New results on permutation polytope

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Abstract

The aim of this paper is to calculate the Ehrhart polynomial for permutation polytope, that is associative permutation group. The permutation group consists of: symmetric group S_n, alternating group A_n, dihedral group D_n, Frobenius group F_n and cyclic group C_n. after calculating the Ehrhart polynomial, explain it with examples.

الخلاصة

الهدف من البحث حساب متعددة الحدود ايرهارت لمتعدد السطوح التبادلي المرتبط بجموعه التبادل. مجموعه التبادل تتكون من مجموعة التناظر, مجموعه التناوب, مجموعه ثنائي الاسطح , مجموعه فروبينوس ومجموعه الحلقية. بعد حساب متعددة حدود ايرهارت توضيحها بأمثله

Key words: Ehrhart polynomial, permutation polytope ,permutation group

كلمات المفتاحية: متعددة الحدود ايرهارت , متعدد السطوح التبادلي ،مجموعه التبادل

Introduction

The history of permtation polytope is mising. About 2000 BC polytopes showed in mathematical in Babylonia, in Egypt and in Sumerian culture. A few of the regular polytopes were appearing by then. A basic question was to explain the volume of truncated pyramids. They were needed to determine the number of bricks for fortification and buildings [Gruber,2007:243].The permutation polytope is the convex hull of n×n permutation matrices[Striker,2007:1]. This mean P(G) = conv(M(G)) is called the permutation polytope to G[Baumeister,2012:425]. An n×n permutation matrix M(G) is a matrix possess set are zero and one, that rows and columns sum to 1[Heuer,2021:8]. We use the cycle ( ) notation for permutations [Bollobas,2002:9]. In fact a permutation then replaces the numbers 1, …, n by themselves in a different order for example $\begin{pmatrix} 1 & 2 & 3 \dots n \\ x_1 & x_n & x_3 \dots x_n \end{pmatrix}$. If we have two permutations on n objects we can define their product to be the permutation obtained by applying the first by the second.The product ab is the permutation which sends i into yi. This is of course a permutation. Symbolically ab=$\begin{pmatrix} 1 & 2 & 3 \dots n \\ x_1 & x_n & x_3 \dots x_n \end{pmatrix}\begin{pmatrix} x_1 & x_n & x_3 \dots x_n \\ y_1 & y_n & y_3 \dots y_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \dots n \\ y_1 & y_n & y_3 \dots y_n \end{pmatrix}$ [Hall,1969:12−13]. In case G = Sn explain nth Birkhoff polytope Bn = P(M(Sn))[Baumeister,2012:425].

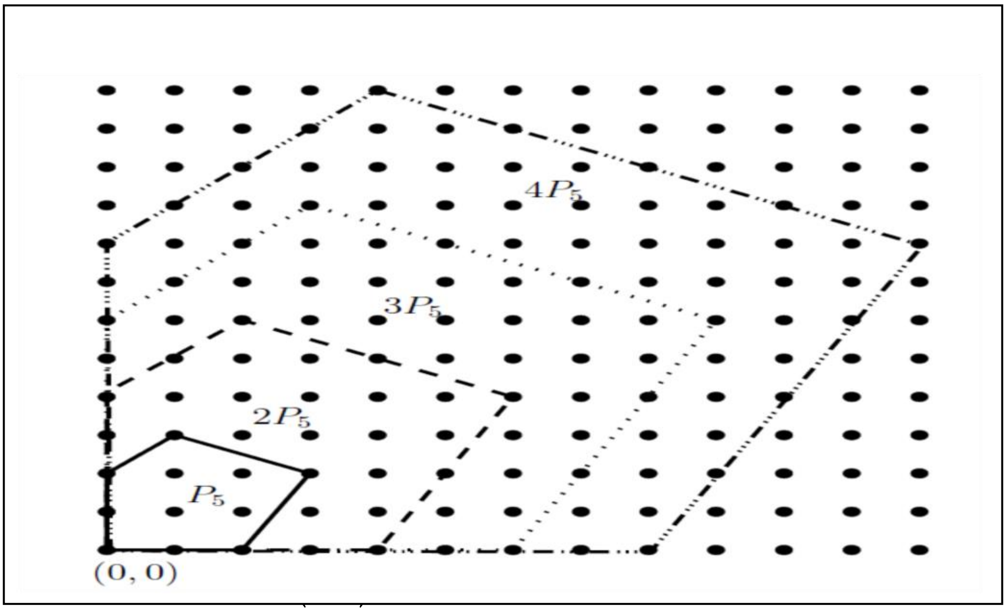
1. The Ehrhart polynomial:

In this part explain Ehrhart polynomials and their relation to the convex polytope together with theorems and methods that found the coefficients of it are given

Defintion(1.1),[Braun,2007:5]:

Let P be a polytope, and tP is expanding P a factor t the Ehrhart polynomial for a convex polytope is defined as in each dimension

$$\mathrm{Lp}(\mathrm{t}) = integer\ lattice\ points\ in\ tp = |\mathrm{tp} \cap \mathrm{Z}^{\mathrm{n}}|$$



Where the polygon tP is resulting of scaling P by a factor of t.

Theorem(1.1),[Braun,2007:6]:

let P convex lattice polygon and t be a positive integer the following equality always holds:

$$\mathrm{Lp(t)=A(p)t^2} + \frac{B(p)}{2}t + 1$$

where B(P) is the number of the boundary points of a polygon P and A(P) represents the area of a polygon P. The following example calculates the Ehrhart polynomial of a polytope. There are ways that calculate the Ehrhart polynomial of polytopes, some methods are found in later.

Example(1.1):

Let us consider three points in two dimensional space Then the convex hull of v_1, v_2 and v_3 is triangle in two dimensional space which are $v_1=(6,1)$, $v_2=(1,7)$ and $v_3=(1,1)$.

We calculate the Ehrhart polynomial of triangle as follows:

$$Lp(t)= A(P)t^2 + \frac{B(p)}{2}t+1$$

The area of Δ, is $A(P) = \frac{1}{2}(6)(5)$ and $B(P)=3$ represents the number of integral points of the triangle: $Lp(t)=15\,t^2 + \frac{3}{2}\,t + 1$

1- The dimension of P is degree of Lp(t) is equal two and area of p is 15.

2-.The boundary points are 3.

3- Last term is 1.

2. Ehrhart polynomial in dimension d,[Beck1,2003:624]:

For the Ehrhart polynomial in dimension d, let P be a d-dimensional lattice polytope such that $P \subset R^d$ and $Lp(t) = \sum_{k=0}^{d} c_k\, t^k$ v the following properties:

1- The dimension of P is degree of $\mathrm{Lp}(\mathrm{t})$

2-The leading term is volume of P.

3-The second term is area of P.

4-$c_0 = 1$.

Definition(2.1),[Beck3,2008:945]:

We define the Dedekind sum by:

$s(a,b) = \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$, a and b are positive integers

$((x))$ is sawtooth explain by

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & if\ x \notin z \\ 0 & if\ x \in z \end{cases}$$

Here {x} = x − $\lfloor x \rfloor$ denotes the fractional part of x.

Remark(2.1),[Beck2,2007:128]:

The Dedekind sum of variable a is a periodic function, with period b, by periodicity of the sawtooth function. That is,

s(a + ib, b) = s(a, b) for all i ∈ Z.

Theorem(2.1),[Beck3,2008:946]:

where a,b and c are pairiwise relatively prime positive integers and P plytope convex hull of (a,0,0), (0,b,0), (0,0,c) and (0,0,0). Then Ehrhart polynomial of P is

$$L_P(t) = \frac{abc}{6}t^3 + \frac{ab+ac+bc+1}{4}t^2 + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right) - s(bc,a) - s(ca,b) - s(ab,c)\right)t + 1$$

Example(2.1): Let us consider the tetrahedron $\Delta \subset R^3$ with vertices (0,0,0), (3,0,0), (0,5,0) and (0,0,7), to find the Ehrhart polynomial for the tetrahedron, we use the formular given by theorem(2.1).

Sol:

$$L_P(t)=\frac{abc}{6}t^3+\frac{ab+ac+bc+1}{4}t^2$$
$$+\left(\frac{3}{4}+\frac{a+b+c}{4}+\frac{1}{12}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}+\frac{1}{abc}\right)\right.$$
$$\left.-s(bc,a)-s(ca,b)-s(ab,c)\right)t+1$$
$$=\frac{105}{6}t^3+\frac{15+21+35+1}{4}t^2$$
$$+\left(\frac{3}{4}+\frac{15}{4}+\frac{1}{12}\left(\frac{35}{3}+\frac{21}{5}+\frac{15}{7}+\frac{1}{105}\right)-s(35,3)\right.$$
$$\left.-s(21,5)-s(15,7)\right)t+1$$
$$=\frac{35}{2}t^3+\frac{72}{4}t^2$$
$$+\left(\frac{18}{4}+\frac{1}{12}\left(\frac{1892}{105}\right)-s(35,3)-s(21,5)\right.$$
$$\left.-s(15,7)\right)t+1$$
$$=\frac{35}{2}t^3+18t^2+\left(\frac{3781}{630}-s(35,3)-s(21,5)-s(15,7)\right)t+1$$

s(a,b)= $s(a \bmod b, b)$

s(35,3)= s(2,3)

s(2,3)= $\sum_{k=0}^{2}\left(\left(\frac{2k}{3}\right)\right)\left(\left(\frac{k}{3}\right)\right)=\left(\left(\frac{2}{3}\right)\right)\left(\left(\frac{1}{3}\right)\right)+\left(\left(\frac{4}{3}\right)\right)\left(\left(\frac{2}{3}\right)\right)$

$$\left(\left(\frac{1}{3}\right)\right)=\frac{1}{3}-0-\frac{1}{2}=\frac{-1}{6},\quad \left(\left(\frac{2}{3}\right)\right)=\frac{2}{3}-0-\frac{1}{2}=\frac{1}{6}$$
$$\left(\left(\frac{4}{3}\right)\right)=\frac{4}{3}-1-\frac{1}{2}=\frac{-1}{6}$$

$\therefore$ s(35,3) $=-\frac{1}{18}$

s(21,5)= s(1,5)

s(1,5) $=\sum_{k=0}^{4}\left(\left(\frac{k}{5}\right)\right)\left(\left(\frac{k}{5}\right)\right)$

$$=\left(\left(\frac{1}{5}\right)\right)\left(\left(\frac{1}{5}\right)\right)+\left(\left(\frac{2}{5}\right)\right)\left(\left(\frac{2}{5}\right)\right)+\left(\left(\frac{3}{5}\right)\right)\left(\left(\frac{3}{5}\right)\right)+\left(\left(\frac{4}{5}\right)\right)\left(\left(\frac{4}{5}\right)\right)$$

$$\left(\left(\frac{1}{5}\right)\right)=\frac{1}{5}-0-\frac{1}{2}=\frac{-3}{10}, \qquad \left(\left(\frac{2}{5}\right)\right)=\frac{2}{5}-0-\frac{1}{2}=\frac{-1}{10}$$

$$\left(\left(\frac{3}{5}\right)\right)=\frac{3}{5}-0-\frac{1}{2}=\frac{1}{10}, \qquad \left(\left(\frac{4}{5}\right)\right)=\frac{4}{5}-0-\frac{1}{2}=\frac{3}{10}$$

$$\therefore s(21,5)=\frac{1}{5}$$

s(15,7)=s(1,7)

$$s(1,7)=\sum_{k=0}^{6}\left(\left(\frac{k}{7}\right)\right)\left(\left(\frac{k}{7}\right)\right)$$

$$=\left(\left(\frac{1}{7}\right)\right)\left(\left(\frac{1}{7}\right)\right)+\left(\left(\frac{2}{7}\right)\right)\left(\left(\frac{2}{7}\right)\right)+\left(\left(\frac{3}{7}\right)\right)\left(\left(\frac{3}{7}\right)\right)+\left(\left(\frac{4}{7}\right)\right)\left(\left(\frac{4}{7}\right)\right)+\left(\left(\frac{5}{7}\right)\right)\left(\left(\frac{5}{7}\right)\right)+\left(\left(\frac{6}{7}\right)\right)\left(\left(\frac{6}{7}\right)\right)$$

$$\therefore s(15,7)=\frac{5}{14}$$

$$L_P(t)=\frac{35}{2}t^3+18t^2+\left(\frac{3781}{630}+\frac{1}{18}-\frac{1}{5}-\frac{5}{14}\right)t+1$$
$$=\frac{35}{2}t^3+18t^2+\left(\frac{3465}{630}\right)t+1$$
$$=\frac{35}{2}t^3+18t^2+\frac{11}{2}t+1$$

3.The Ehrhart polynomial of a Birkhoff polytope,[Beck1,2003:624]:

The Ehrhart polynomial associated with Birkhoff polytope is known for small values. When the polytope associated to the symmetric group, it is called the Birkhoff polytope:

$$B_n = P(S_n)$$

Where n=1,2of the Ehrhart polynomial for a Birkhoff polytope are trivial which are:

$$L_{B_1}(t) = 1$$
$$L_{B_2}(t) = t + 1$$

Remark(3.1),[Beck1,2003:628]:

The relative volume for the Birkhoff polytope is given by n^{n-1}.

Theorem(3.1),[Beck1,2003:627]:

Ehrhart polynomial $L_{B_n}(t)$:

$$L_{B_n}(t) = \frac{1}{(2\pi i)^n} \int_{|z_1|=\varepsilon_1} \cdots \int_{|z_n|=\varepsilon_n} (z_1 \cdots z_n)^{-t-1} \left(\sum_{k=1}^{n} \frac{z_k^{t+n-1}}{\prod_{j\neq k}(z_k - z_j)} \right)^n dz_n \cdots dz_1$$

For any distinct $0 < \varepsilon_1, \ldots, \varepsilon_n < 1$.

Example(3.1),[Beck1,2003:627−628]:

To find the Ehrhart polynomial this polytope when n=3, theorem(3.1) is used:

$$L_{B_3}(t) = \frac{1}{(2\pi i)^3} \int (z_1 z_2 z_3)^{-t-1} \left(\frac{z_1^{t+2}}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2^{t+2}}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3^{t+2}}{(z_3 - z_1)(z_3 - z_2)} \right)^3 d\boldsymbol{z}$$

Where $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < 1$.

We use this fact

$$\frac{z_1^{-t-1}\, z_2^{-t-1}\, z_3^{2t+5}}{(z_3 - z_2)^3 (z_3 - z_1)^3}$$

This equation becomes:

$$L_{B_3}(t) = \frac{1}{(2\pi i)^3}\int z_1^{-t-1}\, z_2^{-t-1} z_3^{-t-1}\left(\frac{z_1^{t+2}}{(z_1-z_2)^3(z_1-z_3)^3} + 3\frac{z_2^{t+2}}{(z_2-z_1)(z_2-z_3)}\cdot\frac{z_1^{t+2}}{(z_1-z_2)^2(z_1-z_3)^2}\right)d\boldsymbol{z}$$

$$L_{B_3}(t) = \frac{1}{(2\pi i)^3}\int \frac{z_1^{2t+5}\, z_2^{-t-1}\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3} - 3\frac{z_1^{t+3} z_2\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^2(z_2-z_3)}\, d\boldsymbol{z}$$

$$L_{B_3}(t) = \frac{1}{(2\pi i)^3}\int \frac{z_1^{2t+5}\, z_2^{-t-1}\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3}\, d\boldsymbol{z} - \frac{3}{(2\pi i)^3}\int \frac{z_1^{t+3} z_2\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^2(z_2-z_3)}\, d\boldsymbol{z} \quad \cdots (1)$$

The first integral yields:

$$\frac{1}{(2\pi i)^3}\int \frac{z_1^{2t+5}\, z_2^{-t-1}\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^3}\, d\boldsymbol{z} = \frac{1}{(2\pi i)^3}\int z_1^{2t+5}\left(\int \frac{z^{-t-1}}{(z_1-z)^3}\, dz\right)^2 dz_1$$

$$= \frac{1}{2\pi i}\int z_1^{2t+5}\left(-\frac{1}{2}(-t-1)(-t-2)\, z_1^{-t-3}\right)^2 dz_1$$

$$= \binom{t+2}{2}^2 \cdots (2)$$

For the second integral yields:

$$-\frac{3}{(2\pi i)^3}\int \frac{z_1^{t+3} z_2\, z_3^{-t-1}}{(z_1-z_2)^3(z_1-z_3)^2(z_2-z_3)}\, d\boldsymbol{z} = -\frac{3}{(2\pi i)^2}\int \frac{z_1^{t+3}\, z_3^{-t}}{(z_1-z_3)^5}\, dz_3 dz_1$$

$$= -\frac{3}{2\pi i}\int z_1^{t+3}\frac{1}{4!}(-t)(-t-1)(-t-2)(-t-3) z_1^{-t-4} dz_1$$

$$= -3\binom{t+3}{4} \cdots (3)$$

Put (2) and (3) in (1), the following is obtained:

$$L_{B_3}(t) = \binom{t+2}{2}^2 - 3\binom{t+3}{4}$$
$$= \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

Vol(B_3) = $n^{n-1} \cdot \frac{1}{8} = 3^2 \cdot \frac{1}{8} = \frac{9}{8}$

4 .The Ehrhart polynomial of polytope of a dihedral group:

Now, the Ehrhart polynomial is obtained:

Theorem(4.1),[Burggraf2,2013:57]:

Let n be an integer, $n \geq 2$

1-where n is odd. Ehrhart polynomial p(D_n) is

$$\sum_{k=0}^{n-2}\binom{2n}{k+1}\binom{t-1}{k} + \sum_{k=n-1}^{2n-2}\left(\binom{2n}{k+1} - \binom{n}{k-n+1}\right)\binom{t-1}{k}$$

And the volume of P(D_n) is $\frac{n}{(2n-2)!}$

2-where n is even and n=2m. The Ehrhart Polynomial of p(D_n) is:

$$\sum_{k=0}^{m-2}\binom{2n}{k+1}\binom{t-1}{k}$$
$$+ \sum_{k=m-1}^{2m-2}\left(\binom{2n}{k+1} - 2\binom{2n-m}{k+1-m}\right)\binom{t-1}{k} +$$
$$\sum_{k=2m-1}^{4m-3}\left(\binom{2n}{k+1} - 2\binom{2n-m}{k+1-m} + \binom{2n-2m}{k+1-2m}\right)\binom{t-1}{k}$$

And the Vol(D_n) is $\frac{n^2}{4(2n-3)!}$

Example(4.1):

To find Ehrhart polynomial of permutation polytope related to dihedral group when n=6.

Now, according to theorem(4.1);

n=2m since n=6 then m=3.

Ehrhart polynomial of p(D_6):

$$\sum_{k=0}^{1}\binom{12}{k+1}\binom{t-1}{k}+\sum_{k=2}^{4}\left(\binom{12}{k+1}-2\binom{9}{k-2}\right)\binom{t-1}{k}+$$
$$\sum_{k=5}^{9}\left(\binom{12}{k+1}-2\binom{9}{k-2}+\binom{6}{k-5}\right)\binom{t-1}{k}$$

$$=\binom{12}{1}\binom{t-1}{0}+\binom{12}{2}\binom{t-1}{1}-\left(\binom{12}{3}-2\binom{9}{0}\right)\binom{t-1}{2}$$
$$-\left(\binom{12}{4}-2\binom{9}{1}\right)\binom{t-1}{3}-\left(\binom{12}{5}-2\binom{9}{2}\right)\binom{t-1}{4}$$
$$+\left(\binom{12}{6}-2\binom{9}{3}+\binom{6}{0}\right)\binom{t-1}{5}$$
$$+\left(\binom{12}{7}-2\binom{9}{4}+\binom{6}{1}\right)\binom{t-1}{6}$$
$$+\left(\binom{12}{8}-2\binom{9}{5}+\binom{6}{2}\right)\binom{t-1}{7}$$
$$+\left(\binom{12}{9}-2\binom{9}{6}+\binom{6}{3}\right)\binom{t-1}{8}$$
$$+\left(\binom{12}{10}-2\binom{9}{7}+\binom{6}{4}\right)\binom{t-1}{9}$$

$$\binom{12}{1}\binom{t-1}{0}=12$$
$$\binom{12}{2}\binom{t-1}{1}=66(t-1)$$

$$\left(\binom{12}{3} - 2\binom{9}{0}\right)\binom{t-1}{2} = 109(t^2 - 3t + 2)$$

$$\left(\binom{12}{4} - 2\binom{9}{1}\right)\binom{t-1}{3} = \frac{159}{2}\ (t^3 - 6t^2 + 11t - 6)$$

$$\left(\binom{12}{5} - 2\binom{9}{2}\right)\binom{t-1}{4} = 30(t^4 - 10t^3 + 35t^2 - 50t + 24)$$

$$\left(\binom{12}{6} - 2\binom{9}{3} + \binom{6}{0}\right)\binom{t-1}{5}$$
$$= \frac{757}{120}(t^5 - 15t^4 + 85t^3 - 225t^2 + 274t - 120)$$

$$\left(\binom{12}{7} - 2\binom{9}{4} + \binom{6}{1}\right)\binom{t-1}{6}$$
$$= \frac{91}{120}(t^6 - 21t^5 + 175t^4 - 735t^3 + 1624t^2$$
$$- 1764t + 720)$$

$$\left(\binom{12}{8} - 2\binom{9}{5} + \binom{6}{2}\right)\binom{t-1}{7}$$
$$= \frac{43}{840}(t^7 - 28t^6 + 322t^5 - 1960t^4 + 6769t^3$$
$$- 13132t^2 + 13068t - 5040)$$

$$\left(\binom{12}{9} - 2\binom{9}{6} + \binom{6}{3}\right)\binom{t-1}{8}$$
$$= \frac{1}{560}(t^8 - 36t^7 + 546t^6 - 4536t^5 + 22449t^4$$
$$- 67284t^3 + 118124t^2 - 109584t + 40320)$$

$$\left(\binom{12}{10} - 2\binom{9}{7} + \binom{6}{4}\right)\binom{t-1}{9}$$
$$= \frac{1}{40320}(t^9 - 45t^8 + 870t^7 - 9450t^6 + 63273t^5$$
$$- 269325t^4 + 723680t^3 - 1172700t^2 + 1026576t$$
$$- 362880)$$

The coefficient of t^9 is $\frac{1}{40320}$ and other coefficients of t^8, t^7, t^7, t^6, t^5, t^4, t^3, t^2, t are computed which are $\frac{3}{4480}$, $\frac{19}{2240}$, $\frac{21}{320}$, $\frac{43}{128}$, $\frac{741}{640}$, $\frac{6653}{2520}$, $\frac{4229}{1120}$, $\frac{2533}{840}$ respecctively

Then the Ehrhart polynomial of p(D_6) is

$$\frac{1}{40320}t^9+\frac{3}{4480}t^8+\frac{19}{2240}t^7+\frac{21}{320}t^6+\frac{43}{128}t^5+\frac{741}{640}\ t^4 +\frac{6653}{2520}\ t^3+\frac{4229}{1120}t^2+\frac{2533}{840}t+1$$

By Ehrhart polynomial the volume is $\frac{1}{40320}$.

Also We can find the volume by vol(D_n) $=\frac{n^2}{4(2n-3)!}$

Vol(D_4)$=\frac{6^2}{4(12-3)!}=\frac{36}{4*9!}=\frac{1}{40320}$

5.Ehrhart polynomial of a permutation polytope of Frobenius group:

The Ehrhart polynomial for Frobenius polytope is depending on dihedral group since Frobenius group is a special case of the dihedral group D_n where n is odd.

Example(5.1):

To find the Ehrhart polynomial for this polytope tp dihedral group D_n when n=3

Sol:

Ehrhart polynomial of P(D_3) is :

$$\sum_{k=0}^{n-2}\binom{2n}{k+1}\binom{t-1}{k}+\sum_{k=n-1}^{2n-2}\left(\binom{2n}{k+1}-\binom{n}{k-n+1}\right)\binom{t-1}{k}$$

$$=\sum_{k=0}^{1}\binom{6}{k+1}\binom{t-1}{k}+\sum_{k=2}^{4}\left(\binom{6}{k+1}-\binom{3}{k-2}\right)\binom{t-1}{k}$$

$$= \binom{6}{1}\binom{t-1}{0} + \binom{6}{2}\binom{t-1}{1} + \left(\binom{6}{3} - \binom{3}{0}\right)\binom{t-1}{2}$$
$$+ \left(\binom{6}{4} - \binom{3}{1}\right)\binom{t-1}{3} + \left(\binom{6}{5} - \binom{3}{2}\right)\binom{t-1}{4}$$
$$= 6 + 15(t-1) + 19\binom{t-1}{2} + 12\binom{t-1}{3} + 3\binom{t-1}{4}$$
$$= 6 + 15t - 15 + \frac{19}{2}(t^2 - 3t + 2) + 2(t^3 - 6t^2 + 11t - 6)$$
$$+ \frac{1}{8}(t^4 - 10t^3 + 35t^2 - 50t + 24)$$
$$= \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

By Ehrhart polynomial the volume is $\frac{1}{8}$.

Also We can find the volume by using vol(D_n) $= \frac{n}{(2n-2)!}$.

Vol(D_3)= $\frac{3}{(6-2)!} = \frac{3}{4!} = \frac{1}{8}$

6. Ehrhart polynomial of a permutation polytope of a cyclic group:

A cyclic group C_n is the group generated by permutation (12…n), Ehrhart polynomial for this polytope P(C_n) is $\binom{t+n-1}{n-1}$ with the volume is $\frac{1}{(n-1)!}$, [Burggraf1,2011:21].

Example(6.1):

To find Ehrhart polynomial of permutation polytope related to a cyclic group when n=3

Sol:

By using Ehrhart polynomial of cyclic group P(C_n) obtained:

$$\binom{t+n-1}{n-1} = \frac{(t+n-1)!}{t!\,(n-1)!}$$
$$= \frac{(t+2)!}{t!\,(2)!} = \frac{(t+2)(t+1)(t)!}{t!\,(2)!} = \frac{t^2+3t+2}{2}$$

$$= \frac{1}{2}\,t^2 + \frac{3}{2}t + 1$$

Vol(P(C_3) = $1/2$

Or by volume of P(C_n) is $\frac{1}{(n-1)!}$

Vol(P(C_3) = $\frac{1}{(3-1)!} = \frac{1}{2}$

Conclusion

We can compute Ehrhart polynomial for a permutation polytopes related to permutation groups

Future work

1-Finding theorem to compute the volume of permutation polytope don't depend on permutation group

2-Compute the vol B_n for $n > 10$. Compute the $L_{B_n}(t) > 9$.

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