Expansion Method For Solving Linear Delay Integro-Differential Equation Using B- Spline Functions

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Abstract

The main goal of this paper lies briefly in submitting and modifying some methods for solving linear delay integro-differential equations (L-DIDEs) containing three types (retarded, neutral and mixed) numerically by employing expansion method (collocation and partition) with the aid of B-spline polynomials as basis functions to compute the numerical solutions of (L-DIDEs). Three numerical examples are given for determining the results of this method.

Keywords: Linear Delay Integro- Differential Equation (L-DIDE), B-Spline Functions.

طريقة التوسيع لحل المعادلات التكاملية – التفاضلية التباطؤية الخطية بأستخدام الدوال التوصيلية

الخلاصة

الهدف الاساسي من هذا البحث هو تقديم وتطوير بعض الطرق لحل المعادلات التكاملية التفاضلية التباطؤية الخطية متضمنة أنواعها الثلاثة (التراجعية, المتعادلة والمختلطة) عدديا بتطبيق طريقة التوسيع والمتضمنةطريقة التجميع والتجزئة مع الدوال التوصيلية كدوال اساسية لحساب الحل العددي لهذه المعادلات واعطيت ثلاثة أمثلة لتوضيح هذه الطرق.

1. Introduction

Delay integro-differential equations (DIDEs) are equations having delay argument. They arise in many realistic models of problems in science, engineering and medicine [6]. Only in the last few years has much effort in behavior of solution of delay differential and delay integro- differential equations, i.e. equations in which the highest order derivative of unknown function appear with delay which is called neutral delay

integro- differential equation as well as retarded differential equation.

The general form of linear delay integro- differential equation is:-

$$f(t-t_1)=g(t)+\int_a^{b(t)} k(t,s)\,f(s-t_2)\,ds \quad ...(1)$$

where $f(t)$ is the unknown function and $g(t)$, $k(t,s)$ are known continuous functions. Eq. (1) is

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classified as Volterra delay integro-differential equation if (b(t)=t), otherwise when (b(t)=b) where (b is constant) it's called Fredholm delay integro- differential equation.

Three types of eq. (1) can be introduced depending on (τ_1 and τ_2) as follows:-

(1) Retarded integro- differential equation if (τ_1=0 and τ_2>0)

$$f`(t) = g(t) + \int_a^{b(t)} k(t,s) f(s - t_2) ds \ldots (2)$$

(2) Neutral integro-differential equation if (τ_1>0 and τ_2=0)

$$f`(t - t_1) = g(t) + \int_a^{b(t)} k(t,s) f(s) ds \ldots (3)$$

(3) Mixed integro- differential equation when (τ_1>0 and τ_2>0)

$$f`(t - t_1) = g(t) + \int_a^{b(t)} k(t,s) f(s - t_2) ds \ldots (4)$$

Many researchers used linear delay integro- differential equation in some subjects such as Hopkins, T [1] found the numerical solution of stochastic DIDEs in population dynamics as well as Xuyang lou [14] studied delay integro differential equation, modeling neural field.

2. B- Spline Polynomials and Properties:-

B-splines are the standard representation of smooth non linear geometry in numerical calculation. Schoendery [12] first introduced the B-spline in 1949. He defines the basis functions using integral convolution. B-spline means spline basis and letter B in B-spline stands for basis.

Definition (1)

Given m+1 knots t_i in [0, 1] with $t_0 < t_1 < t_2 < \ldots < t_m$

A B-spline of degree n is a parametric curve

$$B: [0, 1] \rightarrow R^2$$

Composed of basis B-spline of degree n

$$B(t) = \sum_{i=0}^{m+1} p_i B_{i,n}(t) \qquad , t \in [0,1]$$

The p_i, i=0, 1, …, m+1 are called control points or anchor points or de Boor point. Apolygon can be constructed by connecting the de Boor points with lines starting with p_0 and finishing with p_n this polygon is called the de Boor polygon.

The m-n basis B-spline of degree n can be defined using the cox-de Boor recursion formula.

$$B_{k,0}(t) = \begin{cases} 1 & if\ t_k \le t \le t_{k+1} \\ 0 & otherwise \end{cases}$$

$$B_{k,n} = \frac{t - t_k}{t_{k+n} - t_k} B_{k,n-1}(t) + \frac{t_{k+n+1} + t}{t_{k+n+1} - t_{k+1}} B_{k+1,n-1}(t) \qquad \ldots (5)$$

when the knots are equidistant we say the B-spline is uniform otherwise we call it non- uniform.

The B-spline can be defined in another way like:-

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad \text{for i=0, 1,…, n}$$

where

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

There are n+1 n^{th} degree B-spline polynomials for mathematical convenience, we usually set $B_{k,0}(t)=0$ if k<0 or k>n.

Some important types of B- spline polynomials which are used in this work are introduced as follows:-

2.1 Constant B-spline $B_{k,0}(t)$

The constant B-spline is the simplest spline. It is defined on only one knot span and is not even continues on the knots. It is a just indicator function for the different knot spans.

$$B_{k,0}(t)=\begin{cases} 1 & if\ t_k \le t \le t_{k+1} \\ 0 & otherwise \end{cases}$$

2.2 Linear B-spline $B_{k,1}(t)$

The linear B-spline is defined on two consecutive knot spans and is continues on the knots, but not differentiable.

$$B_{k,n}=\begin{cases} \frac{t-t_k}{t_{k+1}-t_k} & if\ t_k \le t \le t_{k+1} \\ \frac{t_{k+2}-t}{t_{k+2}-t_{k+1}} & t_{k+1\le t\le t_{k+2}} \\ 0 & otherwise \end{cases}$$

0r $B_{0,1}(t)=1-t$, $B_{1,1}(t)=t$

2.3 Quadratic B-spline $B_{k,2}(t)$

Quadratic B-spline with uniform knot-vector is a commonly used form of B-spline. The blending function can easily recalculate, and is equal to each segment in this case:-

$$B_{k,2}(t)=\begin{cases} \frac{1}{2}t^2 \\ -t^2+t+\frac{1}{2} \\ \frac{1}{2}(1-t)^2 \end{cases}$$

or $B_{2,2}(t)=t^2$, $B_{1,2}(t)=2t(1-t)$,

$B_{0,2}(t)=(1-t)^2$

2.4 Cubic B-spline $B_{k,3}(t)$

Cubic B-spline with uniform knot-vector is the most commonly used form of B-spline. The blending function can easily be recalculated and is equal to each segment in this case:-

$B_{3,3}(t)=t^3$, $B_{2,3}(t)=3t^2(1-t)$,

$B_{0,3}(t)=(1-t)^3$

Some important properties of B-spline polynomials are given as follows:-

Property (1):-

Derivatives of the n^{th} degree B-spline polynomial are polynomials of degree n-1. Using the definition of the B-spline polynomial we can show that this derivative can be written as a linear combination of B-spline polynomials [12].

In particular

$$\frac{d}{dt}B_{k,n}(t)=n(B_{k-1,n-1}(t)-B_{k,n-1}(t)) \quad for\ 0\le k\le n \quad \ldots(6)$$

That derivative of a B-spline polynomial can be expressed as the degree of the polynomial, multiplied by the difference of two B-spline polynomial of degree n-1.

Property (2):-

The B-spline polynomials of order n form a basis for the space of polynomials of degree less than or equal to n because they span the space of polynomials- any polynomial of degree less than or equal to n can be written as a linear combination of the B-spline polynomials and they are linearly independent that is if there exist constants c_0, c_1, …, c_n that the identity

$$0=c_0B_{0,n}(t)+c_1B_{1,n}(t)+\ldots+c_nB_{n,n}(t)$$

holds for all t, then all the c_i 's must be zero.

3. Expansion Method:-

This method illustrate important approaches coming frome the field of approximation theory, in which the unknown solution f(t) is expanded in terms of a set of known functions (B-spline polynomials) as follows

$$f(t) \cong f_N(t) = \sum_{i=0}^{N} c_i B_{i,N}(t)$$

The unknown then being the expansion coefficients c_i. An algorithm based on the above approximation is an expansion method in this work collocation and partition methods are considered as expansion method.

3.1 Solution of L-DIDEs Using Collocation Method with the Aid of B- spline:-

First consider the linear retarded integro-differential equation of the form

$$\frac{df(t)}{dt} = g(t) + \int_a^{b(t)} k(t,s)\, f(s-t_2)\,ds \quad \mathbf{L}\,(7)$$

By approximating the unknown function f(t) using B-spline polynomial as a basis function we have

$$f(t) \cong f_N(t_j) = \sum_{i=0}^{N} c_i B_{i,N}(t_j) \quad for\ \ j=0,1,\ldots,N \quad \mathbf{L}\,(8)$$

Substituting eq.(8) into eq.(7) for f(t) and with $t=t_j$, and by using property (1) for $f'(t)$ we get the following formula

$$\sum_{i=0}^{N} N(c_i B_{i-1,N-1}(t_j) - c_i B_{i,N-1}(t_j)) = g(t_j) + \sum_{i=0}^{N} \int_a^{b(t)} k(t_j,s)\, c_j B_{i,N}(s-t_2)\,ds$$

So

$$\sum_{i=0}^{N} c_i [N(B_{i-1,N-1}(t_j) - B_{i,N-1}(t_j)] = \sum_{i=0}^{N} c_i \int_a^{b(t)} k(t_j,s) B_{i,N}(s-t_2)\,ds + g(t_j) \quad \mathbf{L}(9)$$

for j=0,1,…,N

then

$$\sum_{i=0}^{N} c_i [(N(B_{i-1,N-1}(t_j) - B_{i,N-1}(t_j)) - \int_a^{b(t)} k(t_j,s) B_{i,N}(s-t_2)\,ds] = g(t_j)$$

Second consider the linear neutral integro-differential equation of the form

$$\frac{df(t-t_1)}{dt} = g(t) + \int_a^{b(t)} k(t,s)\, f(s)\,ds \qquad \mathbf{L}\,(10)$$

Substituting eq.(8) into eq.(10) we get the equation

$$\sum_{i=0}^{N} c_i [N(B_{i-1,N-1}(t_j - t_1) - B_{i,N-1}(t_j - t_1))] = \sum_{i=0}^{N} c_i \int_a^{b(t)} k(t_j,s) B_{i,N}(s)\,ds + g(t_j) \quad \mathbf{L}(11)$$

then

$$\sum_{i=0}^{N} c_i[(N(B_{i-1,N-1}(t_j-t_1)-B_{i,N-1}(t_j-t_1))-\int_a^{b(t)} k(t_j,s)B_{i,N}(s)ds]=g(t_j)$$

Third consider the L-MIDE's of the form

$$\frac{df(t-t_1)}{dt}=g(t)+\int_a^{b(t)} k(t,s)\,f(s-t_2)ds \quad ...(12)$$

By Substituting eq. (8) into eq.(12) and follows the previous steps we have the following equation as a results:-

$$\sum_{i=0}^{N} c_i[N(B_{i-1,N-1}(t_j-t_1)-B_{i,N-1}(t_j-t_1)]=\sum_{i=0}^{N} c_i\int_a^{b(t)} B_{i,N}(s-t_2)k(t_j,s)ds+g(t_j)$$

then

$$\sum_{i=0}^{N} c_i[(N(B_{i-1,N-1}(t_j-t_1)-B_{i,N-1}(t_j-t_1))-\int_a^{b(t)} k(t_j,s)B_{i,N}(s-t_2)ds]=g(t_j)$$

…(13)

for j=0,1,...,N

These equations involve n+1 unknown coefficients [c_i], then we may select n-1 points {$t_1,t_2,...,t_{N-1}$}in the range of integration and require $f_N(t)$ to satisfy the delay integro – differential equation at just these n-1 points. This method requires us to solve just the system of n+1 linear equations for each type of delay integro differential equations (retarded, neutral and mixed).these system of n+1 Linear equations for the coefficients c_i's can be written in matrix form as follows:-

AC = G

where

$$A=\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N} & \cdots & a_{NN} \end{bmatrix}$$

$$a_{ij}=[(N(B_{i-1,N-1}(t_j-t_h)-B_{i,N-1}(t_j-t_h))-\int_a^{b(t)} k(t_j,s)B_{i,N}(s-t_h)ds]$$

...(14)

k=1,2

the existence of (τ_k)depends on the type of the equation (retarded, neutral and mixed)

and $C=\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix}$, $G=\begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix}$

By solving this system by Gauss elimination procedure to find c_i's for i=0,1,...,N.

3.2 Solution of L-DIDEs Using Partition Method with the Aid of B-spline:-

In this method we divide the domain R into P non over lapping sub domains R_j , j=1,2,...,p,

if the weighting functions are chosen as follows

$$w_j=\begin{cases} 1 & t\in R_j \\ 0 & t\notin R_j \end{cases}$$

then the delay integro –differential equation is satisfied on the average in each of the P sub domains R_j, the required equation for partition method become

$$\int_{R_j} E(t)dt=0 \qquad j=0,1,2,...,N \quad …(15)$$

where E(t)is called residue equation and hopefully it approaches zero on R_j.

or which we have the residue equation for (retarded, neutral and mixed) integro- differential equation respectively as follows:-

for retarded equation we have

$$\bullet E_N(t)=\sum_{i=0}^{N}c_i[(N(B_{i-1,N-1}(t)-B_{i,N-1}(t))-\int_a^{b(t)}k(t,s)B_{i,N}(s-t_2)ds]-g(t)$$

for neutral equation we get

$$\bullet E_N(t)=\sum_{i=0}^{N}c_i[(N(B_{i-1,N-1}(t-t_1)-B_{i,N-1}(t-t_1))-\int_a^{b(t)}k(t,s)B_{i,N}(s)ds]-g(t)$$

for mixed equation we get

$$\bullet E_N(t)=\sum_{i=0}^{N}c_i[(N(B_{i-1,N-1}(t-t_1)-B_{i,N-1}(t-t_1))-\int_a^{b(t)}k(t,s)B_{i,N}(s-t_2)ds]-g(t)$$

Substituting these equations into eq.(15) we get:

$$\bullet \sum_{i=0}^{N}c_i\int_{R_j}[(N(B_{i-1,N-1}(t)-B_{i,N-1}(t))-\int_a^{b(t)}k(t,s)B_{i,N}(s-t_2)ds]dt=\int_{R_j}g(t)dt$$

$$\bullet \sum_{i=0}^{N}c_i\int_{R_j}[(N(B_{i-1,N-1}(t-t_1)-B_{i,N-1}(t-t_1))-\int_a^{b(t)}k(t,s)B_{i,N}(s)ds]dt=\int_{R_j}g(t)dt$$

$$\bullet \sum_{i=0}^{N}c_i\int_{R_j}[(N(B_{i-1,N-1}(t-t_1)-B_{i,N-1}(t-t_1))-\int_a^{b(t)}k(t,s)B_{i,N}(s-t_2)ds]dt=\int_{R_j}g(t)dt$$

These equations give system of (N+1) Linear equations in N+1 unknown coefficients c_i, i=0,1,...,N. Rewriting the above equations in a matrix form as follows:-

MC =G

where

$$M=\begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0N} \\ b_{10} & b_{11} & \cdots & b_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N0} & b_{N} & \cdots & b_{NN} \end{bmatrix}$$

$$b_{ij}=\int_{R_j}[(N(B_{i-1,N-1}(t-t_h)-B_{i,N-1}(t-t_h))-\int_a^{b(t)}k(t,s)B_{i,N}(s-t_h)ds]dt=\int_{R_j}g(t)dt$$

$$C=\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \qquad G=\begin{bmatrix} \int_{R_0} g(t)dt \\ \int_{R_1} g(t)dt \\ \vdots \\ \int_{R_p} g(t)dt \end{bmatrix}$$

by using Gauss elimination procedure we find the values of c_i's.

4. Numerical Examples:-

Example (1):

Consider the following retarded Volterra integro-differential equations:

$$f(t)=(1-\frac{t^4}{3})+\int_0^t tsf(s-1)ds \qquad 0\le t\le 1$$

with exact solution $f(t)=t+1$

Assume the approximate solution is:

$$f_1(t)=c_0(1-t)+c_1t$$

This problem is solved using collection method and partition method with the aid of B-spline functions as basis functions, the solution of f(t) is obtained as shown in table (1),with N= 10, h=0.1 and t_i=ih, i=0,1,...,N.

Example (2):

Consider the following neutral integro-differential equations:

$$f'(t-1)=4(t-1)^3-\frac{t^5}{5}+\int_0^t f(s)ds \qquad 0\le t\le 1$$

with exact solution $f(t)=t^4$

Assume the approximate solution is:

$$f_4(t)=\sum_{i=0}^{4}c_iB_{i,4}(t)$$

$$f_4(t)=c_0(1-t)^4+4c_1t(1-t)^3+6c_2t^2(1-t)^2+4c_3t^3(1-t)+c_4t^4$$

Table (2) lists the results obtained by achieving collection method and partition method with the aid of B-spline function.

Example (3):

Consider the following mixed Fredholem integro-differential equations:

$$f'(t-1) = 2t - \frac{25}{12} + \int_0^1 f(t-1/2)dt \qquad 0 \le t \le 1$$

with exact solution $f(t) = t^2$

Assume the approximate solution is:

$$f(t) = \sum_{i=0}^{2} c_i B_{i,2}(t)$$

The results of this example list in table (3) which obtained by using collection and partition method with the aid of B-spline.

Conclusions

1. The expansion method including (collocation and partition) methods with the aid of B-spline polynomials as a basis function which are used in this paper have proved their effectiveness in solving (L-DIDEs) numerically and finding accurate results.

2. B-spline function depends on N as N increased, the error term is decreased.

3. The results show a marked improvement in the least square errors from which we conclude that.

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Table (1)

ti=ih	Exact Solution	Collection with(B-spline)	Partition with(B-spline)
0	1.0000	1.0000	1.0000
0.1	1.1000	1.1000	1.1000
0.2	1.2000	1.2000	1.2000
0.3	1.3000	1.3000	1.3000
0.4	1.4000	1.4000	1.4000
0.5	1.5000	1.5000	1.5000
0.6	1.6000	1.6000	1.6000
0.7	1.7000	1.7000	1.7000
0.8	1.8000	1.8000	1.8000
0.9	1.9000	1.9000	1.9000
1	2.0000	2.0000	2.0000
L.S.E		0.0000	0.0000

Table (2)

t	Exact solution	Collocation with (B-spline)	Partition with (B-pline)
0	0	0	0
0.1	0.0001	0.0001	0.0001
0.2	0.0016	0.0016	0.0016
0.3	0.0081	0.0081	0.0081
0.4	0.0256	0.0256	0.0256
0.5	0.0625	0.0625	0.0625
0.6	0.1296	0.1296	0.1296
0.7	0.2401	0.2401	0.2401
0.8	0.4096	0.4096	0.4096
0.9	0.6561	0.6561	0.6561
1	1	1	1
L.S.E		0.0000	0.0000

Table (3)

t	Exact solution	Collocation with(B-spline)	Partition with(B-spline)
0	0	0	0
0.1	0.01	0.01	0.01
0.2	0.04	0.04	0.04
0.3	0.09	0.09	0.09
0.4	0.016	0.016	0.016
0.5	0.25	0.25	0.25
0.6	0.36	0.36	0.36
0.7	0.49	0.49	0.49
0.8	0.64	0.64	0.64
0.9	0.81	0.81	0.81
1	1	1	1
L.S.E		0.0000	0.0000