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Using Variable Iterative Method to Calculate the Numerical Approximations Solution for Fractional Differential Equations

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استخدام الطريقة التكرارية المتغيرة لحساب حل التقريبات العددية للمعادلات التفاضلية الكسرية

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ABSTRACT

Background:

Fractional calculus is an area of mathematics that discusses the research and applications of non-integer calculus in ℝ or ℂ. I present the variable iterative method for calculating numerical approximations to solve classical differential equations that was developed to solve linear fractional differential equations.

Materials and Methods:

Were studied using the variable iterative method, where fractional derivatives according to Caputo's derivative were used. Some examples were studied to test the accuracy of the results we obtained. This comparison was presented using the curves for these examples used to compare the absolute error of the approximate solution and the exact solution.

Results:

The results of Table (3) prove that there is acceptable agreement between the approximate and exact solution for Example (2). when n is larger, the closer the approximate solution approaching. Through the results of the three example tables, the effectiveness of numerical methods in obtain acceptable results demonstrated.

Conclusion:

When studying the examples given in the research, when using the variable iterative method and through examples and tables. I noticed the effectiveness of the variable iterative method when used, especially in the second example in the case $\alpha = 0.9$

Key words:

Caputo's derivative, Lagrange factorial, Linear fractional differential equation, The_variational iteration method.

INTRODUCTION

In this research, numerical approximations for solving fractional differential equations for formula:

$$ {}^{c}D_0^{\alpha} - \alpha u(t) = g(t); 0 < \alpha < 1; u(0) = 0; t > 0 $$

Fractional differential equations and fractional integral equations are important because they are used to model many practical problems, such as electromagnetic waves, viscosity, and diffusion equations. Were studied using the variable iterative method, where fractional derivatives according to Caputo's concept were used. I concluded the research by studying numerical examples using the variable iterative method.

Compare the exact solution with the approximate solution, and present this comparison with the help of graphs and error tables.

The variational iteration method

The VIM method is to improve for the Lagrange factorial method. We consider the following differential equation:

$$ L(u) + N(u) = g(x) $$

Where L is a linear operator defined as N , $L = \frac{d^m}{dt^m}; m \in N^*$ a nonlinear effect, g is a known function.

The initial conditions are defined as follows:

$$ u^{(k)}(0) = c_k; k = 0, 1, \dots m - 1 $$

where ck are real coefficients. We construct a corrective function through the following variational iterative method:

$$ U_{n+1} = U_n + \int_0^t \lambda(t)(L(u_n(t) + N\breve{u}_n(t) - g(t))dt \qquad \dots\dots (1) $$

Where λ is the generalized Lagrange factorial, $u_n(t)$ it represents the approximation of the rank n, $\breve{u}_n$ residual change meaning $\delta\breve{u}_n(t) = 0$

The (VIM) is based on the following two steps

First: Determine Lagrange factorial:

Determine the Lagrange factorial λ which may be optimally defined by integral by piecemeal and using the residual variation. After defining λ, the variable formula without residual variation is used to define the approximation $u_{n+1}(x); n \geq 0$ for solution $U(x)$.

Zero can be approximation for any chosen function, we use initial values $u(0), u'(0), u''(0), \dots, u^{(k)}(0)$ to round the chosen zero u_0

in the end, the exact solution is $\lim_{n\to\infty} U_n = U(x)$

We have two cases:

if $m = 1, \lambda(t) = -1; u_0(x) = u(0)$

and if $m = 2, \lambda(t) = t - x;\ u_0(x) = u(0) + xu'(0)$

In the general case:

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$$\lambda(t) = \frac{(-1)^m}{(m-1)^!} (t-x)^{m-1}, u_0(x) = \sum_{k=0}^{m-1} \frac{c_k}{k!} x^k;\ c_k = u^{(k)}(0) \qquad \ldots\ldots (2)$$

Second: Determine the iterative formula:

$$U_{n+1}(x) = U_n(x) + \int_0^x \frac{(-1)^m}{(m-1)^!} (t-x)^{m-1}[L(u_n(t) + N\breve{u}_n(t) - g(t)]dt \qquad \ldots\ldots. (3)$$

Convergence was studied using the variable iteration method for fractional differential equations using Caputo's derivative.

Let the following issue be:

$$\begin{cases} ({}^cD_0^\alpha\, y)\,(t) = f\big(t, y(t)\big); n-1 < \alpha \le n \\ \qquad y^{(k)}(0) = y_0^k, k = 0, 1, \ldots, n-1 \end{cases} \qquad \ldots\ldots (4)$$

Where $y^{(k)}(t), t \in [a,b]$ The derivative of order k for $y(t)$ and $f: [0\,, T] \times R \to R$ is satisfy Lipschitz condition:

$$|f(t,u_1) - f(t,u_2)\,| \le L\,|u_1 - u_2|;\ t > 0, u_1, u_2 \in R$$

Where L is constant of Lipschitz

We define $\|y\|_\infty = \max\limits_{0\le t\le T} |y(t)|$

$({}^cD_0^\alpha\, y)\,(t)$ Caputo's fractional derivative for real numbers α where $n-1 < \alpha \le n$

We define Caputo's fractional derivative from order α to function $f(t)$on the domain [a , b] as follow:

$${}^cD_0^\alpha + f(t) = I_{a+}^{n-a}(D^n f)(\mathrm{x}) = \frac{1}{\Gamma(n-a)} \int_a^x (x-t)^{(n-a-1)} f^{(n)}\, dt \qquad \ldots\ldots (5)$$

Diethelm and others proved that the problem is equivalent to the following Volterra integral equation:

$$y(t) = \sum_{j=0}^{[a]-1} \frac{t^j}{j!}\, {y_0}^{(j)} + \frac{1}{\Gamma(a)} \int_0^t (t-T)^{(a-1)} f\big(T, y(T)\big)dT \qquad \ldots\ldots (6)$$

We put $g(t) = \sum_{j=0}^{[a]-1} \frac{t^j}{j!}\, {y_0}^{(j)}$ We transform the problem into the form:

$$y(t) = g(t) + \frac{1}{\Gamma(a)} \int_0^t (t-T)^{(a-1)} f\big(T, y(T)\big)dT$$

Through the idea of the variable iterative method of the equation as follows:

$$y_{n+1}(t) = g(t) + \frac{1}{\Gamma(a)} \int_0^t (t-T)^{(a-1)} f\big(T, y_n(T)\big)dT \qquad \ldots.. (7)$$

We use initial value

$$y_0(t) = y_0 + y_1 t + y_2\, t^2 + y_3\, t^3 + \cdots + y_{n-1}\, t^{n-1}$$

Then we start repeating. The value y_n is the iteration of order n, approaching the exact solution of the problem by

$$y = \lim_{n\to\infty} y_n\,|$$

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Examples with exact solution of fractional differential equations

The approximate solution was searched using the variable iterative method, and then the exact solution was compared with the approximate solution in each case through graphical representations and a table of absolute errors.

Example (1) Let us have the following fractional differential equation:

$$\begin{cases} D^{\alpha}u(x) + xu = 1.50451\, x^{\frac{3}{2}} + x^3 + x \;; 0 \le x \le 1, \quad 0 < \alpha < 1 \\ u(0) = 1 \end{cases} \qquad \ldots\ldots (8)$$

The exact solution is $u(x) = x^2 + 1$

, $\alpha = 0.5$

Table 1. Numerical solution using the variable iterative method (VIM) in the cases: $n = 10$, $n = 20$, $n = 30$, When $\alpha = 0.5$

xi	Sol Exact	Sol Ap n = 10	Sol Ap n = 20	Sol Ap n = 30
0	1	1	1	1
0.1	1.01	1.00913	1.01003	1.01002
0.2	1.04	1.03968	1.0401	1.03999
0.3	1.09	1.09094	1.08996	1.08999
0.4	1.16	1.16179	1.15987	1.16001
0.5	1.25	1.2516	1.24993	1.25001
0.6	1.36	1.36031	1.36007	1.35999
0.7	1.49	1.48837	1.49018	1.48998
0.8	1.64	1.63659	1.64015	1.63999
0.9	1.81	1.80586	1.80998	1.81001
1	2	1.99689	1.99978	2.00002

Table 2. Estimation of error in the cases: $n = 10$, $n = 20$, $n = 30$ and $\alpha = 0.5$

xi	Err n = 10	Err n = 20	Err n = 30
0	0	0	0
0.1	0.00087339	0.0000268	0.000018595
0.2	0.00032350	0.0000268	0.000014884
0.3	0.00094001	0.0000963	6.167627960E-06
0.4	0.00179166	0.0001277	0.00001422133
0.5	0.001600156	0.0000733	0.00001052215
0.6	0.000307642	0.0000698	8.47748806109E-06
0.7	0.001630798	0.0001757	0.0000179098
0.8	0.003411330	0.0001470	6.404158311E-06
0.9	0.004135065	0.00001940	0.0000149950
1	0.0031139520	0.00022468	0.000024357367

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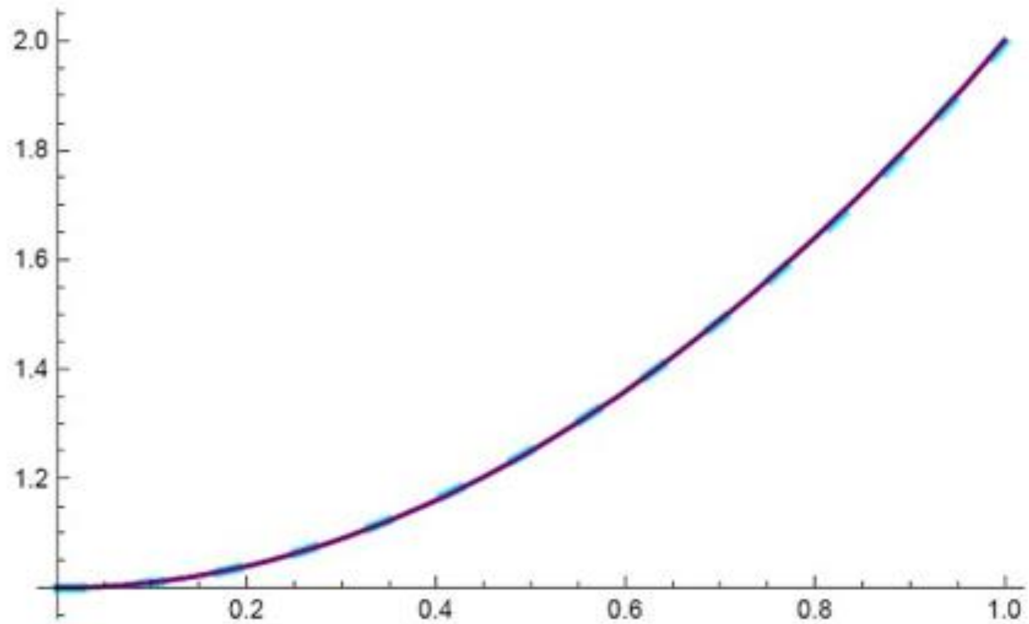
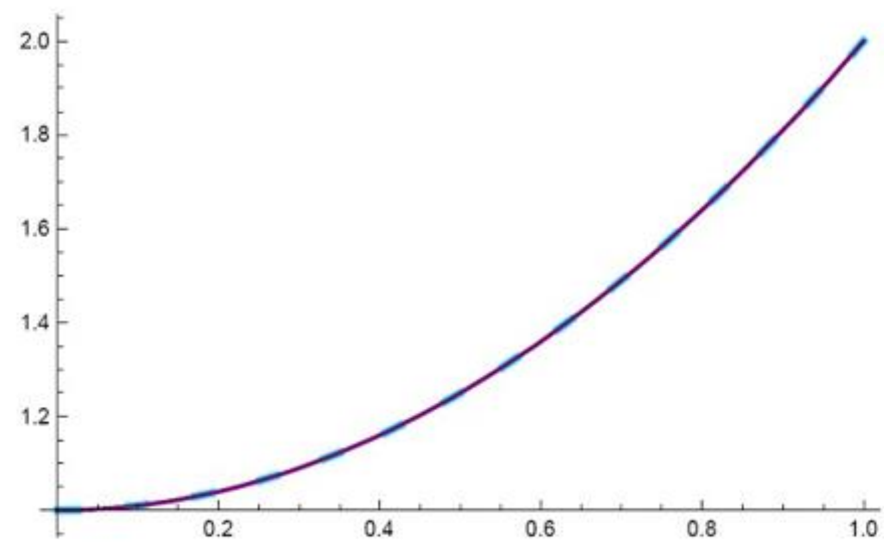
We presented this comparison using the following curves:

n=10



n=20



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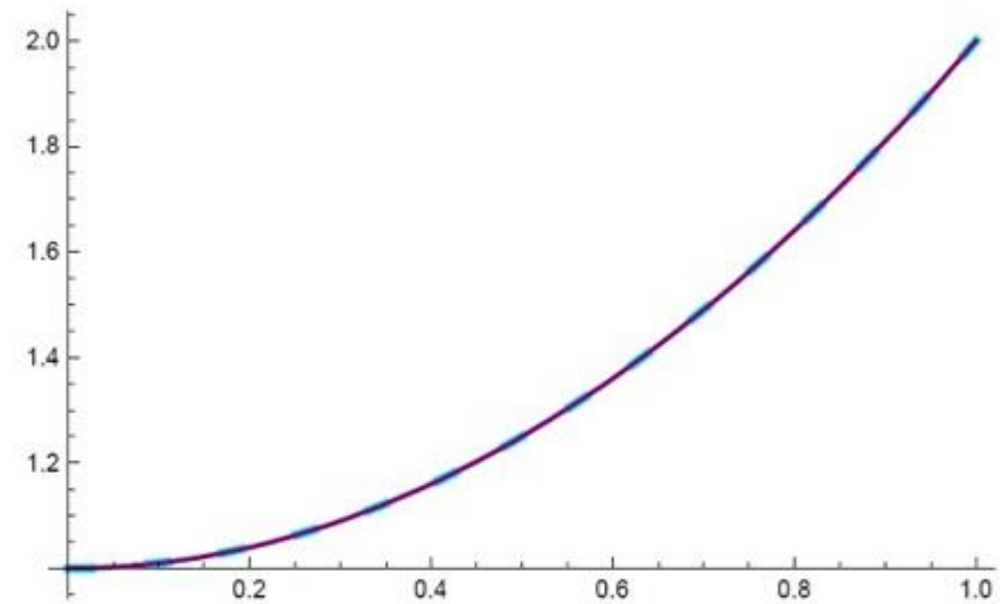
n=30



Results obtained through the *Wolfram Mathematica program* by Using the variable iterative method for different values *n = 10, n = 20, n = 30* with the order of *α=5.0.*

From the results recorded in Table (1) we notice that the larger *n* is, the smaller the error is, that is, the closer the approximate solution is to the exact solution.

Example (2) Let us have the following fractional differential equation:

$$\begin{cases} D^{\alpha}u(x) + xu = 1.91116\, x^{1.1} + x^3 + x \ ;0 \le x \le 1, \quad 0 < \alpha < 1 \\ u(0) = 1 \end{cases} \quad \ldots\ldots (9)$$

The exact solution is $u(x) = x^2 + 1$, *α = 0.9*

Table 3. Estimation of error in the cases: *n = 10, n = 20, n = 30*, and *α = 0.9*

xi	Err n = 10	Err n = 20	Err n = 30
0	0	0	0
0.1	6.897E − 9	8.504E − 14	7.276E − 12
0.2	1.056E − 8	1.554E − 15	6.386E − 11
0.3	3.693E − 8	1.168E − 13	5.196E − 11
0.4	1.693E − 8	6.83E − 13	4.708E − 10
0.5	1.52E − 8	5.029E − 12	7.778E − 11
0.6	8.051E − 9	3.981E − 12	7.311E − 10
0.7	3.947E − 8	7.654E − 12	4.548E − 9
0.8	4.694E − 8	1.062E − 11	5.642E − 9
0.9	1.257E − 9	1.491E − 11	8.135E − 9
1	1.012E 7	4.315E 11	3.44E 8

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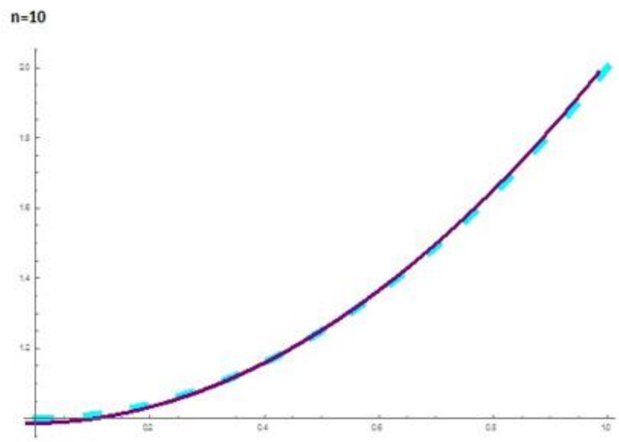
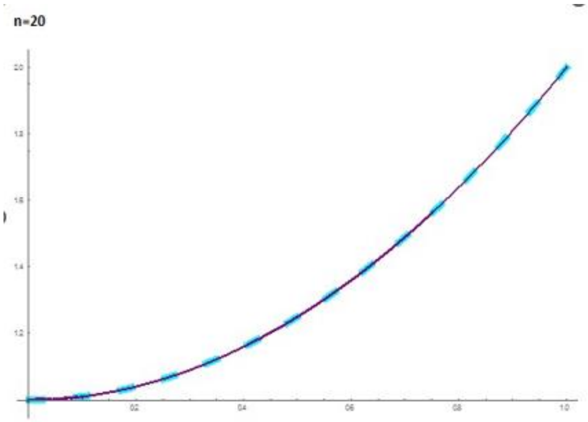
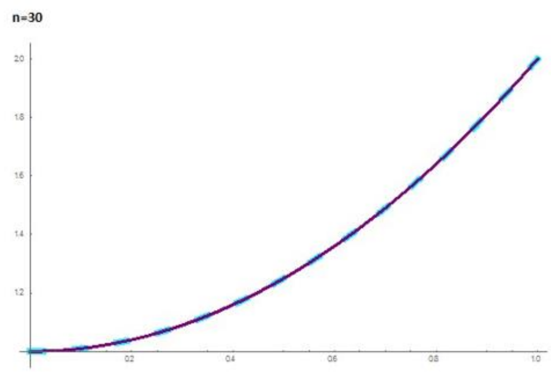
We presented this comparison using the following curves:







DISCUSSION AND CONCLUSIONS

The results of Table (3) represent a comparison between the absolute error of the approximate solution and the exact solution.

For example (2) in various cases $n = 10$, $n = 20$, $n = 30$, with a rank of $\alpha = 0.9$, by using the variable iterative method.

The results of Table (3) prove that there is acceptable agreement between the approximate and exact solution for Example (2). when n is larger, the closer the approximate solution approaching
to matching the exact solution.

Through the results of Table (1), Table (2) and Table (3), we notice the effectiveness of numerical methods in obtaining acceptable results

Through the examples, I noticed the effectiveness of the variable iterative method when used, especially in the second example in the case $\alpha = 0.9$. In the future, I intend to develop other numerical methods to solve differential equations with fractional derivatives with greater accuracy.

CONCLUSIONS

I investigated the possibility of obtaining an approximate solution by studying numerical examples using Variable iterative method by comparing the results. I am obtained, I noticed the effectiveness of the variable iterative method, especially in the second example. It gave acceptable results It can be used and relied upon.

Conflict of interests.

There are non-conflicts of interest

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الخلاصة

المقدمة:

حساب التفاضل والتكامل الكسري هو مجال من مجالات الرياضيات يناقش أبحاث وتطبيقات حساب التفاضل والتكامل غير الصحيح في ℝ أو ℂ. قدمت الطريقة التكرارية المتغيرة لحساب التقريبات العددية لحل المعادلات التفاضلية الكلاسيكية التي تم تطويرها لحل المعادلات التفاضلية الكسرية الخطية، حيث تعتمد الطريقة التكرارية المتغيرة على خطوتين أساسيتين وهما تحديد مضروب لاغرانج وتحديد الصيغة التكرارية.

طرق العمل:

تمت دراسة التقارب بطريقة التكرارات المتغيرة للمعادلات التفاضلية الكسرية بمفهوم كابوتو. تم البحث عن الحل التقريبي باستخدام الطريقة التكرارية المتغيرة، واستخدمت ثلاثة امثلة. تم تقديم هذه المقارنة باستخدام منحنيات هذه الأمثلة المستخدمة لمقارنة الخطأ المطلق للحل التقريبي والحل الدقيق. قمت بالاستعانة بالبرنامج الحاسوبية للحصول على نتائج الجداول المذكورة في البحث وكذلك المنحنيات كبرنامج ماثيماتيكا.

الاستنتاجات:

عند دراسة الأمثلة الواردة في البحث ومن خلال الأمثلة لاحظت فعالية الطريقة التكرارية المتغيرة عند استخدامها لإيجاد الحل التقريبي وخاصة في المثال الثاني فقد اعطت نتائج مقبولة عند مقارنة النتائج مع الاستعانة بالمنحنيات ومقارنة الخطأ المطلق للحل التقريبي والحل الدقيق. فقد لاحظت فعالية الطريقة التكرارية المتغيرة عند استخدامها وخاصة في المثال الثاني في الحالة التي تكون فيها $\alpha = 0.9$ في المثال الثاني.

الكلمات المفتاحية:

المعادلة التفاضلية الكسرية الخطية، الطريقة التكرارية المتغيرة، مشتق كابوتو الكسري، مضروب لاغرانج.

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