

A General Equation for the Flexural – Membrane Behaviour of Rigid – Plastic RC Square Slabs Having Variously Restrained Edges

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Abstract

Uniformly loaded rigid – plastic reinforced concrete (RC) square slabs having six different cases of boundary restraints have been recently analyzed⁽¹⁾ under the combined effect of bending and membrane action, and a separate load – deflection relationship for each slab case has been obtained. In this paper, the load – deflection behaviour of all these six slab cases is expressed in one single compact equation as a function of the slab material properties and the configuration of the slab boundary restraints. The application of the proposed equation in the analysis of a typical RC Square slab indicates that when two or more edges of the slab are restrained against rotation and horizontal translation the live load carrying capacity of the slab can reach up to five times that suggested by the simple yield line theory.

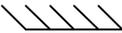
المعادلة العامة للانحناء والفعل الغشائي في البلاطات الخرسانية المربعة اللدنة والجاسئة وذات حافات مختلفة التقييد

الخلاصة

في بحث سابق⁽¹⁾ تم الاخذ بنظر الاعتبار التأثير المشترك للانحناء والفعل الغشائي في تحليل ست حالات من بلاطات خرسانية مسلحة مربعة الشكل، جاسئة – لدنة، ومحملة حملاً منتظماً، وذات حافات مختلفة التقييد، حيث تم اشتقاق علاقة الحمل – الاود لكل حالة بلاطة على انفراد. في هذا البحث تم التعبير عن سلوكية الحمل – الاود لجميع حالات البلاطة الستة بمعادلة عامة واحدة مختصرة بدلالة خواص مواد البلاطة ونوع التقييد عند حافات البلاطة. تم استخدام المعادلة المقترحة في تحليل بلاطة خرسانية مسلحة مربعة الشكل نموذجية، فتبين ان وجود تقييد ضد الدوران والحركة الجانبية عند حافتين او اكثر من البلاطة يزيد من سعة الحمولة الحية للبلاطة حيث تصل هذه الحمولة الى مايزيد عن خمس مرات القيمة المستتبهة من نظرية خط الخضوع المبسطة.

Notations:

A_s	Area of tensile reinforcement per unit width of slab	$f_1, f_2,$	Factors expressing configuration of slab boundary restraints
d	Effective depth of slab	f_3	Thickness of slab
f'_c	Concrete cylinder strength	h	
f_y	Yield strength of steel reinforcement	$k_1 k_3,$	Parameters fixing concrete compressive stress block
		k_2	
		l	Span of square slab
		M_o	Ultimate bending moment of resistance per unit width

	of slab
t	Parameter expressing slab material properties $= r \cdot f_y / f'_c$
w	Uniformly distributed load per unit area
w_d	Uniform dead load of slab per unit area
w_J	Collapse uniform load of slab per unit area estimated by Johansen's simple yield line theory
$w_1)_{Y.L}$	Maximum uniform live load per unit area estimated by simple yield line theory
$w_1)_{mod.Y.L}$	Maximum uniform live load per unit area estimated by the proposed modified yield line equation of the present research (considering membrane action)
a, b	Parameters expressing slab properties
Δ	Vertical deflection at center of slab
r	Steel ratio = A_s/d
$R.C$	Reinforced concrete
	Simply supported edge
	Fixed edge

balanced in – plane membrane forces which develop inside the slab as a result of the applied loading have been found^(9,10) to help in producing higher yield loads with continuing deflection.

In a recently submitted M.Sc thesis⁽¹⁾, the effect of boundary restraints on the ultimate capacity of rigid – plastic RC square slabs has theoretically been established. It has been found that the combination of restrained and unrestrained slab edges give six possible cases of such slabs as shown in Fig. (1). Accordingly, each slab case has been analyzed separately to include the combined effect of bending and membrane action and different load – deflection relationships have been obtained for the six different slab cases.

It is aimed from the present research to express the load – deflection behaviour of all the six cases of RC square slabs in one single compact equation. The equation is general and is a function of the slab material properties and the configuration of the slab boundary restraints.

1-Introduction

Previous studies ^(2, 3) of the phenomenon of membrane action in RC slabs showed that the presence of restraining conditions at the edges of a slab can considerably enhance the load carrying capacity of the slab. Values of ultimate loads which are many times greater than that suggested by the simple Johansen's yield line theory⁽⁴⁾ have been recorded both theoretically^(2,5,6) and experimentally^(2,7,8). Even in cases of unrestrained slabs, the self

2-Analysis of RC Square Slabs Having Variously Restrained Edges:

2.1 Analysis By The Simple Johansen's Yield Line Theory (Ignoring Membrane Action):

It is assumed that the RC square slabs under consideration are isotropically reinforced in the two orthogonal directions and equal amount of reinforcement are provided at positive and negative moment regions such that the

ultimate bending moment of resistance per unit width of the slab is M_o at all slab sections, where

$$M_o = r f_y d^2 \left(1 - \frac{k_2}{k_1 k_3} r \frac{f_y}{f_c} \right) \dots (1)$$

and k_2 , $k_1 k_3$ are parameters defining the shape of the compressive stress block of concrete which are given by Hognested⁽¹¹⁾ as functions of the concrete cylinder strength f_c' (in units of MPa);

$$k_1, k_3 = \frac{27 + 0.35 f_c'}{22 + f_c'} \dots (2a)$$

$$k_2 = 0.5 - \frac{f_c'}{550} \dots (2b)$$

As mentioned earlier, the variation in the restraining conditions at the periphery of a RC square slab gives the six possible cases shown in Fig. (1). Under the applied uniform load these slabs will collapse in the form of the yield line patterns shown in the same figure.

It can be seen that the yield line pattern of slab cases (1) and (6), being the well known cross yield line pattern, has no variable parameter. The yield line pattern of slab case (3) and (4) has one variable parameter (which is x_o in slab case 3 and y_o in slab case 4) while that for slab case (2) and (5) has two variable parameters (which is x_1, x_2 in slab case 2 and x_o, y_o in slab case 5).

By applying the principle of virtual work for each slab case the following values of Johansen's uniform load (w_J) will be obtained;

Slab Case (1)

$$w_J = 48 \frac{M_o}{l^2} \dots (3)$$

Slab Case (2)

$$w_J = \frac{6M_o}{l^2} \cdot \left(\frac{8 + \frac{2}{x_1} + \frac{1}{x_2}}{3 - x_1 - x_2} \right) \dots (4)$$

Differentiating Eq.(4) with respect to x_1 and x_2 each at a time and setting the derivatives equal to zero;

$$\frac{\partial w_J}{\partial x_1} = 0 \dots (5a)$$

$$\frac{\partial w_J}{\partial x_2} = 0 \dots (5b)$$

The solution of Eqs. (5a) and (5b) gives the true values of x_1 and x_2 that fix the actual yield line pattern

$$x_1 = 0.539$$

$$x_2 = 0.381$$

A substitute of these values of x_1 and x_2 into Eq. (4) gives;

$$w_J = 41.352 \frac{M_o}{l^2} \dots (6)$$

Slab Case (3)

$$w_J = \frac{12M_o}{l^2} \left(\frac{4 + \frac{1}{x_o}}{3 - 2x_o} \right) \dots (7)$$

Setting $\frac{\partial w_J}{\partial x_o} = 0$ gives $x_o = 0.4114$

which when substituted into Eq. (6) gives;

$$w_J = 35.444 \frac{M_o}{l^2} \dots (8)$$

Slab Case (4)

$$w_J = \frac{6M_o}{l^2} \left[\frac{2}{y_o} + \frac{1}{(1 - y_o)} \right] \dots (9)$$

Setting $\frac{\partial w_J}{\partial y_o} = 0$ gives $y_o = 0.5858$

which when substituted into Eq. (9) gives;

$$w_J = 34.971 \frac{M_o}{I^2} \dots (10)$$

Slab Case (5)

$$w_J = \frac{6M_o}{I^2} \left[\frac{2}{y_o} + \frac{2}{x_o} + \frac{1}{(1-y_o)} \right] \dots (11)$$

Setting $\frac{\partial w_J}{\partial x_o} = 0$ and $\frac{\partial w_J}{\partial y_o} = 0$ gives $x_o = 0.4521$ and $y_o = 0.5858$ which when substituted into Eq. (11) gives;

$$w_J = 29.351 \frac{M_o}{I^2} \dots (12)$$

Slab Case (6)

$$w_J = 24 \frac{M_o}{I^2} \dots (13)$$

The variation in the value of the collapse uniform load w_J as determined by Johansen's simple yield line theory for the six cases of RC square slabs is shown in Fig. (2).

2.2 Analysis by the Modified Yield Line Theory (Including Membrane Action):

The effect of including membrane action in the yield line analysis of the deferent slab cases under study has been carried out in ref. (1). It was found that, due to the assumption of rigid – plastic consideration, the maximum value of the yield load in the restrained slab cases (1) to (5) occurs at zero deflection and decreases non – linearly with increasing slab deflection. A reverse behaviour was obtained for the unrestrained slab case (6) where the yield load at the start of collapse (i.e at zero deflection) is Johansen's load and increases non –linearly with increasing slab deflection. However, the load – deflection relationships for the different slab

cases were derived separately and were given in complex forms that require lengthy computations before they can be used.

In this paper, these load – deflection equations are re – examined, simplified further and re – arranged in form of ordinary quadratic equations. By doing so, all the equations become identical in form but they differ only in the coefficients of the terms. Therefore, a single compact general equation is proposed to express the load – deflection behaviour of all the six cases of the RC square slabs. The proposed equation is simple to use and illustrates that the load – deflection behaviour of a particular slab is affected by the slab material properties and the configuration of the slab boundary restraints. This equation, written in non – dimensional form, is:

$$\frac{w}{w_J} = 1 + f_1 \frac{a^2}{4b} - f_2 a \left(\frac{a}{2b} + 1 \right) \left(\frac{\Delta}{h} \right) + f_3 b \left(\frac{a}{2b} + 1 \right)^2 \left(\frac{\Delta}{h} \right)^2 \dots(14)$$

- where $\frac{w}{w_J}$ is the ratio of the slab yield load (considering membrane action) to the corresponding Johansen's load (neglecting membrane action)
- $\frac{\Delta}{h}$ is the ratio of the slab central deflection to the thickness of the slab.
- a, b are factors expressing the slab material properties

$$a = \frac{\frac{1}{2} \frac{h}{d} - \frac{2k_2}{k_1 k_3} t}{1 - \frac{k_2}{k_1 k_3} t}$$

$$b = \frac{\frac{k_2}{k_1 k_3} t}{1 - \frac{k_2}{k_1 k_3} t}$$

in which $t = r \frac{f_y}{f_c}$

and $\frac{h}{d}$: is the ratio of the slab thickness to its effective depth
 f_1, f_2, f_3 : are coefficients expressing the configuration of the slab boundary restraints. Therefore their values vary from one slab case to another, as given in Table(1).

The proposed equation (14) is plotted in Figs. (3) and (4) to show respectively the effects of the slab material properties and the configuration of the slab boundary restraints on the load – deflection behaviour of RC square slabs.

Figure (3) shows the effect of varying the slab material properties (represented by the parameter $t = r \frac{f_y}{f_c}$) on the load – deflection characteristics of a typical RC square slab (taken to be slab case 1 as an example). The figure clearly shows that for lower values of t, higher load ratios w/w_j can be obtained at any relative slab deflection (Δ/h).

Figure (4) shows the effect of varying the slab boundary restraints on the load deflection

behaviour of a typical RC square slab having the following properties;

$h = 150^{mm}, d = 125^{mm}, r = 0.2\%, f_y = 400MPa, f_c = 25MPa$

It can be seen from this figure that the increase in the maximum yield load above the simple yield line theory load is more pronounced in slab case 1 (where all the edges of the slab are fixed). As the number of the fixed edges in the slab decreases, the enhancement in the maximum yield load (due to membrane action) will decrease, and when the slab has all edges simply supported the initial collapse load of the slab will simply be the well known Johansen's load.

3- Determination of the Actual Load Carrying Capacity of RC Square Slabs:

Previous studies (2, 3, 8) of the effect of membrane action in elastic – plastic RC restrained slabs show that the real load – deflection relationship of the slab consists of an initial ascending part (representing both the elastic deformations and the subsequent elastic – plastic stage which is characterized by the development of compressive membrane action in the slab) followed by a descending part (representing the spread of full depth cracking and tensile membrane action in the slab). Therefore, the ultimate load is reached at a certain value of deflection and is many times greater than Johansen's load due to the combined effect of bending and membrane action.

The proposed Equation (14), being based on the assumption of considering the slab to behave in a rigid – perfectly plastic manner, predicts a continuous descending load – deflection curve starting from maximum value of load at zero slab deflection. Therefore to make use of equation (14) in estimating the actual load carrying capacity of a particular RC square slab, the value of the slab central deflection corresponding to the slab ultimate load has to be specified first and then can be inserted in the equation to get the actual value of the ultimate load.

Existing experimental tests ^(12, 13) on restrained RC slabs indicated an average value of such deflection to be 0.3 times the slab thickness. Therefore inserting $\Delta/h=0.3$ in Eq. (14) will enable determining the actual load carrying capacity of any RC square slab as illustrated in the following typical examined slab.

3.1 Properties of the Examined Slab:

Fig. (5) shows a RC square slab of dimensions $6m \times 6m$ and thickness $140mm$ (calculated as perimeter of slab/180). The slab carries uniform load distributed over its entire area. All the six cases of boundary restraints described in Fig. (1) will be examined in this slab.

The properties of concrete and steel bars used in the slab are such that $f'_c = 25MPa$ and $f_y = 400MPa$. The slab is assumed to be reinforced with the minimum amount of reinforcement (required for shrinkage and temperature) as

specified by ACI-05 Code ⁽¹⁴⁾, giving

$$A_s = A_{s_{min}} = 0.0018bh =$$

$$0.0018 * 1000 * 140 = 252mm^2/m$$

The slab is isotropically reinforced with this amount of reinforcement in the two directions, which are placed at the bottom face only at positive moment regions (slab central zone) and at the top face only at negative moment regions (zones of fixed edges if present).

Using $f12mm$ steel bars and considering a concrete cover to the reinforcement of $20mm$, the average effective depth of the slab is $d_{ave} = 108mm$. Therefore the steel ratio r at any slab section is;

$$r = \frac{A_s}{d} = \frac{252}{1000 * 108} = 0.002333$$

The constant a and b of the slab section (as defined in Eq. 14) can be calculated as follows;

From Eq. (2a);

$$k_1 k_3 = \frac{27 + 0.35 f'_c}{22 + f'_c} = \frac{27 + 0.35 * 25}{22 + 25} = 0.7606$$

From Eq. (2b);

$$k_2 = 0.5 - \frac{f'_c}{550} = 0.5 - \frac{25}{550} = 0.4545$$

and since

$$t = r \frac{f_y}{f'_c} = 0.002333 * \frac{400}{25} = 0.03733$$

therefore;

$$a = \frac{\frac{1}{2} \frac{h}{d} - \frac{2k_2}{k_1 k_3} \cdot t}{1 - \frac{k_2}{k_1 k_3} \cdot t} = \frac{\frac{1}{2} * \frac{140}{108} - \frac{2 * 0.4545}{0.7606} * 0.03733}{1 - \frac{0.4545}{0.7606} * 0.03733} = 0.6173$$

$$b = \frac{\frac{k_2}{k_1 k_3} \cdot t}{1 - \frac{k_2}{k_1 k_3} \cdot t} = \frac{\frac{0.4545}{0.7606} * 0.03733}{1 - \frac{0.4545}{0.7606} * 0.03733} = 0.0228$$

The yield moment of the slab section (given by Eq. 1) is;

$$M_o = r f_y d^2 \left(1 - \frac{k_2}{k_1 k_3} r \frac{f_y}{f_c} \right) = 0.002333 * 400 * (108)^2 \left(1 - \frac{0.4545}{0.7606} * 0.03733 \right) * 10^{-3} = 10.642 \text{ kN.m/m}$$

By assuming the slab to be a panel in an intermediate floor of a RC residential building, the total dead load of the slab (assuming a unit weight of 24 kN/m³ for concrete and tiles) is;

Self weight of the slab=

$$0.14 * 24 = 3.36 \text{ kN/m}^2$$

Weight of tiles and mortar=

$$0.04 * 24 = 0.96 \text{ kN/m}^2$$

∴ the total dead

$$\text{load } (w_d) = 4.32 \text{ kN/m}^2$$

3.2 Estimation of the live load Carrying Capacity Using the Simple Yield Line Theory;

According to the simple yield line theory, the values of the collapse uniform load for the six cases of the RC square slab under examination are given by Eqs. (3), (6), (8), (10), (12) and (13) respectively. The values of w_j

obtained from these equations are listed in column (3) of Table (2) for the six consecutive cases of the slab. By deducting from these values the total dead load of the slab (which is listed in column 4 of the table) the collapse uniform live load. $w_1)_{Y.L}$ as predicted by the simple yield line theory can be obtained as listed in column (5) of the table.

3.3 Estimation of the live load carrying capacity using the proposed modified yield line equation (14) of the present research (accounting for membrane action):

By inserting $\Delta/h=0.3$ in the proposed equation (14) and using the calculated values of a and b together with the values of the coefficients f_1 , f_2 and f_3 that express the configuration of the slab boundary restraints as listed in table (1), the load ratios w/w_j for the six cases of the examined slab can be determined as listed in column (6) of table (2). By multiplying each value of w/w_j with its corresponding w_j value (from column 3 of the table) the actual ultimate uniform load of each slab case can be obtained as listed in column (7) of the table, and when w_d is subtracted the collapse uniform live load $w_1)_{\text{modi.Y.L}}$ as predicted by the modified yield line theory (accounting for membrane action) can be obtained as listed in column (8). Finally column (9) of table (2) shows the ratio $w_1)_{\text{modi.Y.L}}/w_1)_{Y.L}$ for each slab case.

It can be seen from the last column of table (2) that when two

or more edges of the examined RC square slab are restrained against rotation and horizontal translation (i.e fixed), the actual live load carrying capacity of the slab can reach up to five times that suggested by the simple yield line theory.

4- Conclusions

The main conclusions to be drawn from the present study are:

1. R.C square slabs having different boundary restraints are categorized into six cases. The load – deflection behaviour of all these slab cases, considering membrane action, is expressed in one single compact equation. The equation is a function of two main parameters, namely, the slab material properties and the configuration of the slab boundary restraints.
2. According to the proposed load – deflection equation, RC square slabs having edges restrained against rotation and lateral movement have ultimate loads which are, in many cases, far in excess of those indicated by Johansen's yield line theory due to the development of compressive membrane forces.
3. The enhancement in ultimate load above Johansen's load is more pronounced in slabs having lower values of the slab material parameter t (where $t = r \frac{f_y}{f_c}$) and greater number of fixed edges.
4. The application of the proposed equation in the analysis of a typical R.C square slab indicates that when two or more

edges of the slab are fixed, the live load carrying capacity of the slab will be five times that estimated by the simple yield line theory.

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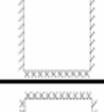
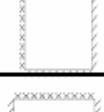
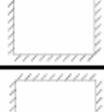
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Table (1) Values of the Coefficients f_1 , f_2 , f_3 in
 the Proposed Equation

Slab Case	Slab Shape	f_1	f_2	f_3
1		1	0.5	0.4166
2		0.9663	0.5015	0.4712
3		1.8572	1.1620	1.5121
4		0.8284	0.4460	0.4852
5		0.5551	0.2905	0.4540
6		0	0	0.3333

Table (2) Estimation of the Uniform Live Load Carrying Capacity of the Examined RCSquare Slab by: (1) The Simple Yield Line Theory (2) The Proposed Modified Yield Line Equation (14) of the Present Research

(1) Slab Case	(2) Shape of Slab	(3) w_j (kN/m ²)	(4) w_d (kN/m ²)	(5) $w_{l,ijl}$ (kN/m ²)	(6) $\frac{w}{w_j} \Delta=0.3h$	(7) w (kN/m ²)	(8) $w_{l,mod,ijl}$ (kN/m ²)	(9) $\frac{w_{l,mod,ijl}}{w_{l,ijl}}$
1		14.19	4.32	9.87	4.01	56.90	52.58	5.33
2		12.22	=	7.90	3.89	47.54	43.22	5.47
3		10.48	=	6.16	3.56	37.31	32.99	5.36
4		10.34	=	6.02	3.47	35.88	31.56	5.24
5		8.68	=	4.36	2.73	23.70	19.38	4.44
6		7.10	=	2.78	1.15	8.16	3.84	1.38

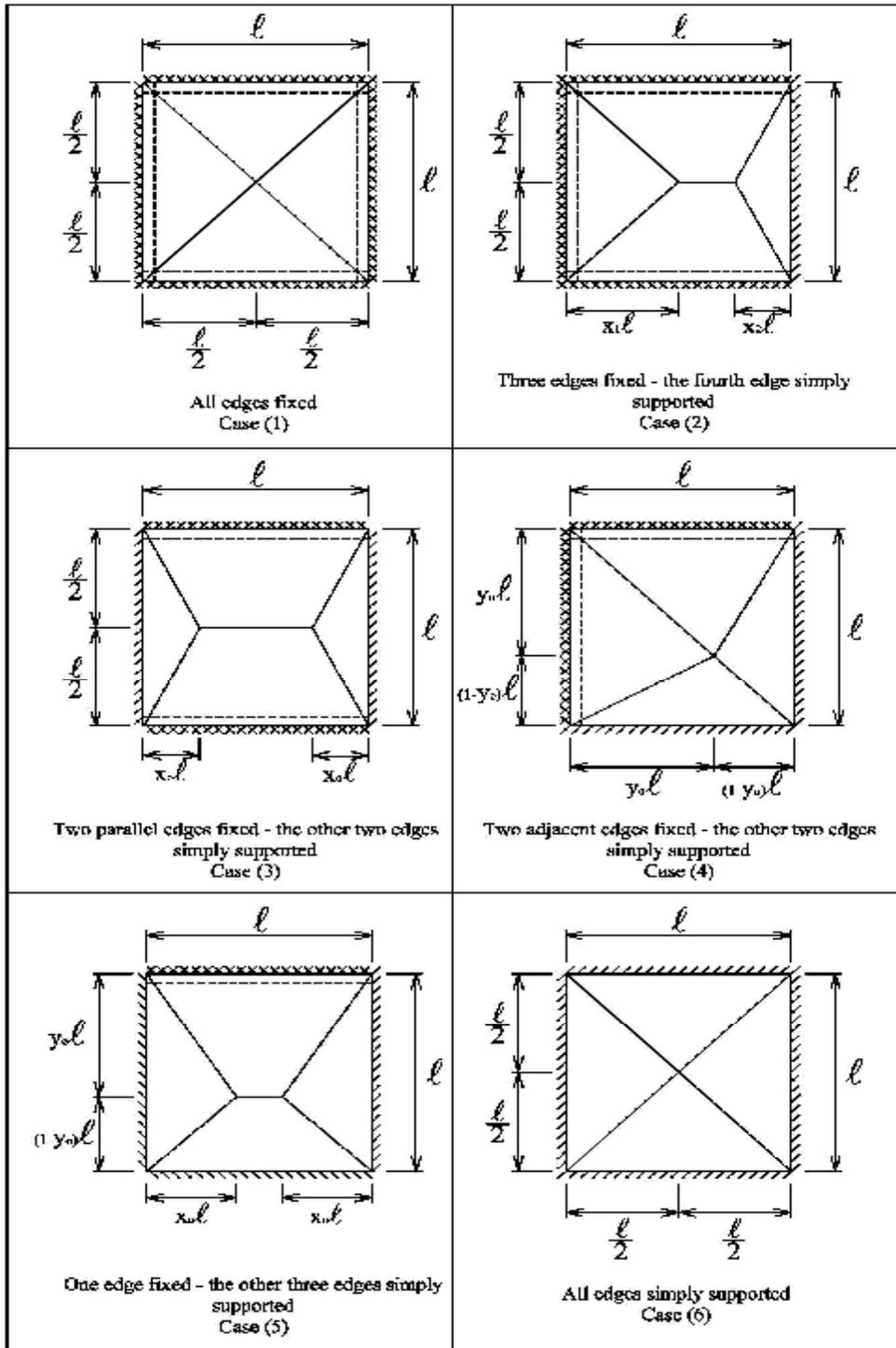


Figure (1) The Six Possible Cases of R.C Square Slabs With Their Collapse Modes

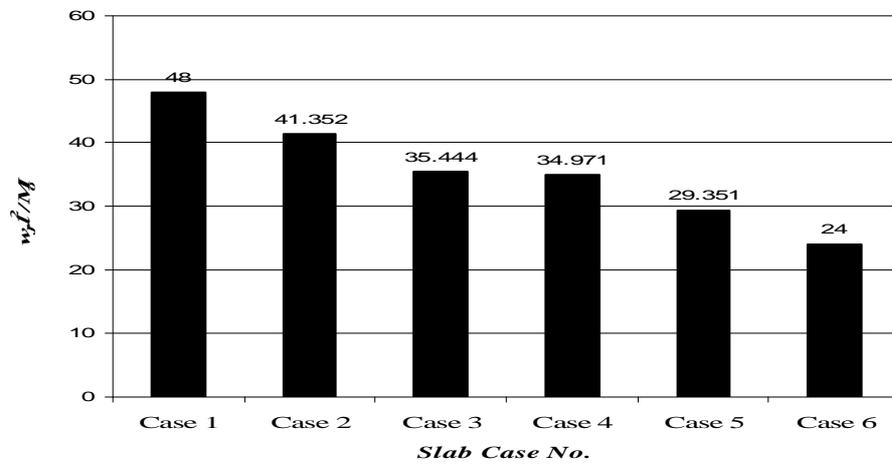


Figure (2) The Value of The Collapse Uniform Load For The Six Possible Cases of R.C Square Slabs

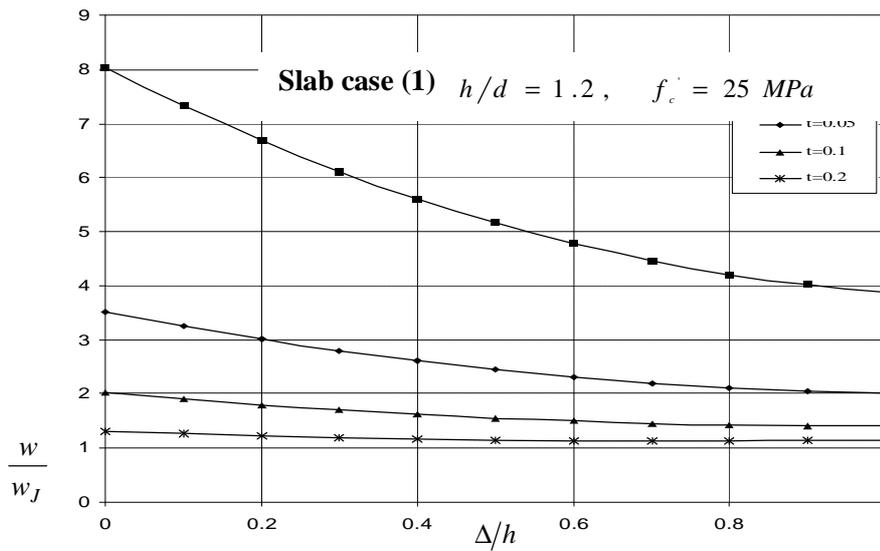
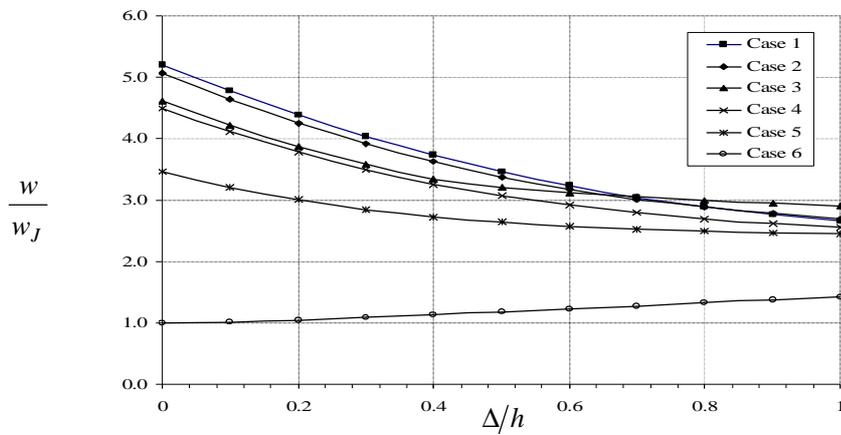


Figure (3) Effect of varying the slab material parameter $t \left(= r \frac{f_y}{f_c} \right)$ on the load – deflection behaviour of a RC square slab of the type (case 1)



$$h/d = 1.2, r = 0.2\%, f_y = 400\text{MPa}, f'_c = 25\text{MPa}$$

Figure (4) Effect of varying the slab boundary restraints on the load – deflection behaviour of a typical RC square slab

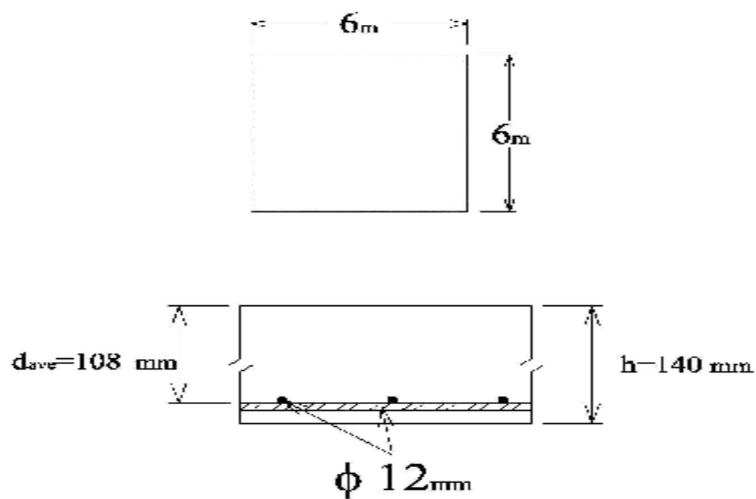


Figure (5) Details of the Examined Slab