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Lagrange Interpolation Polynomial in Special Regions A Review

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مراجعة: متعددة اندراج لاكرانج في مناطق خاصة

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المديرية العامة لتربية–وزارة التربية

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ABSTRACT

It is known that Lagrange interpolation polynomial is considered as one of important tools to approximate the chosen function by a polynomial of degree n with knowing points that belong in it. Also, amount of error between the original function and this polynomial is known in books of approximation and numerical analysis. But studying the effects and amount of error of this polynomial in convex and compact regions has not been studied before. In this article, we will study in some detail the behavior of the Lagrange interpolation polynomial in such regions in terms of the form and its characteristics and the extent of the difference between this polynomial and the function in question. We will notice that this polynomial will be turned into a convex function when applied to a function in a convex region, and the degree of error or the amount of difference between it and the original function gets smaller if we apply our polynomial on the desired function in a convex region. The same is true if the effect of this polynomial is applied on a function in a compact region, and notice that the amount of difference between it and this function is reduced to zero if n goes to infinity.

Key words: Lagrange Interpolation Polynomial, convex set, compact set, convex function, concave function.

مجلـــة جـــامعة بـــابل للعلـــوم الصـــرفة والتطـــبيقية مـــجلة جـــامعة بـــابل للعلـــوم الصـــرفة والتطـــبيقية مـــجلة جـــامعة بـــابل
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1.INTRODUCTION

Lagrange interpolation polynomial is regarded as one of the most important polynomials used in the subject of polynomial interpolation. It has been studied in many books and researches such as **[1]** ,**[2]** ,**[3]** ,**[4[**…etc. but it is not studied in a convex set :moreover, it is not studied in a compact set. In this article we will study a behaver of this polynomial in terms of its form and the extent of the difference between it and the chosen function, and it is appropriate here to mention that in this article, we Symbolize to Lagrange interpolation polynomial with the symbol (L.i.p) for the sake of brevity. The general formula for this polynomial is $L_n(x) = \sum_{i=0}^{n-1} f(x_i) \prod_{j=0}^{n-1} \frac{x-x_i}{x_j-x_i}, i \neq j$ **[1]**, **[5]** as well as .It is clear here, and as we will demonstrate in the papers of this article, some of the properties of this polynomial will change when applied on a function in special regions such as the convex and compact regions. Here it is appropriate to mention that the set say A will be convex if for all $x, y \in A$, $\lambda x + (1-\lambda)y \in A, x, y \in A$ and $\lambda \in [0,1]$, Also we have the right to ask the first question, which is that, Is it possible for the form of the Lagrange interpolation polynomial to change when applied on a function in a convex region? Before answering this question, it is appropriate here to mention the definition of the convex function, which is that the function $f; X \to R$ will be a convex function **[6]** if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2), \forall x_1, x_2 \in X , \lambda \in [0,1]$. Now the answer to our first question is yes, the form of this polynomial (i.e. (L.i.p)) will change when applied it on a function in a convex region, as we will notice in our article, where (L.i.p) becomes, as we will prove later, a convex function. Another question related to this topic should also be asked here: Is the difference between our polynomial and the given function affected or changed when the function is in a convex region? The answer to our second question here is also yes, the difference between the function given in a convex region and our mentioned polynomial will be smaller, and this expresses a characteristic and feature very well, as we will notice. Now, with regard to the behavior of the Lagrange interpolation polynomial in the compact region, as it noticed in our study, the difference between it and the given function will diminish little by little as the degree of this polynomial increases until that difference reaches to zero when the degree of our polynomial goes to infinity as we will notice.

MATERIALS AND METHODS

2.Auxiliary Results

In this part, we study behaver of Lagrange interpolation polynomial (L.i.p.) in the convex set (which, as we mentioned previously, was discussed in primary sources up to advanced sources such as **[7])**, and we will prove that (L.i.p.) will become a convex function, this is what we will notice in theorem 2.2. Now, since every element in the convex set ,say A ,can be written as $\sum_{i=1}^{n-1} \lambda_i x_i \in A, \sum_{i=1}^{n-1} \lambda_i = 1$so, we can continue increasing until to the case that n going to infinity, and with knowing that the set A is a compact set, so what effect does it have here? the answer of this question is while the sum of the infinity numbers is infinite, here when it is belong in a convex set say A, this sum will be belong in this set, as we will see in theorem 2.4.

Now, the following theorem, as stated from its reference, concerns one of the equivalents of the convex function, which we will use to prove the convexity of the Lagrange interpolation polynomial.

2.1 Proposition:[8]

Let f be defined on I. Then f is convex if and only if for $x < y < z$ in I, then: $\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}$.

2.2 Theorem: (convexity of (L.i.p.) in a convex set)

Suppose that A be a convex set and let $\lambda \in [0,1]$ then (L.i.p.) is convex function (i.e. $L_n(\sum_1^n \lambda_i x_i) \leq \sum_1^n \lambda_i L_n(x_i)$).

Proof:

Since A is a convex set and let $x, z \in A$ such that $x \leq z$

By hypothesis of the convex set $\lambda x + (1-\lambda)z \in A$

Suppose that $\lambda x + (1-\lambda)z = y$,it is clear that y is located between x and y

So, $x \leq y \leq z$

As above, the general formula for (L.i.p.) is $L_n(x) = \sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{x-x_i}{x_j-x_i}, i \neq j$

$$\frac{L_n(y)-L_n(x)}{y-x} = \frac{\sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{y-x_i}{x_j-x_i} - \sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{x-x_i}{x_j-x_i}}{y-x}$$

Since $\lambda x + (1-\lambda)z = y$ so $x = \frac{y-(1-\lambda)z}{\lambda}$ and then the last term becomes:

$$= \frac{\sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{y-x_i}{x_j-x_i} - \sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{\frac{y-(1-\lambda)z}{\lambda} - x_i}{x_j-x_i}}{\lambda x + (1-\lambda)z - x}$$

Since $z > y$ then the above term is:

$$\leq \frac{\sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{z-x_i}{x_j-x_i} - \sum_{i=1}^n f(x_i) \prod_{j=1}^n \frac{\frac{y-(1-\lambda)z}{\lambda} - x_i}{x_j-x_i}}{\lambda x + z - \lambda z - x}$$

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Since $z > y$ then $y \geq y - (1-\lambda)z$ so, the last term becomes:

$$\leq \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{\frac{y}{\lambda} - x_i}{x_j - x_i}}{\lambda x + z - \lambda z - x}$$

Since $\lambda < 1$ so we can write it as $\lambda = \frac{a}{b}$ such that $b > a$ so, $\frac{y}{\lambda} > y$, because the aforementioned result is in the numerator and because every fraction is preceded by a negative sign, that is (a subtraction operation), we obtain:

$$\leq \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{\lambda(x - z) + (z - x)}$$

Since $z > y > x$ and $\lambda(x-z) + (z-x) = \lambda x - \lambda z + z - x = x(\lambda - 1) + z(1-\lambda) > y(\lambda - 1) + z(1-\lambda) = \lambda(y-z) + (z-y)$ (note that $\lambda - 1 < 0$) and since the result is in the denominator, So we have:

$$\leq \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{\lambda(y - z) + (z - y)}$$

$$= \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{-\lambda(z - y) + (z - y)}$$

Since $z \geq y$ then $z - y \geq 0$ and $-\lambda(z-y) \leq 0$ and since this amount is in the denominator then,

$$\leq \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{-\lambda(z - y)}$$

Since $0 < \lambda < 1$ then $-\lambda \leq 0$ so we can write $\lambda = \frac{a}{b}$ such that $b > a$ and then $-\frac{b}{a} < -1$, so we will get the following:

$$= \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{-\frac{a}{b}(z - y)}$$

$$= -\frac{b}{a}\left(\frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{z - y}\right)$$

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$$\leq \frac{\sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{z - x_i}{x_j - x_i} - \sum_{i=1}^{n} f(x_i) \prod_{j=1}^{n} \frac{y - x_i}{x_j - x_i}}{z - y} = \frac{L_n(z) - L_n(y)}{z - y}$$

So, (L.i.p.) is a convex polynomial.

2. 3Theorem:[9].[10]

Let (X, d) be a metric space and let $K \subset X$ such that k is a compact

set. Then the following are equivalent:

(i) Every sequence in K has a Cauchy subsequence. (ii) K is totally bounded.

By the above theorem, we can proof the following theorem:

2.4 Theorem:

Let A be a convex and compact set then the infinite sum of elements of the set A is located (belongs) in A.

Proof:

Let $x_1, x_2 \in A$, since A is a convex set then $x_3 = \lambda x_1 + (1 - \lambda)x_2 \in A, \lambda \in [0,1]$

Again, since A is a convex set $x_4 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \in A$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$

And so on, $x_n = \sum_{i=1}^{n} \lambda_i x_i \in A$ where $\sum_{i=1}^{n} \lambda_i = 1$

Now, if $n \longrightarrow \infty$ then, number of elements is become infinity

Now, since A is a compact set every infinite set in A has a limit pointe.

So, $\lim_{n \to \infty} (\sum_{i=1}^{n} \lambda_i x_i) \in A$ And suppose that $\lim_{n \to \infty} (\sum_{i=1}^{n} \lambda_i x_i) = y$

Again, since A is a convex set then $y = \lim_{n \to \infty} (\sum_{i=1}^{n} \lambda_i x_i) \in A$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$ And the proof is complete.

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RESULTS AND DISCUSSION

3.Main Results:

In this part, we will focus on the change in the difference between the original function and the Lagrange interpolation polynomial when applying this polynomial in the convex and compact regions. We will notice that the difference will be smaller in these two regions than if it was applied in regions other than these two regions.

3.1 <u>Theorem:</u>[1],[10]

Let $[a, b]$ be any interval which contains all $n + 1$ points $x_0, x_1, \ldots\ldots, x_n$ Let $f, f', \ldots. f^{(n)}$ exist and be continuous on $[a, b]$ and let $f^{(n+1)}$ exist for $a < x < b$. Then, given any $x \in [a, b]$, there exists a number ξ (depending on x) in (a, b) such that: $f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \ldots.\ (x - x_n)$

3.2 <u>Theorem</u>:

Suppose that A be a convex set and let $x_0, x_1, \ldots\ldots, x_{n-1} \in A$ such that, f defined on the set $x_0, x_1, \ldots\ldots, x_{n-1}$ then the difference between $L_n(x)$, which is defined on the set A and the function f is less in the convex set than in the non-convex set .

<u>Proof:</u>

First, let us take $x_0, x_1 \in A, x_0 > 0$

Since A is a convex set then, let $x_2 = \lambda x_0 + (1 - \lambda)x_1 \in A$ and x_2 between x_0, x_1.

So, $x_0 = \frac{x_2 - (1-\lambda)x_1}{\lambda}$ and $x_1 = \frac{x_2 - \lambda x_0}{1-\lambda}$

Now, let we take the new forms of x_0 and x_1 which get it from its set A which is a convex set and see the difference between it and any value of x:

$$\left(x - \frac{x_2 - (1-\lambda)x_1}{\lambda}\right).\left(x - \frac{x_2 - \lambda x_0}{1-\lambda}\right) = \left(\frac{\lambda x - x_2 + (1-\lambda)x_1}{\lambda}\right).\left(\frac{(1-\lambda)x - x_2 + \lambda x_0}{1-\lambda}\right)$$

Since $-x_2$ is negative, then the last term becomes:

$$\leq \left(\frac{\lambda x + (1-\lambda)x_1}{\lambda}\right).\left(\frac{(1-\lambda)x + \lambda x_0}{1-\lambda}\right) = \left(\frac{\lambda x + x_1 - \lambda x_1}{\lambda}\right).\left(\frac{x - \lambda x + \lambda x_0}{1-\lambda}\right)$$

$$= \left(\frac{\lambda x}{\lambda} - \frac{\lambda x_1 - x_1}{\lambda}\right).\left(\frac{x}{1-\lambda} - \frac{\lambda x - \lambda x_0}{1-\lambda}\right)$$

$$= -\left(\frac{-\lambda x}{\lambda} + \frac{\lambda x_1 - x_1}{\lambda}\right).-\left(\frac{\lambda x - \lambda x_0}{1-\lambda} - \frac{x}{1-\lambda}\right)$$

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$$= \left((x_1 - x) - \frac{x_1}{\lambda}\right).\left(\frac{\lambda}{1-\lambda}(x - x_0) - \frac{x}{1-\lambda}\right)$$

Since $1 - \lambda > \lambda$ then $\frac{\lambda}{1-\lambda} < 1$, then the last term becomes:

$$\leq \left((x_1 - x) - \frac{x_1}{\lambda}\right).\left((x - x_0) - \frac{x}{1-\lambda}\right)$$

Since $\frac{x_1}{\lambda}$ and $\frac{x}{1-\lambda}$ are negative, then the last term becomes:

$$\leq (x_1 - x).(x - x_0)$$

$$= -(x - x_1).(x - x_0)$$

$$\leq (x - x_1).(x - x_0)$$

This proof shows clearly that for every two points in the convex region, under the property of convexity, a third point will be produced located between those two points which gives them a new form, and the difference between an arbitrary point x and those original two points will be smaller in the convex region because of existence a third point between the original points, as in the proof above.

If we repeated this proof with three points, four, or more, we would have obtained the same result

So, $(x - x_1).(x - x_0) \ldots..(x - x_n)$ in a convex set $\leq$ $(x - x_1).(x - x_0) \ldots..(x - x_n)$ in a non-convex set.

So, by theorem 3.1, $f(x) - L_n(x)$is smaller than in convex set, and the proof is complete.

3.3 Theorem:

Suppose that A be a compact set and let $x_0, x_1, \ldots \ldots., x_{n-1} \in A$ such that f defined on the set $x_0, x_1, \ldots \ldots., x_{n-1}$which is located in A,then (L.i.p.) (i.e. $L_n(x)$) Which defined on the set A is equal to the function f.

Proof:

Suppose that $L_0(x)$ be (L.i.p.) which define on the point x_0.

And $L_1(x)$ be (L.i.p.) which defines on the point x_0, x_1

$L_2(x)$ be (L.i.p.) which defines on the point x_0, x_1, x_2 And so on

So, we have a sequence of polynomials: $L_0(x), L_1(x), ., L_n(x), \ldots.$ located in set A.

Since A is a compact set then by theorem 2.3 this sequence is convergent in A

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This means we get a one polynomial only (i.e. $\lim_{n\to\infty} L_n(x) = L_m(x)\ \ , m > n$)

Now, by theorem 3.1 we have:

$$f\ (x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ ....\ (x - x_n)$$

since n in the denominator, it is clear that right-hand side goes to zero if $n \longrightarrow \infty$

So, this sequence is convergent and then $f\ (x) - L_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$

So, $\lim_{n\to\infty} L_n(x) = f(x)$.

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Conflict of interests.

Non conflict of interest

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الخلاصة

من المعلوم أن متعددة إندراج لاكرانج تعد إحدى الأدوات المهمة لتقريب دالة ما بمتعددة حدود من الدرجة n بمعلومية نقاط تنتمي الى تلك الدالة وأن درجة (مقدار الخطأ) بين الدالة الاصلية وهذه المتعددة معروف في كتب التقريب والتحليل العددي لكن دراسة تأثير ودرجة الخطأ لهذه المتعددة على دالة ما في مناطق خاصة مثل المنطقة المحدبة والمرصوصة لم يدرس من قبل. في هذه المقالة سوف ندرس بشيء من التفصيل سلوك متعددة اندراج لاكرانج في مثل هذه المناطق من حيث شكل تلك المتعددة وصفاتها وعلى مقدار الفرق بينها وبين الدالة المعنية وسوف نلاحظ ان هذه المتعددة تتحول الى دالة محدبة عند تطبيقها على دالة ما في منطقة محدبة وأن درجة الخطأ أو مقدار الفرق بينها وبين الدالة الاصلية يكون اصغر فيما لو طبقت هذه المتعددة على دالة ما في تلك المنطقة المحدبة ،كذلك الحال فيما لو طبق تأثير هذه المتعددة على داله ما في منطقة مرصوصة حيث وكما سوف نلاحظ ان مقدار الفرق بينها وبين الدالة المطبق تأثرها عليها يؤول الى الصفر مع ذهاب درجة المتعددة (n) الى المالانهاية .

الكلمات المفتاحية: متعددة اندراج لاكرانج، المجموعة المحدبة، المجموعة المرصوصة، الدالة المحدبة، المجموعة المقعرة.