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On the Wielandt and Bhatt-Dedania Theorems

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Abstract

In this paper, we generalize a norm topology in Wielandt's theorem for unital normed algebras and in Bhatt-Dedania's theorem for Banach algebras, with each element being a zero topological divisor, by using 2-normed algebra and 2-Banach algebra, respectively.

I. Introduction

In 1949, Wielandt proved that the difference between the commutator of any two elements in the unital normed algebra is not equal to the identity [7]. In 1995, Bhatt-Dedania have proven that each element of a complex Banach algebra $\mathfrak{A}$ is a zero topological divisor if $\mathfrak{A}$ was an infinite-dimensional and has admits an orthogonal basis [1]. In this paper, especially in the second section, we prove that if $\mathfrak{A}$ is a 2-normed algebra, then the difference of any two commutator elements is not equal to the identity. In the third section, we prove that each element of the complex 2-Banach algebra $\mathfrak{A}$, is a zero topological divisor, if $\mathfrak{A}$ is an infinite-dimensional, and has admits an orthogonal basis. Note that, in this paper, all fields are considered complex. Now, we recall that, Srivastava, Bhattacharya, and Lal [5],[6], proposed the definition of 2-normed algebra and 2-Banach algebra as follows: Let $\mathfrak{B}$ be a sub-algebra of dimension greater than 1 of an algebra $\mathfrak{A}$, $\|.,.\|$ be a 2-norm in $\mathfrak{A}$ and $z_1, z_2 \in \mathfrak{A}$ be linearly independent, non-invertible and be such that for all $m, n \in \mathfrak{B}$, $\|mn, z_i\| \leq \|m, z_i\| \, \|n, z_i\|$, $i = 1, 2$. Then $\mathfrak{B}$ is said to be 2-normed algebra with respect to z_1, z_2. Also if $\mathfrak{B}$ is a 2-normed algebra and $\{m_k\}_{k\in\mathbb{N}}$ is considered as the sequence in $\mathfrak{B}$ satisfying $\lim_{k,w\to\infty} \|m_k - m_w, z_i\| = 0, i = 1, 2$, if there exists an element m in $\mathfrak{B}$ for which $\lim_{k\to\infty} \|m_k - m, z_i\| = 0$, $i = 1, 2$ then $\mathfrak{B}$ is called a 2-Banach algebra with respect to z_1, z_2. The element m, in the normed algebra $\mathfrak{A}$ is named a zero topological divisor [3] if there is a sequence $\{m_k\}_{k\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\|m_k\| = 1$, $mm_k \to 0$ or $m_k m \to 0$ as $k \to \infty$. An orthogonal basis of a Banach algebra $\mathfrak{A}$ is a sequence $\{m_k\}_{k\in\mathbb{N}}$ in $\mathfrak{A}$ such that each $m \in \mathfrak{A}$ can be represented as $m = \sum \alpha_k m_k$, where α_k's are scalars, and $m_k m_w = \delta_{kw} m_k$, δ_{kw} being the delta of Kronecker [2],[4].If $\{m_k\}_{k\in\mathbb{N}}$ is an orthogonal basis in $\mathfrak{A}$, then $\{m_k\}_{k\in\mathbb{N}}$ a basis of Schauder. Our goal of this paper is to generalize Wielandt's theorem [7], and Bhatt-Dedania's theorem [1] in 2-normed algebra.

Now, we can present the most prominent results.

2. Wielandt theorem in 2-normed algebra

In this section, we generalize the theorem of Wielandt, and as the following:

Theorem 2.1: Let $\mathfrak{A}$ be a 2-normed unital algebra with respect to z_1, z_2 with unit e. If $m, n \in \mathfrak{A}$, then $mn - nm \neq e$.

Proof: By contradiction, let $m, n \in \mathfrak{A}$ such that $mn - nm = e$, so for all $k \in \mathbb{N}$ we get:

$$m^k n - n\, m^k = k m^{k-1} \neq 0 \quad \ldots\ldots (2.1)$$

We will prove (2.1) by induction over $k \in \mathbb{N}$.

when $k = 1$, $mn - nm = 1$, so (2.1) holds.
Now, suppose that for $k = w$ ($w \in \mathbb{N}$), (2.1) is true, then $m^w \neq 0$, and

$$\begin{aligned} & m^{w+1} n - n\, m^{w+1} \\ &= m^w m\, n - m^w\, n\, m \\ &+ m^w\, n\, m - n m^w m \\ &= m^w(mn - nm) + (m^w\, n - n m^w)m \\ &= m^w e + w m^{w-1} m \\ &= m^w e + w m^w = e m^w + w m^w \\ &= (1 + w) m^w \end{aligned}$$

it follows that,

$$w\| m^{w-1}, z_i\| = \| w m^{w-1}, z_i\| = \|m^w n - n\, m^w, z_i\| \leq$$

$$\begin{aligned} & \|m^w n, z_i\| + \|n\, m^w, z_i\| \leq \\ & \|m^w, z_i\|\, \|n, z_i\| + \|n, z_i\|\|m^w, z_i\| = \\ & 2\, \|m^w, z_i\|\|n, z_i\| \leq \\ & 2\, \|m^{w-1}, z_i\|\, \|m, z_i\|\, \|n, z_i\| \end{aligned}$$

It implies, $w \leq 2\, \|m, z_i\|\|n, z_i\|, i = 1,2$ and this is impossible. ■

Corollary 2.2 [7]: In a 2-Hilbert space $\mathcal{H}$, the identity operator cannot be represented by the commutator of two-bounded linear operators in $\mathcal{L}(\mathcal{H})$. Where $\mathcal{L}(\mathcal{H})$ denotes the linear operator on $\mathcal{H}$. ■

Example 2.3: The previous corollary can be seen as one-dimensional form of uncertainty.
Suppose that $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, $q: \mathcal{H} \to \mathcal{H}$ where $q(g)(m) = m g(m)$,
$\mathcal{D}(q) = \{g \in \mathcal{L}^2(\mathbb{R}) : m \to m g(m) \in \mathcal{L}^2(\mathbb{R})\}$, the operator of the coordinate,
$p: \mathcal{H} \to \mathcal{H}$ where $p(g)(m) = -i\acute{g}(m)$ the momentum operator, and
$\mathcal{D}(p) = \{g \in \mathcal{L}^2(\mathbb{R}): g$ absolutely continuous, $\acute{g} \in \mathcal{L}^2(\mathbb{R})\}$.
It additionally implies, $pq - qp = -i\, id_{\mathcal{D}}\ on\ \mathcal{D} = \mathcal{D}(q) \cap \mathcal{D}(p)$
In accordance with the corollary, $\mathcal{D} = \mathcal{L}^2(\mathbb{R})$ can never be the case.

3. Bhatt-Dedania theorem in 2-Banach algebra

An element m in a 2-normed algebra $\mathfrak{A}$ with respect to z_1, z_2 is called a zero topological divisor, if there exist a sequence $\{m_k\}_{k\in\mathbb{N}}$ in $\mathfrak{A}$ such that;
$\|m_k, z_i\| = 1,\ i = 1,2$, $\{m m_k\} \to 0$ or $\{m_k m\} \to 0\ as\ k \to \infty$.
In this section, we generalize the Bhatt-Dedania theorem as follows:

Theorem 3.1: Each one of the elements of a 2-Banach algebra $\mathfrak{A}$ with respect to z_1, z_2 is a zero topological divisor, if $\mathfrak{A}$ has infinite-dimensional and admits an orthogonal basis.

Proof: Assume that $\{m_k\}$ be an orthogonal basis in $\mathfrak{A}$, and let $m \in \mathfrak{A}$, such that $m = \sum \alpha_k m_k$. Since $m_k m_w = \delta_{kw}\, m_k$ for all $k, w \in \mathbb{N}$
it follows that for any $q \in \mathbb{N}$,

$$\|m_q, z_i\| = \|{m^2}_q, z_i\| = \left\|m_q m_q, z_i\right\| \leq$$

$$\|m_q, z_i\|\, \|m_q, z_i\| = \|m_q, z_i\|^2, i = 1,2$$

Hence $\left\|m_q, z_i\right\| \geq 1$, and
$m m_q = (\sum \alpha_k m_k) m_q = \sum \alpha_k m_k\, m_q = \alpha_q m_k$ is convergent to 0 as $k \to \infty$. Now, let $g_q = \frac{m_q}{\|m_q, z_i\|}$, then $\left\|g_q, z_i\right\| = \frac{\|m_q, z_i\|}{\|m_q, z_i\|} = 1$ which implies to, $\left\|m g_q, z_i\right\| \leq \left\|m m_q, z_i\right\| \to 0$. ■

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