(t)g(t)dt$$

$for\ j=0,1,\ldots,M$.

… (13)

The values required integrals in eq.(13) can be evaluated using Bool method in eq.(5) as follows:

Let

$$y_j(t)=$$
$$B_{j,M}(t)\left[\begin{array}{l}\sum_{i=0}^{n} p_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+\\ \sum_{i=1}^{n} q_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)+\\ \sum_{i=0}^{n} r_i(t)B_{a,M}(t-t_i)-\int_a^{b(t)}k(t-x)B_{a,M}(x-t)dx\end{array}\right]$$

… (14)

for $j=0,1,\ldots,M$ and $a=0,1,\mathbf{K},M$.

then

$$\int_D y_j(t)dt = Bool\big(y_j(t), D, N\big) = \frac{2H}{45}\begin{bmatrix} 7y_j(t_0)+32y_j(t_1)+12y_j(t_2)+ \\ 32y_j(t_3)+\cdots+14y_j(t_{N-4})+ \\ 32y_j(t_{N-3})+12y_j(t_{N-2})+ \\ 32y_j(t_{N-1})+7y_j(t_N) \end{bmatrix} \qquad \ldots(15)$$

and

$$\int_D B_{j,M}(t)g(t)dt = Bool(B_{j,M}(t)g(t), D, N) \qquad \ldots(16)$$

So, by evaluating eq.(13), we have (M+1) simultaneous equations with (M+1) unknown coefficients $c_0, c_1, \ldots, c_M$.

Hence, eq.(13) can be written in matrices form as DC=G which they:

$$\left.\begin{aligned} D &= \begin{bmatrix} d_{00} & d_{01} & \ldots & d_{0M} \\ d_{10} & d_{11} & \ldots & d_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ d_{M0} & d_{M1} & \ldots & d_{MM} \end{bmatrix}, \\ C &= \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_M \end{bmatrix} \quad and \\ G &= \begin{bmatrix} Bool(B_{0,M}(t)g(t), D, N) \\ Bool(B_{1,M}(t)g(t), D, N) \\ \vdots \\ Bool(B_{M,M}(t)g(t), D, N) \end{bmatrix} \end{aligned}\right\} \qquad (\ldots 17)$$

where

$$d_{j,a} = Bool\left(B_{j,M}(t)\begin{bmatrix} \sum_{i=0}^{n} p_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t)\right)+ \\ \sum_{i=1}^{n} q_i(t)\left(\frac{M!}{(M-i)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}B_{a+r-i,M-i}(t-t_i)\right)+ \\ \sum_{i=0}^{n} r_i(t)B_{a,M}(t-t_i)-\int_a^{b(t)}k(t-x)B_{a,M}(x-t)dx \end{bmatrix}, D, N \right)$$

for $j = 0,1,\ldots,M$ and $a = 0,1,\ldots,M$. Then, use Gauss elimination method to find the coefficients c_a's , $a = 0,1,\ldots,M$ which satisfy eq.(6) (the approximate solution $y_M(t)$ of eq.(2)).

The solution of three types n[th] order linear DIDE's-CT using Galerkin method with B-Spline functions and Bool method can be summarized by the following algorithm:

DIDECT-GBSB Algorithm :

INPUT

- n : (the order of DIDE-CT).
- N: (the number of intervals of Bool method)
- M: (the order of B-spline function $B_{a,M}(t)$).
- The limits of the integral a & b .
- The function g(t) of DIDE-CT.
- The difference kernel of the DIDE-CT.

OUTPUT

- c_a's , $a = 0,1,\ldots,M$: (the unknown coefficients of eq.(6)).
- $y_M(t)$: (the approximate solution of DIDE-CT)

Step 1: Set $y_M(t) = \sum_{a=0}^{M} c_a B_{a,M}(t)$

Step 2: Define $y_j(t)$ in eq.(14).

Step 3: Compute B-splines

$B_{a,M}(t), a = 0,1,\ldots,M$ in (step 1) as:

(a) Set $a = 0$

(b) For i = a : M do step (c)

(c) Sum=Sum+ $(-1)^{(i-a)}\binom{M}{i}\binom{i}{a}t^i$

(d) $B_{a,M}(t) = \text{Sum}$

(e) Set $a = a + 1$

(f) If $a = M + 1$ then stop and go to (step 4)

Else go to step (b)

Step 4: Set $j = 0$

Step 5: Compute eq.(13)

Step 6: Put j = j+1

Step 7: If j = M+1 then stop and go to (step 8).
Else go to (step 5)

Step 8: Find B-spline functions in (step 5) using eq.(4).

Step 9: Express the (M+1) simultaneous equations in step(5) by matrices form DC=G as eq.(17).

Step 10: Use Gauss elimination method for finding the coefficients c_a's , $a = 0,1,\ldots,M$ which satisfy the solution $y_M(t)$ in (step 1).

6. Text Examples:

Example (1):

Consider the following 1st order linear Retarded Volterra integro-differential equation of convolution type:

$$\frac{dy(t)}{dt} + t\,y(t-\tfrac{1}{2}) = \left(1 + t + t^2\right) + \int_0^t e^{(t-x)}\,y(x-\tfrac{1}{2})\,dx \qquad 0 \le t \le 0.5 \qquad \ldots (18)$$

with initial function:

$$y(t) = t + \tfrac{1}{2} \qquad -0.5 \le t \le 0.$$

The exact solution of eq.(18) is:

$$y(t) = e^t - \tfrac{1}{2} \qquad 0 \le t \le 0.5 .$$

Assume the approximate solution of eq.(18) in the form:

$$y_M(t) = \sum_{a=1}^{M} c_a B_{a,M}(t)$$

When the algorithm (DIDECT-GBSB) is applied, table (1) presents the comparison results between the exact and Galerkin with B-Spline functions for eq.(18) depending on least square error (L.S.E.) where m=10, h=0.05, $t_j = jh$, $j = 0,1,\ldots,m$.

Figure (1) shows the solution of eq.(18) using Galerkin with B-Spline functions and the exact solution.

Example (2):

Consider the following 2nd order neutral Volterra integro-differential equation of convolution type:

$$\frac{d^2y(t-1)}{dt^2} + \frac{dy(t-0.5)}{dt} = \left(-6\sin t - t^3 + 3t^2 + 9t - \frac{21}{4}\right) + \int_0^t \sin(t-x)\,y(x)\,dx \qquad 0 \le t \le 1 \qquad \ldots (19)$$

with initial functions:

$$y(t) = t^3 \qquad t \le 0$$
$$y'(t) = 3t^2 \qquad t \le 0$$

The exact solution of eq.(19) is:

$$y(t) = t^3 \qquad 0 \le t \le 1 .$$

Assume the approximate solution of eq.(19) in the form:

$$y_4(t) = \sum_{a=0}^{4} c_a B_{a,4}(t)$$

When the algorithm (DIDECT-GBSB) is applied, table (2) presents the comparison between the exact and approximated solutions for eq.(19) using Galerkin with B-spline functions for m=10, h=0.1, $t_j = jh$, $j = 0,1,...,m$ with least square error (L.S.E.).

Figure (2) shows the solution of eq.(19) using Galerkin with B-Spline functions and the exact solution.

Example (3):

Consider the following 3rd order mixed Fredholm integro-differential equation of convolution type:

$$\frac{d^3 y(t-1)}{dt^3} - y(t) = \left(-t^4 + \frac{119}{5}t - \frac{719}{30}\right) + \int_0^1 (t-x)\,y(x-1)\,dx \qquad 0 \le t \le 1 \qquad \dots (20)$$

with initial functions :

$$\left.\begin{aligned} y(t) &= t^4 \\ y'(t) &= 4t^3 \\ y''(t) &= 12t^2 \end{aligned}\right\} \quad -1 \le t \le 0 \cdot$$

The exact solution of eq.(20) is:

$$y(t) = t^4 \qquad 0 \le t \le 1$$

Assume the approximate solution of eq.(20) in the form:

$$y_5(t) = \sum_{a=0}^{5} c_a B_{a,5}(t)$$

When the algorithm (DIDECT-GBSB) is applied, table (3) presents the comparison between the exact and approximate solutions of eq.(20) using Galerkin with B-spline functions for m=10, h=0.1, $t_j = jh$, $j = 0,1,...,m$ depending on least square error (L.S.E.).

Figure (3) shows the solution of eq.(20) by using Galerkin with B-Spline functions and the exact solution.

Conclusions

Galerkin method with the aid of B-Spline functions and Bool method have been presented to find the approximated solutions for three types (retarded, neutral and mixed) of n[th] order linear DIDE's-CT. The results show a marked improvement in the least square error (L.S.E.). From solving three test examples, the following points are drawn:

1. Galerkin method with B-spline functions proved their effectiveness in solving n[th] order linear DIDE's-CT where they give accuracy to results of DIDE's-CT.
2. Galerkin method with B-spline functions and Bool method give qualified way for solving 1st order linear DIDE's-CT as well as n[th] order linear DIDE-CT
3. The good approximation of Bool method depends on the size of H, if H is decreased then the number of points (nodes) increases and the L.S.E. approaches to zero

where this gives the advantage in numerical computation.

4. The good approximation solution depends on the number M of B-spline functions where as M increased, the error term approaches to zero.

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Table (1) The solution of Ex.(1).

t	Exact	Galerkin with B-Splines (DIDECT-GBSB) $y_M(t)$	
		M=2	*M=3*
0	0.5000	0.5000	0.5000
0.05	0.5513	0.5512	0.5508
0.10	0.6052	0.6050	0.6046
0.15	0.6618	0.6617	0.6615
0.20	0.7214	0.7212	0.7215
0.25	0.7840	0.7837	0.7845
0.30	0.8499	0.8493	0.8498
0.35	0.9191	0.9179	0.9195
0.40	0.9918	0.9898	0.9917
0.45	1.0683	1.0650	1.0684
0.50	1.1487	1.1435	1.1459
L.S.E.		4.3466e-5	1.1232e-5

Table (2) The solution of Ex.(2).

t	Exact	Galerkin with B-Splines $y_4(t)$
0	0.0000	0.0000
0.1	0.0010	0.0010
0.2	0.0080	0.0080
0.3	0.0270	0.0270
0.4	0.0640	0.0640
0.5	0.1250	0.1250
0.6	0.2160	0.2160
0.7	0.3430	0.3430
0.8	0.5120	0.5120
0.9	0.7290	0.7290
1	1.0000	1.0000
L.S.E.		0.0000

Table (3) The solution of Ex.(3).

t	Exact	Galerkin with B-Spline
0	0.0000	0.0000
0.1	0.0001	0.0001
0.2	0.0016	0.0016
0.3	0.0081	0.0081
0.4	0.0256	0.0256
0.5	0.0625	0.0625
0.6	0.1296	0.1296
0.7	0.2401	0.2401
0.8	0.4096	0.4096
0.9	0.6561	0.6561
1	1.0000	1.0000
L.S.E.		0.0000

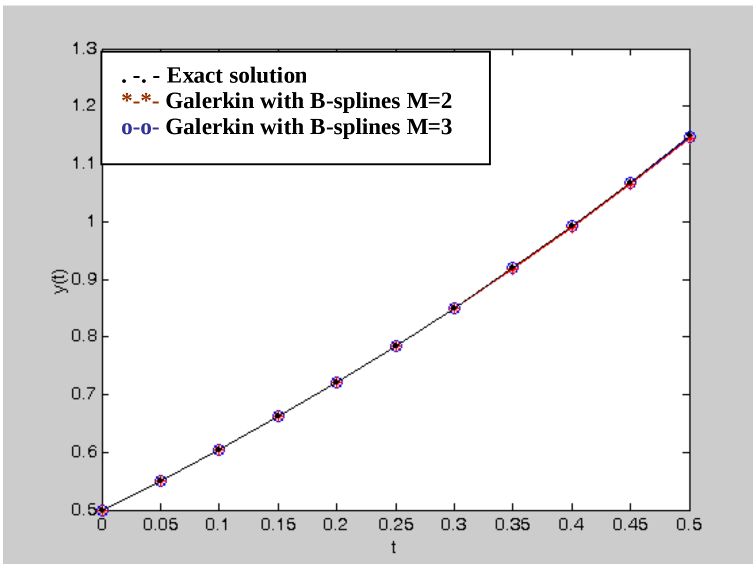

Figure (1) The comparison between the exact and Galerkin with B-Spline functions for eq.(18) in Ex.(1).

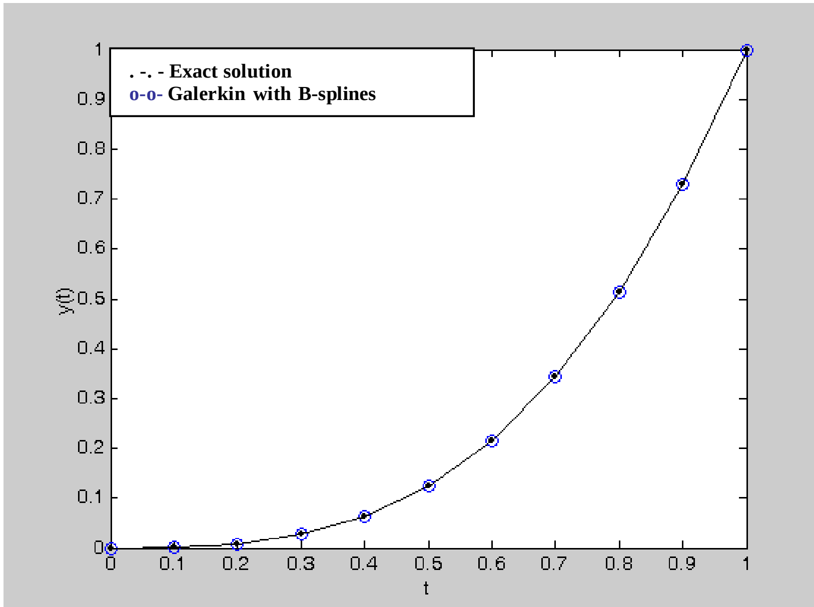

Figure (2) The comparison between the exact and Galerkin with B-Spline functions for eq.(19) in Ex.(2)

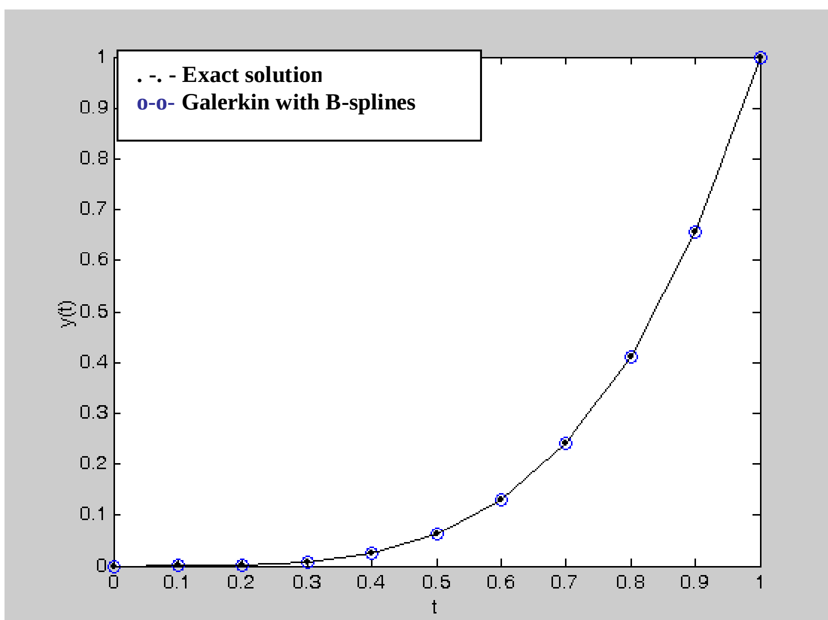

Figure (3) The comparison between the exact and Galerkin with B-Spline functions for eq.(20) in Ex.(3)