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ARTICLE

Fixed Point Theorems in Generalized Banach Spaces With Various Contraction Conditions and Weakly $\alpha-$ Contraction

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Abstract

In this paper we introduce some fixed point theorems type contractions on generalized Banach space and we introduce a class of weakly $\alpha-$ contraction mappings. And we showed that these mappings must have unique fixed points in generalized Banach space.

Keywords: Fixed point, Generalized banach space

1. Introduction

Banach's contraction mapping theorem is well-known as one of the most important conclusions of functional analysis.

A mapping $F: H \to H$ where (H,d) is a metric space, is said to be a contraction if there exists $0 \leq k < 1$ such that, for all $x, y \in H$,

$$d(Fx,Fy) \leq kd(x,y) \tag{1.1}$$

The mapping fulfilling (1) has a unique fixed point if the metric space (H,d) is complete. F is continuity is implied by inequality (1). A natural question is whether contractive conditions can be found that imply the existence of a fixed point in a complete metric space but not continuity.

See [1–3,6,8,9] Many researchers have proven the oneness and uniqueness of the fixed point in many conditions.

Kannan [4,5] Concluded the following conclusion, in which the positive response to the following question was provided.

If $F: H \to H$ where (H,d) is a complete metric space, satisfies the inequality

$$d(Fx,Fy) \leq k\,[d(x,Fx) + d(y,Fy)] \tag{1.2}$$

where $k \in \left[0, \frac{1}{2}\right)$

In 1972, Chatterjea [9] introduced the dual of the Kannan contraction condition.

$$d(Fx,Fy) \leq b[d(x,Fy) + d(y,Fx)], \text{for all } x,y \in H, \tag{1.3}$$

where b $\in$ [0, $\frac{1}{2}$)

For shortcut we put the following code in place of the names:

1.1. U.F.P.: unique fixed point

Definition 1.1. [7]If M nonempty is a linear space having $s \geq 1$, let $\|.\|$ dnotes a functon from linear space M into R that satisfies the following axioms:

1. for all $x \in M$ $\|x\| \geq 0, \|x\| = 0$ if and only if $x = 0$;
2. for all $x, y \in M$, $\|x + y\| \leq s\,[\|x\| + \|y\|]$;
3. for all $x \in M, \alpha \in R$, $\|\alpha x\| = |\alpha|\,\|x\|$;

$(M, \|.\|)$ is called generalized normed linear space. If for $s = 1,$ it reduces to standard normed linear space.

Definition 1.2. [7]A Banach space $(M, \|.\|)$ is a normed vector space such that M is complete under the metric induced by the $\|.\|$.

Definition 1.3. [7]A linear generalized normed space in which every Cauchy sequence is convergent is called generalized Banach space.

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Definition 1.4. [7]Let $(H, \|.\|)$ be a generalized normed space then the sequence $\{u_n\}$ in H is called,

1. Cauchy sequence iff for each $\varepsilon \rangle\ 0$, there exist $n(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq n(\varepsilon)$ we have $\|u_n - u_m\| \langle \varepsilon$.
2. Convergent sequence iff there exist $u \in H$ such that for all $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ we have $\|u_n - u\| \langle \varepsilon$.

Definition 1.5. [10]A mapping $F: H \rightarrow H$ where $(H, \|.\|)$ is generalized Banach space is said to be weakly contractive if

$$\|Fx - Fy\| \leq \|x - y\| - \psi(\|x - y\|), \tag{1.4}$$

where $, y \in H$, $\psi : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x \to \infty} \psi = \infty$.
If we take $\psi(x) = z\,x$ where $0 < z < 1$ then (4) reduces to (3).

Lemma 1.1. .[6]Let $(H, \|.\|)$ be a generalized Banach space with a real number $s \geq 1$, and F self-mapping on H, assume that $\{u_n\}$ is a sequence in H defined by $u_{n+1} = Fu_n$ if,

$$\|u_{n-}u_{n+1}\| \leq \alpha\, \|u_{n-1} - u_n\|, \text{for all } n \in \mathbb{N} \tag{1.5}$$

where $\alpha \in [0, 1)$, $0 \leq s\alpha < 1$. Then $\{u_n\}$ is a Cauchy sequence and is a converges to some $u^* \in$ H as $n \rightarrow +\infty$.

2. Main result

Theorem 2.1. Let $(H, \|.\|)$ be a generalized Banach space with a real number $s \geq 1$ and $F: H \rightarrow H$ such that,

$$\|Fu - Fv\| \leq \alpha[\, \|u - Fu\| + \|v - Fv\|] + \beta[\|u - Fv\| + \|v - Fv\|], \tag{2.1}$$

where $\alpha, \beta > 0$ such that $\alpha + \beta s < \frac{1}{2}$ for all $u, v \in H$. Then F has a U.F.P.

Proof: Let u_0 arbitrary in H and we'll show that $\{u_n\}_{n=0}^{\infty}$ is Cauchy sequence, such that,

$$u_n = Fu_{n-1} = F^n u_0, \text{for all} \quad n \in \mathbb{N}, \tag{2.2}$$

$$\|u_n - u_{n+1}\| = \|Fu_{n-1} - Fu_n\|$$

$$\leq \alpha[\, \|u_{n-1} - Fu_{n-1}\| + \|u_n - Fu_n\|] + \beta[\|u_{n-1} - Fu_n\| + \|u_n - Fu_{n-1}\|]$$

$$= \alpha[\, \|u_{n-1} - u_n\| + \|u_n - u_{n+1}\|] + \beta[\|u_{n-1} - u_{n+1}\| + \|u_n - u_n\|]$$

$$\|u_n - u_{n+1}\| \leq \alpha\|u_{n-1} - u_n\| + \alpha\|u_n - u_{n+1}\| + \beta\|u_{n-1} - u_{n+1}\|$$

$$\leq \alpha\|u_{n-1} - u_n\| + \alpha\|u_n - u_{n+1}\| + \beta s[\|u_{n-1} - u_n\| + \|u_n - u_{n+1}\|]$$

$$\leq \alpha\|u_{n-1} - u_n\| + \alpha\|u_n - u_{n+1}\| + \beta s\|u_{n-1} - u_n\| + \beta s\|u_n - u_{n+1}\|$$

$$(1 - (\alpha + \beta s))\ \|u_n - u_{n+1}\| \leq (\alpha + \beta s)\ \|u_{n-1} - u_n\|$$

$$\|u_n - u_{n+1}\| \leq k\, \|u_{n-1} - u_n\|, \text{in which } k = \frac{\alpha + \beta s}{1 - (\alpha + \beta s)} < 1 \tag{2.3}$$

By lemma (1.1) we can draw the conclusion that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$. As $(H, \|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some $u^* \in H$ as $n \rightarrow \infty$.

We show that, u^* is the fixed point of F.

$$\|u^* - Fu^*\| \leq s\,[\,\|u^* - u_{n+1}\| + \|u_{n+1} - Fu^*\|\,]$$

$$\leq s\,[\,\|u^* - u_{n+1}\| + \|Fu_n - Fu^*\|\,]$$

$$\|u^* - Fu^*\| \leq s\,\|u^* - u_{n+1}\| + s\,\alpha[\,\|u_n - Fu_n\| + \|u^* - Fu^*\|]$$

$$+ s\,\beta[\|u_n - Fu^*\| + \|u^* - Fu_n\|]$$

$$= s\,\|u^* - u_{n+1}\| + s\,\alpha[\,\|u_n - u_{n+1}\| + \|u^* - Fu^*\|]$$

$$+ s^2\beta\,[\|u_n - u^*\| + \|u^* - Fu^*\|] + s\beta\,\|u^* - u_{n+1}\|$$

$$= s\,\|u^* - u_{n+1}\| + s^2\,\alpha\|u_n - u^*\| + s^2\,\alpha\|u^* - u_{n+1}\| + s\,\alpha\,\|u^* - Fu^*\|$$

$$+ s^2\beta\,\|u_n - u^*\| + s^2\beta\,\|u^* - Fu^*\| + s\beta\,\|u^* - u_{n+1}\|$$

$$(1 - s^2\beta - s\,\alpha)\,\|u^* - Fu^*\| \leq s(1 + s\,\alpha + \beta)\,\|u^* - u_{n+1}\|$$

$$+ s^2(\alpha + \beta)\|u_n - u^*\|$$

$$\|u^* - Fu^*\| \leq \frac{s(1 + s\,\alpha + \beta)}{(1 - s^2\beta - s\,\alpha)}\,\|u^* - u_{n+1}\| + \frac{s^2(\alpha + \beta)}{(1 - s^2\beta - s\,\alpha)}\,\|u_n - u^*\| \tag{2.4}$$

In (2.4) taking $\lim_{n \to \infty}$ we get $\lim_{n \to \infty} \|u^* - Fu^*\| = 0 Fu^* = u^*$.

We proved u^* is the fixed point of F.

Now, we have to show that, u^* is U.F.P. of F.

Suppose that v^* is another fixed point of F then,

$Fv^* = v^*$ and $\|u^* - v^*\| = \|Fu^* - Fv^*\|$
$\leq \alpha[\ \|u^* - Fu^*\| + {*}\|v^* - Fv\|]$

$+\beta[{*}\|u^* - Fv^*\| + \|v^* - Fu\|]$

$= \alpha[\|u^* - u^*\| + {*}\|v^* - v\|] + \beta[\|u^* - v^*\| + \|v^* - u^*\|]$

$\|u^* - v^*\| \leq 2\ \beta\|u^* - v^*\|$, which is a contradiction. There fore $\|u^* - v^*\| = 0 \quad u^* = v^*$,
hence u^* is the U.F.P.

Example 2.1.1. Let $= \{i, \ j, \ k\}$, and let, $\|.\| : H \times H \to [0, +\infty)$ be a mapping that fulfills, the business condition (2.1), for all $x, y \in H, \|x - y\| = 0$, where $x = y$,

$$\|i-j\| = \|j-i\| = \frac{1}{3}, \|i-k\| = \|k-i\| = \frac{1}{6}, \|k-j\| = \|j-k\| = \frac{5}{6}.$$

Then $(H, \|.\|)$ be a generalized Banach space with a coefficient $s = \frac{5}{4} > 1$. Consider mapping $F : H \to H$, define by $F(i) = i, F(j) = i, F(k) = j$

Let $\alpha = \frac{1}{4}$ and $= \frac{1}{6}, + s\ \beta \langle \frac{1}{2}$, now we will verify the condition (2.1).

It have the following case to, $\|\mathrm{Fu} - \mathrm{Fv}\| = 0$ the condition (2.1) holds.
$\|Fu - Fv\| \neq 0$, we have the following there cases,

$$\|Fu - Fv\| \leq \alpha[\|u - Fu\| + \mathrm{v}\ \|v - F\|] + \beta[\|u - Fv\| + \|v - Fv\|]$$

Case 1. $u = i, v = j$, we can get $\|Fu - Fv\| = 0$, then

$$\|Fu - Fv\| \leq \alpha[\ \|u - Fu\| + \|v - Fv\|] + \beta[\|u - Fv\| + \|v - Fv\|]$$

$$0 \leq \frac{1}{4}\left[0 + \frac{1}{3}\right] + \frac{1}{6}\left[0 + \frac{1}{3}\right] = \frac{1}{12} + \frac{1}{18} = \frac{5}{36}$$

therefore, the condition (2.1) holds.

Case 2. $u = i, v = k$, we can get $\|Fu - Fv\| = \frac{1}{3}$, then

$$\|Fu - Fv\| \leq \alpha[\ \|u - Fu\| + \|v - Fv\|] + \beta[\|u - Fv\| + \|v - Fv\|]$$

$$\frac{1}{3} \leq \frac{1}{4}\left[0 + \frac{5}{6}\right] + \frac{1}{6}\left[\frac{1}{6} + \frac{5}{6}\right] = \frac{5}{24} + \frac{1}{6} = \frac{9}{24}$$

therefore, the condition (2.1) is holds.

Case 3. $u = j, v = k$, we can get $\|Fu - Fv\| = \frac{1}{3}$, then

$$\|Fu - Fv\| \leq \alpha[\ \|u - Fu\| + \|v - Fv\|] + \beta[\|u - Fv\| + \|v - Fv\|]$$

$$\frac{1}{3} \leq \frac{1}{4}\left[\frac{1}{3} + \frac{5}{6}\right] + \frac{1}{6}\left[0 + \frac{5}{6}\right] = \frac{7}{24} + \frac{5}{36} = .,43$$

thus, the condition (2.1) holds.

We proved that condition (2.1) is fulfilled in all case.

Then F has a U.F.P., $u^* = \mathrm{i}$ such that $(F(i) = i)$.

2.1. The following are the corollaries of Theorem 2.1

Corollary 2.1. Let $(H, \|.\|)$ be a generalized Banach space with a real number $s \geq 1$, and $F : H \to H$, such that,

$$\|Fu - Fv\| \leq \alpha\, max[\|u - Fu\|, \|v - Fv\|] + \beta\, min[\|u - Fv\|, \|v - Fv\|] \tag{2.5}$$

where $\alpha, \ \beta > 0$ such that $0 \leq \alpha, \beta < 1$, for all $u, v \in H$, then F has a U.F.P.

Definition 2.2. (weak α− contraction)
Let $(H, \|.\|)$ be a G.B.S. with a rail number $s \geq 1$, and $F : H \to H$ is said to weakly $\alpha-$ contraction, for all x, $y \in H$,

$$\|Fx - Fy\| \leq \frac{\alpha}{\mathrm{s}}\ [\|x - Fy\| + \mathrm{x}\ \|y - F\|] - \psi(\|x - Fy\|, \|y - Fx\|), \tag{2.9}$$

and $0 \leq \alpha < \frac{1}{2}$.
Where $\psi : (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}^+$ is a continuous mapping in order for $\psi(x, y) = 0$ if and only if $x = y = 0$.
If we take $\psi\ (x, y) = 0$, where $0 \leq \alpha < \frac{1}{2}$ then (2.9) reduces to (2.8).

Theorem 2.2. Let $F : H \to H$ where $(H, \|.\|)$ is a G.B.S. be a weak $\alpha-$ contraction. Then F has a U.F.P.

Proof. Let u arbitrary in H and we will show that $\{u_n\}_{n=0}^{\infty}$ is Cauchy sequence, such that

$$un = Fun - 1 = F^n u_0, for\ all \quad n \in \mathbb{N},$$

we assume $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$,
putting $x = u_{n-1}$ and $y = u_n$ in (2.9), for all $n = 0, 1, 2 \ldots.$

$$\|u_n - u_{n+1}\| = \|Fu_{n-1} - Fu_n\|$$

$$\leq \frac{\alpha}{s}\ [1\|u_{n-1} - Fu_n\| + \|u_n - Fu_{n-}\|] - \psi\ (\|u_{n-1} - Fu_n\|, \|u_n - Fu_{n-1}\|)$$

$$= \frac{\alpha}{s}\ [\ \|u_{n-1} - u_{n+1}\| + \|u_n - u_n\|\ ] - \psi\ (\|u_{n-1} - u_{n+1}\|, \|u_n - u_n\|\ )$$

$$= \frac{\alpha}{s}\ \|u_{n-1} - u_{n+1}\| - \psi\ (\|u_{n-1} - u_{n+1}\|, 0\ )$$

$$\|u_n - u_{n+1}\| \leq \frac{s\alpha}{s}\ [\|u_{n-1} - u_n\| + 1\|u_n - u_{n+}\|]$$

$$\|u_n - u_{n+1}\| \leq \alpha\ [\|u_{n-1} - u_n\| + 1\|u_n - u_{n+}\|]$$

$$(\alpha\, 1 -)\|u_n - u_{n+1}\| \leq \alpha\ \|u_{n-1} - u_n\|$$

$$\|u_n - u_{n+1}\| \leq \frac{\alpha}{1-\alpha}\|u_{n-1} - u_n\|, \text{where } \frac{\alpha}{1-\alpha}\langle 1$$

From By lemma (1.1) we can draw the conclusion that $\{u_n\}$ is a Cauchy sequence in (H, $\|.\|$). As (H, $\|.\|$) is a G.B.S., $\{u_n\}$ is a converges to some $u^* \in H$ as $n \to \infty$.

We'll prove that $Fu^* = u^*$.

$$\|u^* - Fu^*\| \leq s[\ \|u^* - u_{n+1}\| + *\|u_{n+1} - Fu\|]$$

$$\|u^* - Fu^*\| \leq s[\ \|u^* - u_{n+1}\| + \|Fu_n - Fu^*\|\ ]$$

$$\leq s\left[\|u^* - u_{n+1}\| + \frac{\alpha}{s}[\|u_n - Fu^*\| + \|u^* - Fu_n\|] - \psi(\|u_n - Fu^*\|, \|u^* - Fu_n\|)\right] \quad (2.10)$$

by taking the limit as $n \to +\infty$, using (2.10) and continuity of ψ we obtain that,

$$\|u^* - Fu^*\| \leq \frac{s\alpha}{s}\ \|u^* - Fu^*\| - \psi\ (0, \|u^* - Fu^*\|\ )$$
$$\leq \alpha\ \|u^* - Fu^*\|$$

which is a contradiction ($0 \leq \alpha < \frac{1}{2}$), $\|u^* - Fu^*\| = 0$, hence $Fu^* = u^*$.

Now show that u^* is U.F.P., suppose that, u^* and $v^*$are different fixed points of F.

$$\|u^* - v^*\| = \|Fu^* - Fv\|$$

$$\leq \frac{\alpha}{s}\ [\ \|u^* - Fv^*\| + \|v^* - Fu^*\|\ ] - \psi\ (\|u^* - Fv^*\|, \|v^* - Fu^*\|)$$

$$= \frac{\alpha}{s}[\|u^* - v^*\| + \|v^* - u^*\|] - \psi\ (\|u^* - v^*\|, \|v^* - u^*\|)$$

$$\|u^* - v^*\| \leq \frac{2\alpha}{s}\|u^* - v^*\| - \psi(\|u^* - v^*\|, *\|v^* - u\|).$$

Which by property of $\psi = 0$, which is a contradiction.

On him $\|u^* - v^*\| = 0$, that is $u^* = v^*$.

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