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ORIGINAL STUDY

Different Methods to Estimate Stress-Strength Reliability Function for Modified Exponentiated Lomax Distribution

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ABSTRACT

When ensuring the reliability of device or the suitability of a material, it is necessary to take into consideration the stress cases in the operating environment. This means that the uncertainty about the reality environmental stress must be taken into as random. The stress-strength (*S-S*) model treated the stress and strength variables as random. In the simplest form of stress-strength model, y represents the stress put on the unit by the operating environment, and the strength of the unit represented by x. A unit is able to perform its required function if its stress imposed on it is less than the strength of the unit. In this paper, the stress-strength reliability estimation for the modified exponentiated Lomax distribution, which is generalization of the Lomax distribution, with an unknown shape parameter and a known scale parameters is studied using different methods. These methods include the maximum likelihood method, Bayesian estimation method under a quadratic loss function, and the least squares method for complete data. The estimators are compared based on Markov Chain Monte Carlo (*MCMC*) simulations using R-Studio, evaluated by the mean square error (*MSE*) criteria. The simulation results show that the maximum likelihood estimators are the best in two cases: the first is when the sample sizes are equal and the second is when the shape parameter of the strength variable is greater than the shape parameter of the stress variable. While least squares estimators are the beast if the strength sample size is smaller than the stress sample size. Finally when the strength sample size is greater than the stress sample size, then the best estimators differ between the maximum likelihood estimators and Bayesian estimators. Bayesian estimators become the best when the shape parameter of stress variable is larger than the shape parameter of the strength variable.

Keywords: Stress-strength (S-S) reliability, Modified exponentiated Lomax distribution, Maximum likelihood estimator, Bayesian estimator, Least squars estimator

1. Introduction

The Lomax distribution family can be used in the study of stress-strength reliability in many fields such as industrial maintenance, risk management and performance evaluation, this distribution helps in analyzing data and decision making to improve the efficiency [2], one of the extended distributions of Lomax distribution is the exponentiated Lomax distribution was presented by [3] through the addition of a shape parameter λ to the cumulative distribution function in the following form:

$$F(x;\alpha,\theta,\lambda) = \left(1-(1+\alpha x)^{-\theta}\right)^{\lambda};\ x>0 \quad \alpha,\theta,\lambda>0$$

and the probability density function is:

$$f(x;\alpha,\theta,\lambda) = \alpha\theta\lambda\left(1-(1+\alpha x)^{-\theta}\right)^{\lambda-1} \times (1+\alpha x)^{-(\theta+1)};\quad x>0 \quad \alpha,\theta,\lambda>0 \qquad (1)$$

modified this distribution [11] by assuming that $\alpha = 1$ and $\theta = 2$. This is called the modified exponentiated

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Lomax distribution (*MELD*), which is used in this paper. The cumulative distribution function and the probability density function are as follows:

$$F(x;\lambda) = \left(1-(1+x)^{-2}\right)^{\lambda}; \quad x>0 \quad \lambda>0 \tag{2}$$

and

$$f(x;\lambda) = \begin{cases} 2\lambda\left(1-(1+x)^{-2}\right)^{\lambda-1}(1+x)^{-3}; & x>0, \lambda>0 \\ 0; & O.W. \end{cases} \tag{3}$$

This distribution has been applied in many fields, such as actuarial science and biological engineering as he pointed [8]. In the context of stress-strength models, stress refers to the loads and forces exerted on a machine, while strength refers to the machine's ability to resist these loads and perform its required function. The concept of stress-strength models was proposed by [10]. Failure occurs if the stress is greater than the strength. If y represents stress and x represents strength, the main aim is to estimate the probability of failure or reliability of this model Stress-Strength models are applied in many fields, including civil engineering, biomechanics and geological application. One of these applications is the study of two data sets of coating the roofing iron sheets by [18] In this paper, the aim is to find the estimation of the reliability of the stress-strength model, defined as $p(y<x)$. This model has many applications in different fields; for more details, refer to [22], which discusses two methods for estimating reliability: traditional and Bayesian.

The main objective of this paper is to compare three estimation methods of the stress-strength (*S-S*) reliability model $p(y<x)$ using the mean square error criteria. This paper contains eight sections. Section 2 describes the stress-strength model $p(y<x)$ and computes the reliability of (*MELD*). Section 3 covers the maximum likelihood estimation for the stress-strength (*S-S*) reliability. Section 4 presents the Bayesian estimation of (*S-S*) reliability under a quadratic loss function. Section 5 introduces the least squares method for estimation. Section 6 provides a simulation study. Section 7 discusses the simulation results to identify the best estimator. Finally, the conclusion is presented in Section 8.

2. Stress-strength (*S-S*) model

Reliability refers to the ability of a component or system to perform its function accurately without failure within a specified period of time. Reliability has been used in many fields especially in engineering fields based on statistical models. There are studies that aim to evaluate the probability of failure in engineering structures which is called structural reliability [4] or Reliability Allocation [20] others have used accelerated life testing technology to analyze the durability of engineering products or components which helps reduce the time and the cost [1, 6]. Finally reliability can be used to determine the probability of failure or success of a component or system based on the relationship between the strength it possesses and the stress it is exposed to during operation [16] this is called stress-strength reliability which can used in this paper. For the stress-strength parameter of the modified exponentiated Lomax distribution (*MELD*), let x and y represent the strength and stress, respectively, observed from (*MELD*). We can express the stress-strength reliability as follows [5, 13]:

$$R = p(y<x) = \int_0^{\infty}\int_0^{\infty} f(x;\lambda_1)\, f(y;\lambda_2)\, dy\, dx$$
$$= \int_0^{\infty} F_y(x)\, f(y;\lambda_2)\, dx$$

where $F_y(x) = (1-(1+x)^{-2})^{\lambda_2}$ and $f(x;\lambda_1)$ defined in Eq. (3) then where:

$$R = \int_0^{\infty} 2\lambda_1\left(1-(1+x)^{-2}\right)^{\lambda_2}\left(1-(1+x)^{-2}\right)^{\lambda_1-1} \times (1+x)^{-3}\, dx$$

By simplifying the relation above we have the (*S-S*) reliability as follows:

$$R = \left(1+\frac{\lambda_2}{\lambda_1}\right)^{-1} \tag{4}$$

3. Maximum likelihood estimation

The likelihood function of (*MELD*) can be expressed in the following form for a complete random strength sample of sizes n_1 [14, 15].

$$L(\underline{X}|\lambda_1) = \prod_{i=1}^{n_1} f(x_i;\lambda_1)$$
$$= 2^{n_1}\lambda_1^{n_1}\prod_{i=1}^{n_1}(1-(1+x_i)^{-2})^{\lambda_1-1}\prod_{i=1}^{n_1}(1+x_i)^{-3}$$
$$= \mathcal{A}\,\lambda_1^{n_1}\prod_{i=1}^{n_1}(1-(1+x_i)^{-2})^{\lambda_1-1} \tag{5}$$

where

$$\mathcal{A} = 2^{n_1} \prod_{i=1}^{n_1} (1+x_i)^{-3}$$

To find the maximum likelihood estimation for the parameter λ_1 of the strength function, we use the natural logarithm of the likelihood function:

$$lnL\left(\underset{\sim}{X}|\lambda_1\right) = \ln(\mathcal{A}) + n_1 \ln(\lambda_1) + (\lambda_1 - 1)\sum_{i=1}^{n_1} ln\left(1-(1+x_i)^{-2}\right)$$

$$\frac{\partial lnL\left(\underset{\sim}{X}|\lambda_1\right)}{\partial \lambda_1} = \frac{n_1}{\lambda_1} + \sum_{i=1}^{n_1} ln\left(1-(1+x_i)^{-2}\right)$$

Then

$$\hat{\lambda}_{1ML} = \frac{-n_1}{\sum_{i=1}^{n_1} ln\left(1-(1+x_i)^{-2}\right)} \tag{6}$$

Eq. (6) represent the M.L.E. for the λ_1 (the parameter of strength sample) In the same way, the maximum likelihood estimator for λ_2 (the parameter of stress sample) of sizes n_2 can be found:

$$\hat{\lambda}_{2ML} = \frac{-n_2}{\sum_{j=1}^{n_2} ln\left(1-(1+y_j)^{-2}\right)} \tag{7}$$

Using the invariance properly of maximum likelihood estimation, the M.L.E. for the stress-strength (*S-S*) reliability (*R*) is of the following form:

$$\hat{R}_{ML} = \left(1 + \frac{\hat{\lambda}_{2ML}}{\hat{\lambda}_{1ML}}\right)^{-1} \tag{8}$$

Where $\hat{\lambda}_{1ML}$ and $\hat{\lambda}_{2ML}$ are defined in Eqs. (6) and (7) respectively.

4. Bayesian estimation

In the section the Bayes estimate using the quadratic loss function (*QLF*) of the stress-strength (*S-S*) reliability will be obtained. The Bayes estimates are considered under the assumption that the parameter (λ_1, λ_2) follow an independent Gamma distribution. It is assumed that [8, 9, 12, 19], $\lambda_j \sim Gamma(a_j, b_j)$ for $j = 1, 2$. Then the prior density of λ_j can be written as:

$$g(\lambda_j) = \lambda_j^{a_j - 1} e^{-\lambda_j b_j} \quad for \quad j = 1, 2 \tag{9}$$

Where the hyper parameters, a_j, b_j are known and non-negative.

Then the posterior distribution for the strength parameter λ_1 is:

$$p\left(\lambda_1|\underset{\sim}{X}\right) \propto L\left(\underset{\sim}{X}|\lambda_1\right) g(\lambda_1)$$

Using Eqs. (3) and (7) we have:

$$p\left(\lambda_1|\underset{\sim}{X}\right) \propto \lambda_1^{n_1} \prod_{i=1}^{n_1} \left(1-(1+x_i)^{-2}\right)^{\lambda_1 - 1} \lambda_1^{a_1-1} e^{-\lambda_1 b_1}$$
$$\propto \lambda_1^{n_1+a_1-1} e^{-\lambda_1 b_1} \tag{10}$$

Then $(\lambda_1|\underset{\sim}{X}) \sim Gamma(n_1 + a_1, d_1)$ where $d_1 = -\sum_{i=1}^{n_1} ln(1-(1+x_i)^{-2}) + b_1$, and the posterior distribution for the stress parameter λ_2 is:

$$p\left(\lambda_2|\underset{\sim}{Y}\right) \propto \lambda_2^{n_2+a_2-1} e^{-\lambda_2 b_2} \tag{11}$$

where $d_2 = -\sum_{j=1}^{n_2} ln(1-(1+x_j)^{-2}) + b_2$, and the "joint posterior distribution" for λ_1 and λ_2 is:

$$p(\lambda_1, \lambda_2|\underset{\sim}{X}, \underset{\sim}{Y}) = p\left(\lambda_1|\underset{\sim}{X}\right)\ p\left(\lambda_2|\underset{\sim}{Y}\right)$$
$$= \frac{d_1^{n_1+a_1} d_2^{n_2+a_2}}{\Gamma(n_1+a_1)\ \Gamma(n_2+a_2)} \times \lambda_1^{n_1+a_1-1} \lambda_2^{n_2+a_2-1} e^{-\lambda_1 d_1}\ e^{-\lambda_2 d_2} \tag{12}$$

The quadratic loss function for the reliability (*S-S*) is of the form by Sindhu and Islam [21].

$$L\left(\hat{R}_B, R\right) = \frac{\left(\hat{R}_B - R\right)^2}{R} \tag{13}$$

To find the Bayesian estimation for the stress-strength (*S-S*) reliability under quadratic loss function, the risk function should be minimized as much as possible. For the minimization of the risk function, we have:

$$Risk\left(\hat{R}\right) = E\left(L\left(\hat{R}_B, R\right)\right) = E\left(\frac{\left(\hat{R}_B - R\right)^2}{R}\right)$$
$$= \int_0^\infty \int_0^\infty \frac{\left(\hat{R}_B - R\right)^2}{R} p\left(\lambda_1, \lambda_2|\underset{\sim}{x}, \underset{\sim}{y}\right) d\lambda_1 d\lambda_2 \tag{14}$$

$$\frac{\partial E\left(L\left(\hat{R}_B, R\right)\right)}{\partial \hat{R}_B} = 0$$

$$2\int_0^\infty \int_0^\infty \left(\frac{\hat{R}_B}{R} - 1\right) \frac{1}{R} p\left(\lambda_1, \lambda_2|\underset{\sim}{x}, \underset{\sim}{y}\right) d\lambda_1 d\lambda_2 = 0$$

$$\hat{R}_B \int_0^{\infty}\int_0^{\infty} \frac{1}{R^2} p\left(\lambda_1, \lambda_2 | \underset{-}{x}, \underset{-}{y}\right) d\lambda_1 d\lambda_2 - \int_0^{\infty}\int_0^{\infty} \frac{1}{R} p\left(\lambda_1, \lambda_2 | \underset{-}{x}, \underset{-}{y}\right) d\lambda_1 d\lambda_2 = 0$$

$$\hat{R}_B = \frac{\int_0^{\infty}\int_0^{\infty} \frac{1}{R} p\left(\lambda_1, \lambda_2 | \underset{-}{x}, \underset{-}{y}\right) d\lambda_1 d\lambda_2}{\int_0^{\infty}\int_0^{\infty} \frac{1}{R^2} p\left(\lambda_1, \lambda_2 | \underset{-}{x}, \underset{-}{y}\right) d\lambda_1 d\lambda_2}$$

This mean that:

$$\hat{R}_B = \frac{E\left(\frac{1}{R} | \underset{-}{x}, \underset{-}{y}\right)}{E\left(\frac{1}{R^2} | \underset{-}{x}, \underset{-}{y}\right)} \tag{15}$$

where R is defined in Eq. (4), then:

$$E\left(\frac{1}{R} | \underset{-}{x}, \underset{-}{y}\right) = \int_0^{\infty}\int_0^{\infty} \left(1 + \lambda_2 \lambda_1^{-1}\right) p(\lambda_1, \lambda_2 | \underset{-}{x}, \underset{-}{y}) \times d_1^{n_1+a_1} d_2^{n_2+a_2} \lambda_1^{n_1+a_1-1} \lambda_2^{n_2+a_2-1} e^{-\lambda_1 d_1} e^{-\lambda_2 d_2} d\lambda_1 d\lambda_2$$

by solving the integral and simplifying the result we get:

$$E\left(\frac{1}{R} | \underset{-}{x}, \underset{-}{y}\right) = \frac{(n_2 + a_2) d_1}{(n_1 + a_1 - 1) d_2} + 1 \tag{16}$$

by using the same method as before we can compute:

$$E\left(\frac{1}{R^2} | \underset{-}{x}, \underset{-}{y}\right) = 1 + \frac{2\, n_2 + a_2}{d_1^{-1} d_2 (n_1 + a_1 - 1)} + \frac{(n_2 + a_2 + 1)(n_2 + a_2) d_1^2}{(n_1 + a_1 - 1)(n_1 + a_1 - 2) d_2^2} \tag{17}$$

Substituting the Eqs. (16) and (17) in Eq. (15), we get:

$$\hat{R}_B = \frac{\frac{(n_2+a_2)d_1}{(n_1+a_1-1)d_2} + 1}{1 + \frac{2\, n_2+a_2}{d_1^{-1} d_2 (n_1+a_1-1)} + \frac{(n_2+a_2+1)(n_2+a_2)d_1^2}{(n_1+a_1-1)(n_1+a_1-2)d_2^2}}$$

5. Least squares estimation

This method was proposed by [23] and it has been used by many researchers such that [17] to estimate the parameters of some distributions. Let $x_1{:}n_1, x_2{:}n_2, x_3{:}n_3, \ldots, x_n{:}n_n$ be the order statistics of random sample of sizes n1 form ($MELD(\alpha, \theta, \lambda)$) and let $y_1{:}n_1, y_2{:}n_2, y_3{:}n_3, \ldots, y_n{:}n_n$ be order statistics of random sample size n_2 form (MELD(λ_i; $\underset{-}{Y}$)). The least squares estimation of the unknown parameters λ_1, λ_2 can be obtained by minimizing the following function with respect to λ_1, λ_2 as follows:

$$LS(\lambda_1, \lambda_2) = \sum_{i=1}^{n_1} \left(F(x_i : n_1) - \frac{i}{n_1 + 1}\right)^2 + \sum_{j=1}^{n_2} \left(F(y_j : n_2) - \frac{j}{n_2 + 1}\right)^2 \tag{18}$$

By substituting the cumulative distribution function defined in Eq. (2) in Eq. (18) [5, 11]:

$$LS(\lambda_1, \lambda_2) = \sum_{i=1}^{n_1} \left(\left(1 - (1 + x_i : n_1)^{-2}\right)^{\lambda_1} - \frac{i}{n_1 + 1}\right)^2 + \sum_{i=1}^{n_1} \left(\left(1 - (1 + y_j : n_2)^{-2}\right)^{\lambda_2} - \frac{j}{n_2 + 1}\right)^2$$

$$\frac{\partial LS(\lambda_1, \lambda_2)}{\partial \lambda_1} = 2 \sum_{i=1}^{n_1} \left(1 - (1 + x_i : n_1)^{-2}\right)^{\lambda_1} \times ln\left(1 - (1 + x_i : n_1)^{-2}\right) \left(1 - (1 + x_i : n_1)^{-2}\right)^{\lambda_1} - \frac{i}{n_1 + 1} \tag{19}$$

$$\frac{\partial LS(\lambda_1, \lambda_2)}{\partial \lambda_2} = 2 \sum_{i=1}^{n_1} \left(1 - (1 + y_j : n_2)^{-2}\right)^{\lambda_2} \times ln\left(1 - (1 + y_j : n_2)^{-2}\right) \left(1 - (1 + y_j : n_2)^{-2}\right)^{\lambda_2} - \frac{j}{n_2 + 1} \tag{20}$$

To find the least square estimators for λ_1, λ_2, Eqs. (19) and (20) are solved using the Newton-Raphson method. Thus, $\hat{\lambda}_{1LS}, \hat{\lambda}_{2LS}$ are obtained, and then estimated reliability function $R(S\text{-}S)$ is calculated according to the following formula:

$$\hat{R}_{LS} = \left(1 + \frac{\hat{\lambda}_{2LS}}{\hat{\lambda}_{1LS}}\right)^{-1} \tag{21}$$

6. Simulation study

In this section, a Monte Carlo simulation is conducted to estimate the unknown parameters λ_1, λ_2 of the expoentiated Lomax distribution ($MELD$) and to compute stress-strength reliability. The maximum

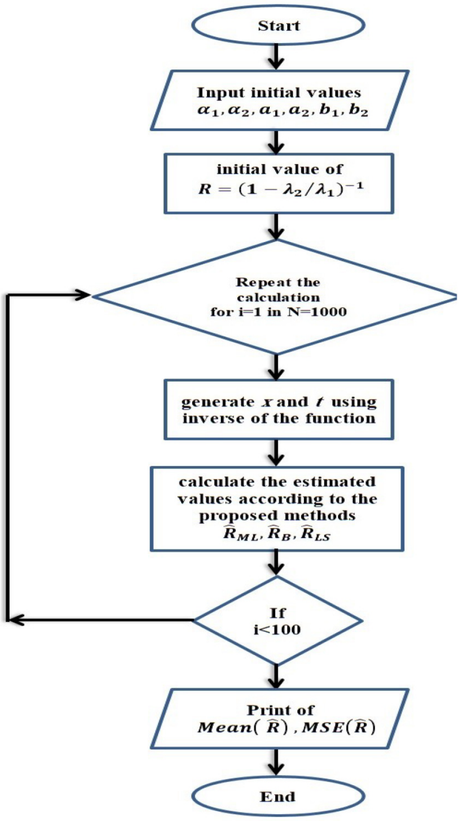

Fig. 1. Algorithm flowchart.

likelihood estimation (ML), Bayesian estimation under quadratic loss function ($BQLF$) [7], and the least squares estimation (LS) methods are evaluated using the mean square error criteria (MSE) with different sample size (25,50,150) where ($\lambda_1 = \lambda_2 = 0.1,\ 0.6,\ 0.9$), ($a_1 = 1.7, a_2 = 1.2$) and ($b_1 = 0.99, b_2 = 0.7$) for 1000 replicates. The simulation study is conducted using R-Studio to compute the reliability estimators through the following three steps:

Generating the random values for the two random variables ($\underline{x}, \underline{y}$) by using the inverse of the distribution function according to the following formula:

$$x_i = \left(1 - u_i^{\frac{1}{\lambda}}\right)^{-\frac{1}{2}} - 1;\ 0 < u_i < 1 \tag{22}$$

Where u is a random variable that distributed as continuous uniform distribution $u_i \sim U(0, 1)$.

Compute the mean of the estimated reliability as follows:

$$\hat{R}_i = \frac{\sum_{i=1}^{N} R_i}{N} \tag{23}$$

Table 1. Reliability estimation when $\lambda_1 = 0.6, \lambda_2 = 0.9$ $R = 0.4, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.402663	0.387377	0.388414	$\hat{R}_{GQLF}$
MSE	0.004369	0.004350	0.004488	
(50,50)				
Mean	0.403843	0.396013	0.384350	$\hat{R}_{BQLF}$
MSE	0.002325	0.002283	0.002336	
(150,150)				
Mean	0.399423	0.396784	0.413772	$\hat{R}_{BQLF}$
MSE	0.000696	0.000691	0.000698	
(25,50)				
Mean	0.404697	0.394412	0.400601	$\hat{R}_{LS}$
MSE	0.003234	0.003250	0.003149	
(25,150)				
Mean	0.403919	0.397170	0.42586	$\hat{R}_{LS}$
MSE	0.002658	0.002566	00.25615	
(50,25)				
Mean	0.396510	0.383774	0.426846	$\hat{R}_{MLE}$
MSE	0.003497	0.003616	0.003796	
(50,150)				
Mean	0.402566	0.398306	0.428401	$\hat{R}_{LS}$
MSE	0.001560	0.001533	0.001513	
(150, 25)				
Mean	0.395815	0.384620	0.383671	$\hat{R}_{MLE}$
MSE	0.002363	0.002477	0.002490	
(150,50)				
Mean	0.398152	0.391962	0.389028	$\hat{R}_{BQLF}$
MSE	0.001475	0.001467	0.001573	

Table 2. Reliability estimation when $\lambda_1 = 0.6, \lambda_2 = 0.1$ $R = 0.8571429, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.9214349	0.917854	0.783991	$\hat{R}_{BQLF}$
MSE	0.0114465	0.011226	0.012684	
(50,50)				
Mean	0.9639194	0.963107	0.967569	$\hat{R}_{BQLF}$
MSE	0.0159511	0.015934	0.016019	
(150,150)				
Mean	0.9974757	0.997457	0.999895	$\hat{R}_{BQLF}$
MSE	0.0200867	0.020085	0.020129	
(25,50)				
Mean	0.9606749	0.959748	0.996813	$\hat{R}_{LS}$
MSE	0.0156505	0.015689	0.015620	
(25,150)				
Mean	0.9972851	0.997262	0.987954	$\hat{R}_{LS}$
MSE	0.0201064	0.020108	0.020103	
(50,25)				
Mean	0.9148838	0.911167	0.916269	$\hat{R}_{MLE}$
MSE	0.0104861	0.010700	0.010795	
(50,150)				
Mean	0.9971154	0.997093	0.991564	$\hat{R}_{LS}$
MSE	0.0200510	0.020052	0.020049	
(150, 25)				
Mean	0.9156633	0.916215	0.916684	$\hat{R}_{MLE}$
MSE	0.0106712	0.010841	0.010872	
(150,50)				
Mean	0.9608007	0.95995	0.962765	$\hat{R}_{MLE}$
MSE	0.0156215	0.015659	0.156829	

Table 3. Reliability estimation when $\lambda_1 = 0.1, \lambda_2 = 0.6$ $R = 0.1428571, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.0801339	0.080168	0.080139	$\hat{R}_{BQLF}$
MSE	0.0108112	0.010811	0.010812	
(50,50)				
Mean	0.0354731	0.035287	0.035663	$\hat{R}_{BQLF}$
MSE	0.0161058	0.016104	0.016106	
(150,150)				
Mean	0.0033805	0.003574	0.003487	$\hat{R}_{BQLF}$
MSE	0.0199900	0.019990	0.019991	
(25,50)				
Mean	0.0792548	0.079318	0.079848	$\hat{R}_{LS}$
MSE	0.0109178	0.010916	0.010914	
(25,150)				
Mean	0.0766278	0.076962	0.076710	$\hat{R}_{LS}$
MSE	0.0110536	0.011051	0.011039	
(50,25)				
Mean	0.0805872	0.080918	0.080187	$\hat{R}_{MLE}$
MSE	0.0105715	0.010574	0.010576	
(50,150)				
Mean	0.0394471	0.039657	0.039411	$\hat{R}_{LS}$
MSE	0.0154952	0.015497	0.015492	
(150, 25)				
Mean	0.0030872	0.003006	0.003023	$\hat{R}_{MLE}$
MSE	0.0200146	0.020019	0.020018	
(150,50)				
Mean	0.0022888	0.002280	0.002293	$\hat{R}_{BQLF}$
MSE	0.0201120	0.020110	0.020113	

Table 4. Reliability estimation when $\lambda_1 = 0.1, \lambda_2 = 0.9$ $R = 0.1, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.0540309	0.056824	0.0536892	$\hat{R}_{BQLF}$
MSE	0.0055547	0.005435	0.0056643	
(50,50)				
Mean	0.0282497	0.028864	0.0273946	$\hat{R}_{BQLF}$
MSE	0.0076057	0.007537	0.0076069	
(150,150)				
Mean	0.0016322	0.001695	0.001599	$\hat{R}_{BQLF}$
MSE	0.0098534	0.009848	0.0098578	
(25,50)				
Mean	0.0561333	0.055866	0.0556757	$\hat{R}_{LS}$
MSE	0.0054004	0.005400	0.0054004	
(25,150)				
Mean	0.0592993	0.059482	0.0591892	$\hat{R}_{LS}$
MSE	0.0051036	0.005103	0.0050950	
(50,25)				
Mean	0.0293769	0.028937	0.0294936	$\hat{R}_{MLE}$
MSE	0.0075227	0.007537	0.0075348	
(50,150)				
Mean	0.0266031	0.026511	0.0264716	$\hat{R}_{LS}$
MSE	0.0076973	0.007696	0.0076960	
(150, 25)				
Mean	0.0021683	0.002209	0.0021938	$\hat{R}_{MLE}$
MSE	0.0097939	0.009794	0.0097920	
(150,50)				
Mean	0.0018941	0.001867	0.0018883	$\hat{R}_{BQLF}$
MSE	0.0098382	0.009837	0.0098379	

Table 5. Reliability estimation when $\lambda_1 = 0.9, \lambda_2 = 0.1$ $R = 0.9, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.9449105	0.941886	0.9453795	$\hat{R}_{BQLF}$
MSE	0.0054944	0.005311	0.0054873	
(50,50)				
Mean	0.9728077	0.972068	0.9719535	$\hat{R}_{BQLF}$
MSE	0.0076937	0.007517	0.0076940	
(150,150)				
Mean	0.9987453	0.998734	0.9985836	$\hat{R}_{BQLF}$
MSE	0.0098816	0.009882	0.0098817	
(25,50)				
Mean	0.9751604	0.974329	0.9755249	$\hat{R}_{LS}$
MSE	0.0078940	0.007913	0.0078901	
(25,150)				
Mean	0.9990969	0.999080	0.9991008	$\hat{R}_{LS}$
MSE	0.0099287	0.009927	0.0099267	
(50,25)				
Mean	0.9409184	0.938010	0.9415873	$\hat{R}_{MLE}$
MSE	0.0051087	0.005220	0.0052367	
(50,150)				
Mean	0.9986837	0.998666	0.9987025	$\hat{R}_{LS}$
MSE	0.0098849	0.009884	0.0098839	
(150, 25)				
Mean	0.9407342	0.938051	0.9402264	$\hat{R}_{MLE}$
MSE	0.0050816	0.005185	0.0051253	
(150,50)				
Mean	0.975594	0.975015	0.9752622	$\hat{R}_{MLE}$
MSE	0.0078939	0.007890	0.0078918	

Table 6. Reliability estimation when $\lambda_1 = 0.9, \lambda_2 = 0.6$ $R = 0.6, a_1 = 1.7, a_2 = 1.2, b_1 = 0.99, b_2 = 0.7$.

Samples Size(n_1, n_2)	$\hat{R}_{MLE}$	$\hat{R}_{BQLF}$	$\hat{R}_{LS}$	Best
(25,25)				
Mean	0.5922679	0.575484	0.573579	$\hat{R}_{BQLF}$
MSE	0.0055568	0.004979	0.005549	
(50,50)				
Mean	0.6003591	0.591919	0.599810	$\hat{R}_{BQLF}$
MSE	0.0023264	0.002204	0.002384	
(150,150)				
Mean	0.5996672	0.596840	0.596539	$\hat{R}_{BQLF}$
MSE	0.0007589	0.000751	0.000759	
(25,50)				
Mean	0.6000496	0.588649	0.601026	$\hat{R}_{LS}$
MSE	0.0036451	0.003749	0.003599	
(25,150)				
Mean	0.6027698	0.595034	0.602666	$\hat{R}_{LS}$
MSE	0.0026445	0.002588	0.002547	
(50,25)				
Mean	0.5967427	0.582983	0.595937	$\hat{R}_{MLE}$
MSE	0.0031363	0.003454	0.003460	
(50,150)				
Mean	0.6012285	0.596469	0.601302	$\hat{R}_{LS}$
MSE	0.0014423	0.001442	0.001442	
(150, 25)				
Mean	0.5954787	0.583690	0.591647	$\hat{R}_{MLE}$
MSE	0.0026699	0.002932	0.002949	
(150,50)				
Mean	0.5978719	0.591368	0.592388	$\hat{R}_{MLE}$
MSE	0.0013971	0.001476	0.001499	

Comparing the three estimation methods is done using the mean square error (*MSE*) criteria:

$$MSE = \frac{\sum_{i=1}^{N}\left(\hat{R}_i - R\right)^2}{N} \tag{24}$$

Where N the number of replication in each experiment, set to 1000. Below is the algorithm flowchart (Fig. 1) of the simulation steps to find the estimated value of the reliability function.

The hyperparameters of the prior distribution were selected by Empirical Bayesian method using the package (ebayesthresh) of R-Studio. The following six tables contain the results of three reliability estimators ($\hat{R}_{MLE}, \hat{R}_{BQLF}, \hat{R}_{LS}$).

7. Discussion

The results in the above tables show the reliability values 0.4, 0.8571, 0.143, 0.1, 0.9, and 0.6 respectively.

1. When $n_1 = n_2$, the Bayesian estimation under quadratic loss function of the reliability is the best estimator.
2. When $n_1 \neq n_2$, the following results are: if $n_1 < n_2$ the least square method gave the best estimator and if $n_1 > n_2$ the preference varies between the maximum likelihood method and Bayesian method, where $\lambda_1 > \lambda_2$ the maximum likelihood method is the best and where $\lambda_1 < \lambda_2$ the Bayesian method is the best.

8. Conclusion

In this paper, three estimation methods were presented to find the estimation for the reliability function of the stress-strength (*S-S*) model $p(y < x)$ when each of x and y follows exponented Lomax distribution with different shape parameters for complete data. The estimation methods included maximum likelihood, Bayesian method under quadratic loss function, and least square method. The simulation results confirm that the Bayesian estimator under quadratic loss function is the best estimator for the equal sample sizes. If the sample sizes are different, the preference will vary among the three methods as indicated in the discussion section.

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