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Extended Idempotent Divisor Graph of Zn

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Abstract

Associate graph Л(R) is said to be idempotent divisor graph with vertices set in R*=R-{0} and two non- zero distinct vertices a and b are adjacent if and only if a.b=e, where e an idempotent element non equal 1. In this work we study the extended idempotent divisor graph of Zn is denoted by $\overline{Л(R)}$, with vertices set in R* that is for any two distinct vertices a and b are adjacent if there are positive integers t_1, t_2 such that $a^{t_1}.\ b^{t_2}$=e, where $a^{t_1},\ b^{t_2} \neq 0$ and an idempotent element e not equal 1, and we found the order and the size for some kinds of the idempotent divisor graph of the rings Zn. Also, we found the Hosoya polynomial and the Wiener index for these graphs.

Keywords:
extended idempotent divisor graph, zero divisor graph, idempotent divisor graph, Zn, degree of vertices, size of a graph, Hosoya polynomial and the Wiener index.

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I. Introduction

Throughout this paper we assume a ring R is a commutative ring with identity not equal zero and Z_n denoted an integer ring modulo n. the set of all unite (zero divisor resp.) elements in a ring R denoted by $U(R)((Z(R)\ resp.)$. The concept zero divisor graph gave relationship between two important branches in mathematics graph and ring theory, it is studied by Beck in 1988 **[3]**, whenever find the coloring of commutative ring**.** Later Anderson and Livingston in 1999 **[1]** modify this graph and dented by Γ(R) with vertices $Z(R)^* = Z(R) - \{0\}$ and two elements are adjacent in Γ(R) if and only if $x.y = 0$ in R .Many authors gave different definition **(see for example [2,5,6])**. In 2016 Bennis, Mikram and Taraza introduced the extended zero-divisor graph of R, denoted by $\overline{\Gamma(R)}$ and two distinct vertices x and y are adjacent if and only if, there are positive integers t_1 and t_2 such that $x^{t_1}.y^{t_2} = 0$ and $x^{t_1}, y^{t_2} \neq 0$ **[4].** Recently Mohammad and Shuker introduced a new graph called idempotent divisor graph with vertices set in $R^* = R - \{0\}$, and two non- zero distinct vertices a and b are adjacent if and only if $a.b = e$, for some non-unit idempotent element e, this graph denoted by Л(R) **[12].** Clearly $\Gamma(R) \subset Л(R)$ when R non-local ring and $\Gamma(R) = Л(R)$ when R local ring. In **[13]** Mohammad and abd-almohy gave the extended idempotent divisor graphs and denoted $\overline{Л(R)}$. In this paper we study the extended idempotent divisor graph for the set of integers module n. it contains three sections. In section two we gave the size and the order for rings as the local and reduced rings and the product of a field with local ring and the product of local ring into a reduced ring and we found the size of these graphs and degree of their vertices. In section three we find the Hosoya and the winer index for these rings.

In a graph theory "The complete graph K_n is a graph in which every two distinct vertices are adjacent. The complement of a graph G is a graph $\bar{G}$ on the same vertices such that two distinct vertices of $\bar{G}$ adjacent if and only if they are not adjacent in the graph G. the diameter of a graph G is the greatest distance between any two vertices of a graph G denoted by $diam(G)$**.** The simple graph G is a graph without loops neither multiple edge. A connected graph G is the graph that has a path between every pair of vertices. We denote m is the size of a graph G it is the number of its edges, while n denote as the order of a graph G is the number of its vertices. The center of a graph is the set of all vertices of minimum eccentricity. For more dateless see for example" **[7,9].**

In a ring theory "a ring R is a local if it has only one maximal ideal and $\mathrm{Z_q}$ is a fild of order q. A reduced ring is a ring that has no non-zero nilpotent elements. The idempotent element is an element such that: $\mathrm{a}^2 = \mathrm{a}$. For a nilpotent element x of a ring R, $v(x)$ denotes the order of nilpotency of x; that is the smallest positive integer t such that $\mathrm{x^t} = 0$. the degree of nilpotency of a ring R defined to be the supremum of the orders of nilpotency of its nilpotent elements and it is denoted by $v(R)$. It well known that any finite non local ring R can be written as a direct product of a local ring, while every reduced ring can be written as a direct product of a field. "[**8**].

II. Extended Idempotent Divisor Graph of Z_n

In this section we investigated the idempotent divisor graph of Z_n, also we found an order and size this graph. First, we investigated when Z_n local ring.

It well now if the rings Z_n local, then $n = p^\alpha$, where p prime and α positive integer as well as in **[12]** show if R local ring, then $Л(\mathrm{R}) = \Gamma(\mathrm{R})$.

Theorem 2.1:-let $R = Z_{p^\alpha}$,where p is a prime number and α a positive integer greater than or equal two, then $\overline{Л(\mathrm{R})}$ is a complete graph of order $|Z(R^*)|$.

Proof:-since $R = Z_{P^\alpha}$,then R local ring ,then by **[12]** we have

$Л(\mathrm{R}) = \Gamma(\mathrm{R})$, and $V\big(Л(\mathrm{R})\big) = \mathrm{Z(R)}^*$. We claim that for any $x, y \in Z(R)^*$, $x^{v(x)-1}.y^{v(y)-1} = 0$. Let $x = k_1 p, y = k_2 p$,where k_1, k_2 positive integers $p \nmid k_1, k_2$,and $1 \leq i,i \leq \alpha - 1$, then $x^{v(x)-1}.y^{v(y)-1} = (k_1 p)^{v(x)-1}.(k_2 p)^{v(y)-1} = (l_1.p^{\alpha-1}).(l_2.p^{\alpha-1}) \equiv 0[mod\ p^\alpha]$, where $l_1 = k_1^{v(x)-1}, l_2 = k_2^{v(y)-1}$. So, by **Theorem 3.3[4]**,$\overline{\Gamma(\mathrm{Z_{p^\alpha}})}$ is a complete graph. Consequently $\overline{Л(\mathrm{Z_{p^\alpha}})}$ is a complete graph.

■

Example 2.2:- let $R = Z_{27}$,then $Л(\mathrm{R}) = \Gamma(\mathrm{R})$ In the extended idempotent graph, we have

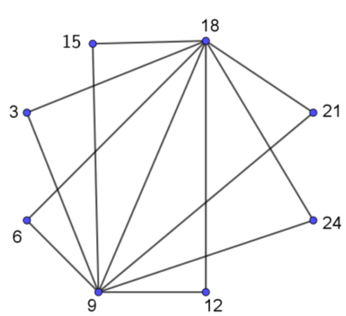

Fig 2.1 $Л(\mathrm{Z_{27}})$

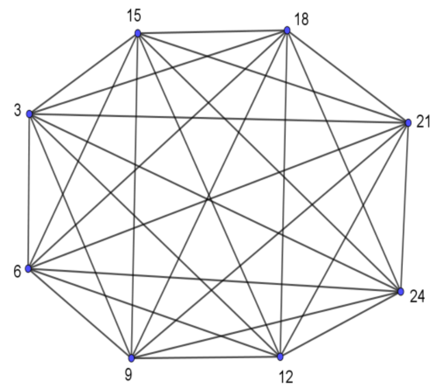

Fig 2. 2 $\overline{Л(\mathrm{Z_{27}})}$

Now, we study when the ring Z_n reduced ring. It well now if Z_n reduced, then $n = p_1 p_2 \dots p_m$, where p_i distinct prime numbers for all $i = 1,2, \dots, m$, as well as the rings $Z_{p_1.p_2 \dots\dots p_m} \cong Z_{p_1} \times Z_{p_2} \times \dots \dots Z_{p_m}$, when p_i distinct prime numbers. The next theorem we give the general form when Z_n reduced ring.

Theorem2.3:-let $R \cong Z_{p_1.p_2 \dots\dots p_m} \cong Z_{p_1} \times Z_{p_2} \times \dots \dots Z_{p_m}$,where $p_1, p_2, \dots p_m$are distinct prime numbers ,then $\overline{Л(\mathrm{R})} = K_{|Z(R)^*|} + \overline{K}_{|U(R)|}$

Proof:- let $x = (x_1, x_2, \dots x_m)$,$y = (y_1, y_2, \dots y_m)$, any two elements in R, then there are three cases:

Case1: if $x, y \in Z(R)^*$,then there exist two positive integers l_1, l_2 such that

$$x_i^{l_1} = \begin{cases} 0_i & if\ x_i = 0_i \\ 1_i & if\ x_i \neq 0_i \end{cases}, \quad y_i^{l_2} = \begin{cases} 0_i & if\ y_i = 0_i \\ 1_i & if\ y_i \neq 0_i \end{cases}$$

Which implies that

$$x_i y_i = \begin{cases} 0_i & if\ x_i\ or\ y_i = 0_i \\ 1_i & if\ x_i\ and\ y_i \neq 0_i \end{cases},$$

$x.y = (t_1, t_2, \dots., t_m) \neq (1_1, 1_2, \dots \dots, 1_m)$, where $t_i \in \{0_i, 1_i\}$. Since $x, y \in Z(R)^*$, then $x.y$ is an idempotent element not equal one.

Case2:$x \in Z(R)^*, y \in U(R)$, then there exist two positive integers l_1, l_2 such that

$x^{l_1}.y^{l_2} = (s_1, s_2, \dots. s_m)$, for every $s_i \in \{0_i, 1_i\}$, then x and y are adjacent in $\overline{Л(\mathrm{R})}$

Case 3: if $x, y \in U(R)$, then

$x^{l_1}.y^{l_2} = (1_1, 1_2, \dots. 1_m)$, which is a contradiction, therefore $x$and y are not adjacent in $\overline{Л(\mathrm{R})}$

From above cases we conclude $\overline{Л(\mathrm{R})} = K_{|Z(R)^*|} + \overline{K}_{|U(R)|}$.

■

Corollary2.4:- let $R \cong Z_{p_1.p_2 \dots\dots p_m} \cong Z_{p_1} \times Z_{p_2} \times \dots \dots Z_{p_m}$,where $p_1, p_2, \dots p_m$ are distinct prime numbers ,then we have

$$\deg(v)_{v \in \overline{Л(\mathrm{R})}} = \begin{cases} v \in Z(R)^*: |R^*| - 1 \\ v \in U(R): |Z(R)^*| \end{cases}$$, so we get the order of $\overline{Л(\mathrm{R})}$ is :

$\mathrm{n} = \mathrm{order}(\overline{Л(\mathrm{R})}) = \prod_{i=1}^{m} p_i - 1$, and the size of$\overline{Л(\mathrm{R})}$:

$$\mathrm{m} = size(\overline{Л(\mathrm{R})}) = \frac{1}{2}[(|R^*| - 1)(|Z(R)^*|) + |Z(R)^*||U(R)|]$$

Corollary2.5:-let $R \cong Z_{p_1.p_2......p_m} \cong Z_{p_1} \times Z_{p_2} \times \dots \dots Z_{p_m}$,where $p_1, p_2, \dots p_m$are distinct prime numbers ,then $Cent(\overline{Л(R)}) = Z(R)^*$,and $diam(\overline{Л(R)}) = 2$. ■

Example 2.6:- $R \cong Z_{10}$, then $Л(R)$

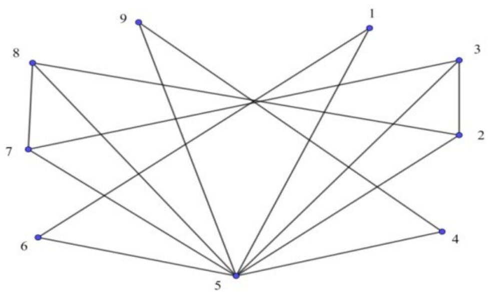

Fig. 2.3 $Л(Z_{10})$

In the extended idempotent graph, we have

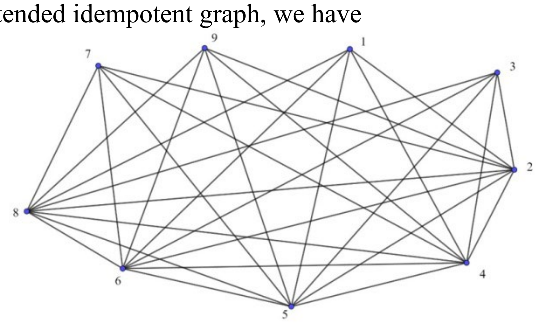

Fig. 2.4 $\overline{Л(Z_{10})}$

Now, we study mixed cases when Z_n direct product field and local.

Theorem 2.7:-let $R \cong Z_{p_1 p_2^\alpha} \cong Z_{p_1} \times Z_{p_2}^\alpha$,where p_1, p_2 prime numbers and $\alpha \geq 2$, then $\deg(v)_{v \in \overline{Л(R)}}$ =

$$\begin{cases} v = (u_1, 0) := |R^*| - 1 \\ v = (0, u_2) := |Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z_{p_1}^*||U(Z_{p_2}^\alpha)| + 1 \\ \quad |U(Z_{p_2}^\alpha)| - 1 \\ v = (0, s_2) := |Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z(Z_{p_2}^\alpha)^*| - 1 \\ v = (u_1, s_2) := |R^*| - 1 \\ v = (u_1, u_2) := |Z_{p_1}^*| + |Z(Z_{p_2}^\alpha)^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| \end{cases}$$

Where $u_1 \in Z_{p_1}^*$, $s_2 \in Z(Z_{p_2}^\alpha)^*$, $u_2 \in U(Z_{p_2}^\alpha)$,and the *order* of $(\overline{Л(R)})$:

$n = p_1 \times p_2^\alpha - 1$, where $\alpha \geq 2$ and the *size* of $(\overline{Л(R)})$:

$m = \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1 + |Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)| - 1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R| - 3) + |Z(Z_{p_2^\alpha})|(1 - |Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]$

Proof:- the order of $(\overline{Л(R)})$:, $n = p_1 \times p_2^\alpha - 1$, where $\alpha \geq 2$. Now let $v = (r, z) \in R^*$,be any element in R^*; where $r \in Z_{p_1}$,$z \in Z_{p_2^\alpha}$,then we can distinguish the vertices into disjoint subsets:

$A = \{(u_1, 0): u_1 \in |Z_{p_1}^*|\}$, $B = \{(0, u_2): u_2 \in U(Z_{p_2}^\alpha)\}$, $C = \{(0, s_2): s_2 \in Z(Z_{p_2}^\alpha)^*\}$,$D = \{(u_1, s_2): u_1 \in |Z_{p_1}^*|, s_2 \in Z(Z_{p_2}^\alpha)^*\}$, $E = \{(u_1, u_2): u_1 \in |Z_{p_1}^*|, u_2 \in U(Z_{p_2}^\alpha)\}$, clearly $|A| = |Z_{p_1}^*|$,$|B| = |U(Z_{p_2}^\alpha)|$,$|C| = |Z(Z_{p_2}^\alpha)^*|$,$|D| = |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*|$, $|E| = |Z_{p_1}^*||U(Z_{p_2}^\alpha)|$

First :if $v = (u_1, 0) \in A$,since by Remark 2.2**[13]**,there exists a positive integer t such that :$(u_1, 0)^t = (1,0)$, then $(u_1, 0)$ adjacent with every other vertices in the graph $\overline{Л(R)}$, so

$\deg(v)_{v \in A} = |R^*| - 1$

Second :if $v = (0, u_2) \in B$, again by Remark 2.2**[13]**,there exists a positive integer l such that :$(0, u)^l = (0,1)$,then v adjacent with every element in A,D,E and B ,so

$$\deg(v)_{v \in B} = |A| + |D| + |E| + |B| - 1 = |Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z_{p_1}^*||U(Z_{p_2}^\alpha)| + |U(Z_{p_2}^\alpha)| - 1$$

Third :if $v = (0, s_2) \in C$, since for any $s_1, s_2 \in Z(Z_{p_2}^2)^*$, $s_1^{v(s_1)-1}.s_2^{v(s_2)-1} = 0$, element in A, C and D, so

$$\deg(v)_{v \in C} = |A| + |D| + |C| - 1 = |Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z(Z_{p_2}^\alpha)^*| - 1$$

Fourth :if $v = (u_1, s_2) \in D$,then v adjacent with every other vertices in the graph $\overline{Л(R)}$,so that :

$\deg(v)_{v \in D} = |R^*| - 1$

Fifth :if $v = (u_1, u_2)$,since v not adjacent for any element in C and E ,so that :

$\deg(v)_{v \in E} = |A| + |B| + |D| = |Z_{p_1}^*| + |Z(Z_{P_2}^\alpha)^*| + |Z_{p_1}^*||Z(Z_{P_2}^\alpha)^*|$

Now, to find the *size* of the graph $\overline{Л(R)}$:since

$m = \frac{1}{2}[\sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) + \sum_{v \in C} \deg(v) + \sum_{v \in D} \deg(v) + \sum_{v \in E} \deg(v)]$

$$= \frac{1}{2}[(|R^*| - 1)(|Z_{p_1}^*|) + (|Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z_{p_1}^*||U(Z_{p_2}^\alpha)| + |U(Z_{p_2}^\alpha)| - 1)(|U(Z_{p_2}^\alpha)|) + (|Z_{p_1}^*| + |Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*| + |Z(Z_{p_2}^\alpha)^*| - 1)(|Z(Z_{p_2}^\alpha)^*|) + (|R^*| - 1)(|Z_{p_1}^*||Z(Z_{p_2}^\alpha)^*|) + (|Z_{p_1}^*| + |Z(Z_{P_2}^\alpha)^*| + |Z_{p_1}^*||Z(Z_{P_2}^\alpha)^*|)(|Z_{p_1}^*||U(Z_{p_2}^\alpha)|)]$$

Put $|R^*| = |R| - 1$, $|(Z_{p_1})^*| = |Z_{p_1}| - 1$, $|Z(Z_{p_2^\alpha})^*| = |Z(Z_{p_2^\alpha})| - 1$, and simplify the size we get :

$m = \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1 + |Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)| - 1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R| - 3) + |Z(Z_{p_2^\alpha})|(1 - |Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]$

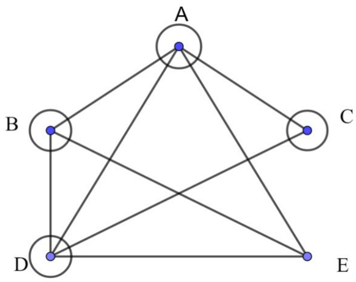

Fig. 2.5 $\overline{Л(Z_{p_1p_2^{\alpha}})}$

At the beginning of this section, we give a generalized Theorem 2.7, when R direct product local ring with reduced ring.

Theorem 2.8:- let $R \cong Z_{p_1^{\alpha}.p_2.......p_m} \cong Z_{p_1}^{\alpha} \times Z_{p_2} \times \ldots.. \times Z_{p_m}$,where p_i distinct prime numbers, α an integer number, Then :

$\deg(v)_{v\in\overline{Л(R)}}$

$$= \begin{cases} v = (z_1', x_i) := \left|Z(Z_{p_1^{\alpha}})^*\right| + \left|Z(Z_{p_1^{\alpha}})\right| \sum_{k=1}^{n-2} \binom{n-1}{k} \\ \quad + \left|Z(Z_{p_1^{\alpha}})\right|\left|(Z_{p_i})^*\right| - 1 \\ v = (z_1, x_i) := |R^*| - 1 \\ v = (z_1, x_i) := |R^*| - 1 \\ v = (u_1, x_i) := |R^*| - \left|Z(Z_{p_1^{\alpha}})^*\right| - 1 \\ v = (u_1, x_i) := |R^*| - \left|Z(Z_{p_1^{\alpha}})^*\right| - 1 \\ v = (u_1, x_i) := |R^*| - \left|Z(Z_{p_1^{\alpha}})^*\right| - 1 \end{cases}$$

for every $z_1' \in Z(Z_{p_1^{\alpha}})^*, z_1 \in Z(Z_{p_1^{\alpha}}),\ u_1 \in U(Z_{p_1^{\alpha}}), u_2 \in (Z_{p_i})^*$

The Order of $(\overline{Л(R)})$;$n = p_1^{\alpha} \times p_i - 1;\ 2 \le i \le m,$

The size of $(\overline{Л(R)})$: $m = \frac{1}{2}[(|Z(Z_{p_1^{\alpha}})| - 1)^2(|Z_{p_i}| - 1) + (|Z(Z_{p_1^{\alpha}})| - 1)(|Z(Z_{p_1^{\alpha}})| - |U(Z_{p_1^{\alpha}})||Z_{p_i}| - 3) + (|R| - 2)(|Z_{p_i}| - 1)(|Z(Z_{p_1^{\alpha}})| + |U(Z_{p_1^{\alpha}})| + 1) + \sum_{k=1}^{n-2}\binom{n-1}{k}\left((|R| - 2)(|Z(Z_{p_1^{\alpha}})| + |U(Z_{p_1^{\alpha}})|) + (|Z(Z_{p_1^{\alpha}})| - 1)(|Z(Z_{p_1^{\alpha}})| - |U(Z_{p_1^{\alpha}})|)\right)]$

Proof:- The Order of $(\overline{Л(R)})$; $n = p_1^{\alpha} \times p_i - 1$,

Now; let $v = (r_1, r_2, \ldots r_m) \in R^*$,where $r_1 \in Z_{p_1^{\alpha}}, r_i \in Z_{p_i}$; $2 \le i \le m$, then we can distinguish the vertices into disjoint subsets :

$A = \{(z_1', 0_2, 0_3, \ldots \ldots, 0_m): z_1 \in Z(Z_{p_1^{\alpha}})^*\}$, $B = \{(z_1, x_2, x_3, \ldots, x_m): z_1 \in Z(Z_{p_1^{\alpha}}), x_i \in (Z_{p_i})^*; 2 \le i \le m\}$, , $C = \{(z_1, x_2, x_3, \ldots, x_m): z_1 \in Z(Z_{p_1^{\alpha}}), x_i \in Z_{p_i}\} - (A \cup B), 2 \le i \le m$, $D = \{(u_1, 0_2, 0_3, \ldots \ldots, 0_n): u_1 \in U(Z_{p_1^{\alpha}})\}$,

$E = \{(u_1, x_2, x_3, \ldots, x_m): u_1 \in U(Z_{p_1^{\alpha}}), x_i \in (Z_{p_i})^*, 2 \le i \le m\}$,

$F = \{(u_1, x_2, x_3, \ldots x_m): : u_1 \in U(Z_{p_1^{\alpha}}), x_i \in U(Z_{p_i})\} - (D \cup E); 2 \le i \le m$. Then by Remark 2.2**[13]**

$u_1^{l_1}, u_2^{l_2} = 1, z_1^{v(z_1)}, z_2^{v(z_2)} = 0$, where $v(z_1), v(z_2), l_1, l_2$ are positive integers

It is obvious that, $|A| = |Z(Z_{p_1^{\alpha}})^*|, |B| = |Z(Z_{p_1^{\alpha}})||(Z_{p_i})^*|$, $|C| = |Z(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k}$, $|D| = |(Z_{p_i})^*|$, $|E| = |U(Z_{p_1^{\alpha}})||(Z_{p_i})^*|$, $|F| = |U(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k}$,

every element in A adjacent with every element in A, B and C, so

$\deg(v)_{v\in A} = |Z(Z_{p_1^{\alpha}})^*| + |Z(Z_{p_1^{\alpha}})||(Z_{p_i})^*| + |Z(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k} - 1$. Also, every element in B adjacent with every element in R^*, so $\deg(v)_{v\in B} = |R^*| - 1$

. Every element in C adjacent with every element in R^*, so$\deg(v)_{v\in C} = |R^*| - 1$. While every element in D adjacent with every element in $R^* - A$. Therefore

$\deg(v)_{v\in D} = |R^*| - |A| - 1 = |R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1$. Every element in E adjacent with every element in $R^* - A$. Whence $\deg(v)_{v\in E} = |R^*| - |A| - 1 = |R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1$. Finally, since every element in F adjacent with every element in $R^* - A$, then we get $\deg(v)_{v\in F} = |R^*| - |A| - 1 = |R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1$.To find the size of $\overline{Л(R)}$. Since m =size$(\overline{Л(R)}) = \frac{1}{2}\sum_{v\in\overline{Л(R)}} \deg(v)$, then

$m = \frac{1}{2}[\sum_{v\in A}\deg(v) + \sum_{v\in B}\deg(v) + \sum_{v\in C}\deg(v) + \sum_{v\in D}\deg(v) + \sum_{v\in E}\deg(v) + \sum_{v\in F}\deg(v)]$

$m = \frac{1}{2}[(|Z(Z_{p_1^{\alpha}})^*| + |Z(Z_{p_1^{\alpha}})||(Z_{p_i})^*| + |Z(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k} - 1)|Z(Z_{p_1^{\alpha}})^*| + (|R^*| - 1)(|Z(Z_{p_1^{\alpha}})||(Z_{p_i})^*|) + (|R^*| - 1)(|Z(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k}) + (|R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1)|(Z_{p_i})^*| + (|R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1)|U(Z_{p_1^{\alpha}})||(Z_{p_i})^*|)(+(|R^*| - |Z(Z_{p_1^{\alpha}})^*| - 1)(|U(Z_{p_1^{\alpha}})|\sum_{k=1}^{n-2}\binom{n-1}{k}))]$

Put $|R^*|=|R|-1$, $|Z(Z_{p_1^\alpha})^*|=|Z(Z_{p_1^\alpha})|-1$, $|(Z_{p_i})^*|=|Z_{p_i}|-1$, and simplify the size we get :

$m=\frac{1}{2}[(|Z(Z_{p_1^\alpha})|-1)^2(|Z_{p_i}|-1)+(|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})||Z_{p_i}|-3)+(|R|-2)(|Z_{p_i}|-1)(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|+1)+\sum_{k=1}^{n-2}\binom{n-1}{k}\left((|R|-2)(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|)+(|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})|)\right)]$

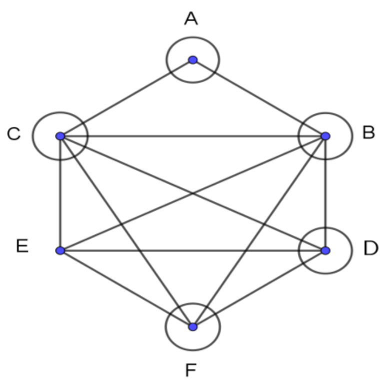

Fig. 2.6 $\overline{\text{Л}(Z_{p_1^\alpha.p_2\ldots\ldots p_m})}$

III. Hosoya Polynomial and Wiener Index of Extended Idempotent Divisor Graph Of

Recall that:

The **Hosoya polynomial[11]** of a graph $\overline{\text{Л}(R)}$ is:

$H(\overline{\text{Л}(R)},x)=\sum_{k=0}^{diam(\overline{\text{Л}})} d(\overline{\text{Л}(R)},k)x^k$, such that $d(\overline{\text{Л}(R)},k)$ be the number of pairs of vertices that are at distance k apart in a connected graph $\overline{\text{Л}}$., for k=0,1,2,..., diam($\overline{\text{Л}(R)}$).

The **Wiener index[14]** of a graph $\overline{\text{Л}(R)}$ is the derivative of the Hosoya polynomial of the graph$\overline{\text{Л}}$(R) with respect to x and putting x=1, that is $W(\text{Л})=\frac{d}{dx}H(\overline{\text{Л}(R)},x)\Big|_{x=1}$

In this section we find Hosoya polynomial and Wiener index of extended idempotent divisor graph of $Z_n$for all cases in section two.

Lemma 3.1[10]:- let G be a connected graph of order n, then

$\sum_{k=0}^{diam(G)} d(G,k)=\frac{1}{2}n(n+1)$

It well now that if a graph G complet , then:

$H(G,x)=a_0+a_1x=n+\frac{n(n-1)}{2}.x$. So when Z_n local ring we are done. Therefore we begin when Z_n a reduced ring.

Theorem3.2:- let $R\cong Z_{p_1.p_2\ldots\ldots p_m}\cong Z_{p_1}\times Z_{p_2}\times\ldots\ldots Z_m$,where $p_1,p_2,\ldots p_m$are prime numbers ,then

$H(\overline{\text{Л}(R)},x)=a_0+a_1x+a_2x^2$, where

$a_0=\prod_{i=1}^m p_i-1$, $a_1=\frac{1}{2}[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]$

$a_2=\frac{1}{2}[\prod_{i=1}^m {p_i}^2-\prod_{i=1}^m p_i-((|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|)]$

Proof:- since R reduced ring ,the diam($\overline{\text{Л}(R)}$)=2, so $H(\overline{\text{Л}(R)},x)=a_0+a_1x+a_2x^2$, it will note that $a_0=d(\overline{\text{Л}(R)};0)$ = order (R) ,so by Theorem2.3, $a_0=\prod_{i=1}^m p_i-1$

$a_1=d(\overline{\text{Л}(R)};1)$ = size(R) , again by theorem 2.3,

$a_1=\frac{1}{2}[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]$

And by Lemma3.1, we have

$a_2=d(\overline{\text{Л}(R)};2)=\frac{n(n+1)}{2}-a_1-a_0$, so that

$a_2=\frac{1}{2}\prod_{i=1}^m p_i(\prod_{i=1}^m p_i-1)-(\prod_{i=1}^m p_i-1)-\frac{1}{2}[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]=$
$=\frac{1}{2}[\prod_{i=1}^n {p_i}^2-3\prod_{i=1}^n p_i+2-((|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|)$

Corollary3.3:- let $R\cong Z_{p_1.p_2\ldots\ldots p_n}\cong Z_{p_1}\times Z_{p_2}\times\ldots\ldots Z_{p_n}$,where $p_1,p_2,\ldots p_n$are prime numbers ,then the Wiener index of $\overline{\text{Л}(R)}$ is :

$W(\overline{\text{Л}(R)})=\frac{1}{2}[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]+(|R^*|)^2-|R^*|-((|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|)$.

Proof:- $W(\overline{\text{Л}(R)})=\frac{d}{dx}H(\overline{\text{Л}(R)};x)|_{x=1}$

$W(\overline{\text{Л}(R)})=0+\frac{1}{2}x[[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]+2x\frac{1}{2}[\prod_{i=1}^n {p_i}^2-3\prod_{i=1}^n p_i+2-((|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|)]]|_{x=1}$

$=\frac{1}{2}[(|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|]+\prod_{i=1}^n {p_i}^2-3\prod_{i=1}^n p_i+2-((|R^*|-1)(|Z(R)^*|)+|Z(R)^*||U(R)|)$. ■

Theorem 3.4:- let $R\cong Z_{p_1p_2^\alpha}\cong Z_{p_1}\times Z_{p_2}^\alpha$,where p_1,p_2 prime numbers and $\alpha\geq 2$, then $H(\overline{\text{Л}(R)},x)=a_0+a_1x+a_2x^2$, where

$a_0=p_1\times p_2^\alpha-1$,

$$a_1=\frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|)+|U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1)+|Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})|+|R|-3)+|Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|)+2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]$$

$$a_2=\frac{1}{2}[(p_1\times p_2^\alpha)^2-3(p_1\times p_2^\alpha)+2-[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|)+|U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1)+|Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})|+|R|-3)+|Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|)+2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]].$$

Proof :- -the diameter of the $\overline{Л(R)}$ is: diam($\overline{Л(R)}$)=2, so that

$H(\overline{Л(R)},x) = a_0 + a_1x + a_2x^2$,

$a_0 = d(\overline{Л(R)};0)$ = order ($\overline{Л(R)}$), so by Theorem2.7,

$a_0 = p_1 \times p_2^\alpha - 1$

$a_1 = d(\overline{Л(R)};1)$=size($\overline{Л(R)}$), again by Theorem2.7, $a_1 = \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]$,

And by Lemma3.1, we have

$a_2 = d(\overline{Л(R)};2) = \frac{n(n+1)}{2} - a_1 - a_0$, so that

$$a_2 = \frac{1}{2}(p_1 \times p_2^\alpha)(p_1 \times p_2^\alpha - 1) - (p_1 \times p_2^\alpha - 1) - \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]$$

$=\frac{1}{2}[(p_1 \times p_2^\alpha)^2 - 3(p_1 \times p_2^\alpha) + 2 - [|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]]$. ■

Corollary 3.5:- let $R \cong Z_{p_1p_2^\alpha} \cong Z_{p_1} \times Z_{p_2}^\alpha$,where p_1,p_2 prime numbers and $\alpha \geq 2$, then the Wiener index of $\overline{Л(R)}$ is : $W(\overline{Л(R)}) = \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|] + [(p_1 \times p_2^\alpha)^2 - 3(p_1 \times p_2^\alpha) + 2 - [|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]]$

Proof :- $W(\overline{Л(R)}) = \frac{d}{dx}H(\overline{Л(R)};x)|_{x=1}$

$W(\overline{Л(R)}) = 0 + [x\frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|] + 2x\frac{1}{2}[(p_1 \times p_2^\alpha)^2 - 3(p_1 \times p_2^\alpha) + 2 - [|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]]]|_{x=1}$.

$W(\overline{Л(R)}) = \frac{1}{2}[|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|] + [(p_1 \times p_2^\alpha)^2 - 3(p_1 \times p_2^\alpha) + 2 - [|Z_{p_1}||U(Z_{p_2}^\alpha)||Z(Z_{p_2^\alpha})|)(1+|Z_{p_1}|) + |U(Z_{p_2}^\alpha)|(|U(Z_{p_2}^\alpha)|-1) + |Z_{p_1}||Z(Z_{p_2^\alpha})|(|Z(Z_{p_2^\alpha})| + |R|-3) + |Z(Z_{p_2^\alpha})|(1-|Z(Z_{p_2^\alpha})|) + 2|Z_{p_1}||(U(Z_{p_2}^\alpha))^2|]]$. ■

Theorem 3.6:- let $R \cong Z_{p_1^\alpha.p_2.......p_m} \cong Z_{p_1}^\alpha \times Z_{p_2} \times ..... \times Z_{p_m}$,where p_i distinct prime numbers, α an integer number, Then $H(\overline{Л(R)},x) = a_0 + a_1x + a_2x^2$, where

$a_0 = n = p_1^\alpha \times p_i - 1$,

$$a_1 = \frac{1}{2}[(|Z(Z_{p_1^\alpha})|-1)^2(|Z_{p_i}|-1) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3) + (|R|-2)(|Z_{p_i}|-1)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|+1) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})|)\Big)]$$

Proof:-since the diameter of $\overline{Л(R)}$ is: diam($\overline{Л(R)}$)=2, so that

$H(\overline{Л(R)},x) = a_0 + a_1x + a_2x^2$,

$a_0 = d(\overline{Л(R)};0)$ = order ($\overline{Л(R)}$), so by Theorem2.8,

$a_0 = p_1^\alpha \times p_i - 1$,

$a_1 = d(\overline{Л(R)};1)$=size($\overline{Л(R)}$), again by Theorem2.8, $a_1 = \frac{1}{2}[(|Z(Z_{p_1^\alpha})|-1)^2(|Z_{p_i}|-1) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3) + (|R|-2)(|Z_{p_i}|-1)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|+1) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})|)\Big)]$

$a_2 = d(\overline{Л(R)};2) = \frac{n(n+1)}{2} - a_0 - a_1$, so that by Lemma 3.1 we have

$a_2 = \frac{1}{2}(p_1^\alpha \times p_i)(p_1^\alpha \times p_i - 1) - (p_1^\alpha \times p_i - 1) - \frac{1}{2}[(|Z(Z_{p_1^\alpha})|-1)^2(|Z_{p_i}|-1) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3) + (|R|-2)(|Z_{p_i}|-1)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|+1) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})|)\Big)]$

$=\frac{1}{2}[(p_1^\alpha \times p_i)^2 - 3p_1^\alpha \times p_i + 2 - [(|Z(Z_{p_1^\alpha})|-1)^2(|Z_{p_i}|-1) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3) + (|R|-2)(|Z_{p_i}|-1)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|+1) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)(|Z(Z_{p_1^\alpha})| + |U(Z_{p_1^\alpha})|) + (|Z(Z_{p_1^\alpha})|-1)(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})|)\Big)]]$

Corollary 3.5:- let $R \cong Z_{p_1^\alpha.p_2.......p_m} \cong Z_{p_1}^\alpha \times Z_{p_2} \times ..... \times$

Z_{p_m} ,where p_i distinct prime numbers, α an integer number, then the Wiener index of $\overline{Л(R)}$ is :

Proof :- $W(\overline{Л(R)}) = \frac{d}{dx} H(\overline{Л(R)}; x)|_{x=1}$

$$W(\overline{Л(R)}) = 0 + \frac{1}{2}x\Big[\big(|Z(Z_{p_1^\alpha})|-1\big)^2\big(|Z_{p_i}|-1\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3\big) + (|R|-2)\big(|Z_{p_i}|-1\big)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|+1\big) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})|\big)\Big)\Big] + 2x\frac{1}{2}[(p_1^\alpha \times p_i)^2 - 3p_1^\alpha \times p_i + 2 - [\big(|Z(Z_{p_1^\alpha})|-1\big)^2\big(|Z_{p_i}|-1\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3\big) + (|R|-2)\big(|Z_{p_i}|-1\big)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|+1\big) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})|\big)\Big)]]]|_{x=1}$$

$$= \frac{1}{2}\Big[\big(|Z(Z_{p_1^\alpha})|-1\big)^2\big(|Z_{p_i}|-1\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3\big) + (|R|-2)\big(|Z_{p_i}|-1\big)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|+1\big) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})|\big)\Big)\Big] + [(p_1^\alpha \times p_i)^2 - 3p_1^\alpha \times p_i + 2 - [\big(|Z(Z_{p_1^\alpha})|-1\big)^2\big(|Z_{p_i}|-1\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})| - |U(Z_{p_1^\alpha})||Z_{p_i}|-3\big) + (|R|-2)\big(|Z_{p_i}|-1\big)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|+1\big) + \sum_{k=1}^{n-2}\binom{n-1}{k}\Big((|R|-2)\big(|Z(Z_{p_1^\alpha})|+|U(Z_{p_1^\alpha})|\big) + \big(|Z(Z_{p_1^\alpha})|-1\big)\big(|Z(Z_{p_1^\alpha})|-|U(Z_{p_1^\alpha})|\big)\Big). \blacksquare$$

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References

[1] Anderson, D. F., & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. Journal of algebra, 217(2), 434-447.

[2] Anderson, D. F., & McClurkin, G. (2020). Generalizations of the zero-divisor graph. International Electronic Journal of lgebra, 27(27), 237-262.

[3] Beck, I. (1988). Coloring of commutative rings. Journal of algebra, 116(1), 208-226.

[4] Bennis, D., Mikram, J., & Taraza, F. (2016). On the extended zero divisor graph of commutative rings. Turkish Journal of Mathematics, 40(2), 376-388.

[5] Bennis, D., El Alaoui, B., Fahid, B., Farnik, M., & L'hamri, R. (2021). The i-extended zero-divisor graphs of commutative rings. Communications in Algebra, 49(11), 4661-4678.

[6] Behboodi, M., & Rakeei, Z. (2011). The annihilating-ideal graph of commutative rings II. Journal of Algebra and its Applications, 10(04), 741-753.

[7] Bondy, J. A., & Murty, U. S. R. (1976). Graph theory with applications (Vol. 290). London: Macmillan.

[8] Burton, D. M. (1967). Introduction to modern abstract algebra. Addison-wesley publishing company

[9] Gross, J. L., & Yellen, J. (2003). Handbook of graph theory. CRC press.

[10] Gutman, I. (1993). Some properties of the Wiener polynomial. Graph Theory Notes New York, 125, 13-18.

[11] Hosoya, and Haruo. "On some counting polynomials in chemistry." Discrete applied mathematics 19.1-3 (1988): 239-257.

[12] Mohammad, H. Q., & Shuker, N. H. (2022). Idempotent divisor graph of commutative ring. Iraqi journal of science, 63(2) 645-651

[13] Mohammad, , H. Q., & Abd-almohy, , S. M.. Extended Idempotent Divisor Graph of Commutative Rings. To appear in " Iraqi journal of science", 66(1).

[14] Wiener, H. (1947). Structural determination of paraffin boiling points. Journal of the American chemical society, 69(1), 17-20.