Grillage Analysis of Plates for Vibration and Stability

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Abstract

This study deals with the problem of approximating the dynamic and static analysis of plates by using equivalent grid-framework model. The emphasis, for plate analysis, is on the stability and vibration analysis. Numerical results are presented for many example problems, and they indicate that the adopted method is reasonably accurate. For vibration analysis of plates using beam-column analogy the percentage of error is depends on mesh size.

Keywords: Grillage analogy, plate analysis, vibration and stability.

تحليل الصفائح شبكيًا للاهتزاز و الاستقرارية

الخلاصة

هذه الدراسة تتناول حل مسائل الصفائح بصورة تقريبية سكونياً و حركياً، و ذلك باستخدام نموذج شبكي مكافئ. نتائج عددية قد قدمت باستخدام العديد من المسائل و التي توضح ان الطريقة المتبعة في هذه الدراسة دقيقة و مقبولة. بعض المسائل المتعلقة بالطريقة المتبعة قد تم التنويه عنها من خلال الامثلة المعطاة. اما في تحليل الصفائح اهتزازيا وجد ان دقة الحل تعتمد على حجم العناصر الشبكية المستخدمة.

Notation:

- A Cross-sectional area
- D Flexural rigidity of plate
- E Young's modulus
- f Natural frequency, Hz
- F_i Frequency functions, (i = 1 to 6)
- g Gravitational acceleration
- G Shear modulus of the material
- J Torsional constant
- L Member length
- S_i Stability functions, (i = 1 to 6)
- t Plate thickness
- θ Rotation
- λ Frequency parameter
- μ Mass per unit length of beam's span
- ν Poisson's ratio
- ρ Specific weight of element
- ω Frequency of structure, rad / sec

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1. Introduction

The analysis of a plate under in-plane or out-of-plane loads has been attempted in various forms: series solutions [(1)], the finite element technique [(2, 3)], the finite strip method [(4)], the finite difference technique [(5)], and grillage analyses [(6, 7, 8)].

In this study a grillage method is adopted based on a formulation presented by Yettram and Husain [(15)] which can be most useful when dealing with plate problems where an exact series solution is difficult to obtain, e.g., plates with complex boundary conditions.

The main difficulty in designing a grid of orthogonally connected beams to simulate a plate is due to the Poisson's effect. This has a considerable effect on deflections and moment distributions in a plate, whereas it has no effect on grids that consist of unidirectional beams. So, in order to simulate a plate by an equivalent grid, the latter has to be designed in such a manner that its flexural behavior in all directions must be coupled. This coupling will be such that the grid deflection in any direction will produce curvatures in all other directions governed by the plate bending relationships.

In deriving the stiffness properties of the beam-column analog, the following assumptions are made:

1. The material is perfectly elastic.
2. The deflections are small relative to the thickness of the continuum.
3. The thickness of the plate is small relative to its other dimensions.
4. The elements used are plane and rectangular in shape.
5. The beams used for the analog are only fictitious beams used to construct the stiffness matrix of the element.
6. The moment intensities on the plate element are constant along any edge.

The last assumption is true for infinitesimal elements, and the accuracy of the results will depend on the size of the mesh used.

The model consists of side and diagonal beams as in Fig. 1b. The cross-sectional properties of the members are obtained by equating the rotations of the nodes of the grid with those of an element of equal size, when both are subjected to statically equivalent moments and torques. A rectangular grid model with five cross-sectional properties will define uniquely a rectangular element of a plate. These properties are chosen to be the flexural and torsional rigidities of the side beams and the flexural rigidity of the diagonals.

A computer program (STAVIBPS) was developed here using continuous mass method[(9)] and Wittrick-Williams method[(10)] for solving for eigenvalues[(11)].

2. Evaluation of the Cross-Sectional Properties

By considering the dynamic stiffness matrix of each beam in the grid-framework model (Fig. 1), which is given in the Appendix for vibration-stability analysis and making appropriate substitutions of dynamic-stability functions given also in the Appendix, the governing matrix equation for the grid model will be

$$\{F\} = [K]\{d\} \qquad (1)$$

where $\{F\}$ and $\{d\}$ are force and displacement vectors and are given as

$$\{F\}=\lfloor M_{1x} \quad M_{1y} \quad Q_1 \quad M_{2x} \quad M_{2y} \quad Q_2 \quad M_{3x} \quad M_{3y} \quad Q_3 \quad M_{4x} \quad M_{4y} \quad Q_4 \rfloor \quad (2)$$

$$\{d\}=\lfloor q_{1x} \quad q_{1y} \quad w_1 \quad q_{2x} \quad q_{2y} \quad w_2 \quad q_{3x} \quad q_{3y} \quad w_3 \quad q_{4x} \quad q_{4y} \quad w_4 \rfloor \quad (3)$$

In these matrices θ_{ix}, θ_{iy}, and w_i are the rotations about the x- and y-directions and the transverse displacement; M_{ix}, M_{iy}, and Q_i are the moments about the x- and y-directions and the transverse shearing force, for a node i = 1, 2, 3, 4 and [K] is the dynamic stiffness matrix for plate element where the nonzero elements of this matrix are [15]

$$k_{1,1}=k_{4,4}=k_{7,7}=k_{10,10}=\frac{1}{l}\left(\frac{GJ_c}{k}a_1\cot a_1 + EI_sF_2(l_s)+\frac{EI_d}{r^3}F_2(l_d)\right)$$

$$k_{1,2}=-k_{4,5}=-k_{7,8}=k_{10,11}=-\frac{EI_d}{r^3 l}kF_2(l_d)$$

$$k_{1,3}=k_{4,6}=-k_{7,9}=k_{10,11}=\frac{E}{l2}\left(I_sF_4(l_s) + \frac{I_d}{r^3}F_4(l_d)\right)$$

$$k_{2,2}=k_{5,5}=k_{8,8}=k_{11,11}=\frac{1}{l}\left(\frac{EI_c}{k}F_2(l_c) + GJ_s a_1\cot a_1 + \frac{EI_d}{r^3}k^2F_2(l_d)\right)$$

$$k_{2,3}=-k_{5,6}=k_{8,9}=-k_{11,12}=-\frac{E}{l2}\left(\frac{I_c}{k2}F_4(l_c) + \frac{kI_d}{r^3}F_4(l_d)\right)$$

$$k_{2,5}=k_{8,11}=\frac{EI_c}{kl}F_1(l_c)$$

$$k_{2,6}=k_{8,12}=-\frac{EI_c}{k^2l^2}F_3(l_c)$$

$$k_{2,8}=k_{5,11}=-\frac{GJ_s}{l}a_1\csc a_1$$

$$k_{2,11}=k_{5,8}=\frac{EI_d}{r^3 l}k^2F_1(l_d)$$

$$k_{2,12}=-k_{5,9}=-\frac{EI_d}{r^3l^2}kF_3(l_d)$$

$$k_{3,3}=k_{6,6}=k_{9,9}=k_{12,12}=\frac{E}{l^3}\left(\frac{I_c}{k}F_6(l_c) + I_sF_6(l_s)+\frac{I_d}{r^3}F_6(l_d)\right)$$

$$k_{3,5}=k_{9,11}=-\frac{EI_c}{k^2l^2}F_4(l_c)$$

$$k_{3,6}=k_{9,12}=\frac{EI_c}{k^3l^3}F_5(l_c)$$

$$k_{3,7}=k_{6,10}=\frac{EI_s}{l^2}F_4(l_s)$$

$$k_{3,9}=k_{6,12}=\frac{EI_s}{l^3}F_5(l_s)$$

$$k_{3,10}=k_{6,7}=\frac{EI_d}{r^3l^2}F_4(l_d)$$

$$k_{3,11}=-k_{6,8}=-\frac{EI_d}{r^3l^2}kF_4(l_d)$$

$$k_{3,12}=k_{6,9}=\frac{EI_d}{r^3l^2}F_5(l_d) \qquad \ldots(4)$$

This is considered to be the dynamic stiffness matrix for plate element. For vibration with including the effect of in-plane forces, the frequency functions, $F_i(\lambda)$, i = 1, 2, …, 6, above are replaced with stability-frequency functions $F_i(a,b)$ as given in the Appendix. For stability analysis [12] the stability functions $S_i(\beta)$, i = 1, 2, …, 4, are used and the elements of dynamic stiffness matrix are modified by replacing $F_1(\lambda)$ to $F_3(\lambda)$ with $S_1(\beta)$ to $S_3(\beta)$, $F_4(\lambda)$ with - $S_3(\beta)$, $F_5(\lambda)$ with $S_4(\beta)$ and $F_6(\lambda)$ with - $S_4(\beta)$.

For zero frequencies, the frequency functions $F_i(\lambda)$, are reduced to the linear static case of plate-grid-framework that is given by Yettram and Husain [7], which are used in the derivation of properties of the constituting elements of grid model.

For the grid to simulate the plate element, corresponding rotations must be equal for the two systems (Fig. 1); thus,

$q_1 = q_6$, $q_2 = q_7$, $q_3 = q_8$, $q_4 = q_9$ and $q_5 = q_{10}$ …(5)

where θ_1 to θ_5 are the rotations of the plate element when subjected to unidirectional moment intensities M_1 and M_2, as in Fig. 1c and Fig. 1e and self-equilibrating torque intensities H along each edge (Fig. 1g), and they can be given as

$$q_1 = \frac{k\,l\,M_1}{E\left(\frac{t^3}{12}\right)},$$

$$q_2 = \frac{n\,l\,M_1}{E\left(\frac{t^3}{12}\right)},$$

$$q_3 = \frac{lM_2}{E\left(\frac{t^3}{12}\right)},$$

$$q_4 = \frac{n\,k\,l\,M_2}{E\left(\frac{t^3}{12}\right)}, \text{ and}$$

$$q_5 = \frac{k\,l\,H(1+n)}{E\left(\frac{t^3}{12}\right)} \quad \dots (6)$$

in which l and kl are the side lengths of the element, t is its thickness, v is Poisson's ratio and E is the elastic modulus of the material.

The rotations θ_6 to θ_{10} are the corresponding rotations of the grid-framework subjected to the same moment and torque intensities and they are

$$q_6 = \frac{M_1 l^2 k}{2E}\frac{r^3 I_s + I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s}$$

$$q_7 = \frac{M_1 l^2 k^2}{2E}\frac{I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s}$$

$$q_8 = \frac{M_2 l^2 k}{2E}\frac{r^3 I_c + k^3 I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s}$$

$$q_9 = \frac{M_2 l^2 k^3}{2E}\frac{I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s}$$

$$q_{10} = \frac{Hr^3 kl^2}{2E\left[r^3\left(\frac{GJ_c}{E}\right) + 2kI_d\right]} \quad \dots (7)$$

As in Fig. 1b, the side beams of length l have equal second moments of area I_s and equal torsion factors GJ_s / E, and the side beams of length kl have equal second moments of area I_c and equal torsion factors GJ_c / E. The diagonals of length rl have second moments of area I_d and no torsional stiffness.

Of Eqs. 5 the second and fourth are identical and the first three then provide, when expanded and solved, the second moments of area of the grid members as

$$I_s = \frac{(k^2 - n)l}{2k(1-n^2)}\frac{t^3}{12},$$

$$I_c = \frac{(1-k^2 n)l}{2(1-n^2)}\frac{t^3}{12},$$

$$I_d = \frac{n\,r^3 l}{2k(1-n^2)}\frac{t^3}{12} \quad \dots (8)$$

The last of Eq. 5 gives the torsion factor

$$\frac{GJ_c}{E} = \frac{(1-3n)l}{2(1-n^2)}\frac{t^3}{12} \quad (9)$$

Substitution of the values θ_{10} and w_4 into the second of Eq. 1 yields the remaining cross-sectional property

$$\frac{GJ_s}{E} = \frac{(1-3n)r\,l}{2(1-n^2)}\frac{t^3}{12} \qquad (10)$$

When $k = 1$ and $v = 0$

$$I_s = I_c = \frac{1}{2}\frac{lt^3}{12},$$

$I_d = 0$, and

$$\frac{GJ_s}{E} = \frac{GJ_c}{E} = \frac{1}{2}\frac{lt^3}{12} \qquad \ldots(11)$$

For this case the grid reduces to one consisting of side beams only, and of equal flexural and torsional rigidities.

3. Plates Subjected to In-Plane and Out-of-Plane Loading

The beam-column analog can also be used to determine the interaction between out-of-plane and in-plane loads through using the stability functions given in the Appendix. The critical load of the plate under any system of axial compressive loads can be determined by considering the effect of axial forces on the side beams only and neglecting the axial forces in the diagonal beams.

The in-plane distributed pressure is replaced by a statically equivalent system of concentrated loads at the edge nodes of the analog (Fig. 2). The effect of the in-plane compressive or tensile loads is introduced simply by substituting the values of the stability functions, corresponding to such loads, into the stiffness matrix of the corresponding member of the analog.

4. Dynamic Analysis using Beam-Column Analogy

When considering the dynamic behavior there must be additional requirement that the mass per unit area of the grillage must be the same as that of the plate. In order to satisfy this condition, the total mass of the plate element must be the same as that of its equivalent beam-column analog element.

Here the diagonal beams will be assumed of zero mass. If each side member of the beam-analog element is assumed to be of constant mass per unit length, thus for an element of dimensions *l* and *kl*, Fig. 3, the mass per unit length of the member of length *l* can be written as [13]

$$m = kltr/4g \qquad (12)$$

and that for the member with length *kl* as:

$$m = ltr/4g \qquad (13)$$

in which t, ρ and g are the thickness, material specific weight of the plate and gravitational acceleration respectively. The values given in Eqs. 12 and 13 are those for boundary members and should be doubled for inside members.

If beside the harmonic excitation, the plate is subjected to in-plane static loads, these loads must be applied as statically equivalent lumped loads along the members of the beam-analog as explained previously.

The dynamic-stability functions, as given in the Appendix, are expressed in terms of the static and dynamic parameters β and λ. If β and λ are calculated, the dynamic-stability functions can be directly determined. The dynamic behavior of the plate in the presence of in-plane static loads can thus be obtained by substituting these values of the dynamic-stability functions into the stiffness matrices of the members of the beam-analog

5. Natural Frequencies Computations

The frequency of the discrete coordinate system may be given as follows

$$\left| [K] - w^2 [M] \right| = 0 \qquad (14)$$

The formulation of Eq. 14 is an important mathematical problem known as a linear eigenvalue problem. There are many numerical methods [12] dealing with eigenvalue and eigenvector problems.

In the continuous mass method [9] adopted here, the eigenvalue is of the type

$$[K(w)]\{D\} = 0 \qquad (15)$$

where the dynamic stiffness matrix $[K(w)]$ is no longer a linear function of ω^2 and Eq. 15 is known as a nonlinear eigenvalue problem. The dynamic stiffness matrix $[K(w)]$ has in general a transcendental dependence on ω^2.

The solution of Eq. 15 needs methods different from those used to solve the linear eigenvalue problem given in Eq. 14. A powerful method presented by Wittrick and Williams [10] is explained by Ali[15] and adopted here.

6. Applications

In order to check the accuracy of the beam-column analogy when used for the analysis of vibration and stability of plates, several examples have been worked out numerically. The examples are chosen to represent different boundary conditions.

6.1 Example 1

Table 1 shows a comparison between the results obtained using the proposed grillage analogy and those obtained using both Mohsin-Sadek's beam-analog and exact series solution, as given by Timoshenko[1], for a square plate with central concentrated load under different boundary conditions.

6.2 Example 2

To determine the interaction between out-of-plane and in-plane loads, a simply supported plate under the action of a uniform pressure, N_x, as well as an out-of-plane central load, *P*, is studied here, Fig. 4. The in-plane distributed pressure is replaced by equivalent system of concentrated loads at the edge nodes of the analog.

Fig. 5 shows the effect of the in-plane compressive load, represented by the non-dimensional factor, γ, ($g = N_x / \left(4p^2 D / a^2\right)$) on the central deflection with comparison with Mohsin and Sadek's [8] solution using their model of beam-analog. The critical value of this load is that at which $\delta \rightarrow \infty$ and, when extrapolated from Fig. 5, it is found by Mohsin and Sadek to be that corresponding to $\gamma = 0.97$. Using Wittrick-Williams method for solving the same case, it is found that $\gamma = 0.9996$ which is closer to the exact solution given by Timoshenko[1] $\gamma_{exact} = 1.0$.

6.3 Example 3

This example is a square cantilever plate and represents a problem of considerable practical interest and one for which no exact solution is available. The first five natural frequencies are calculated using the beam-column analogy method and compared with:

1. The values obtained using the finite element technique with 12 terms quartic element.
2. Beam-analog method used by Mohsin and Sadek [13].
3. The experimental results.

In order to check the rate of convergency, the analysis has been done for mesh sizes of 6 × 6 and 4 × 4. The specimen used, with dimensions 300 × 300 × 2.23 mm, is made of steel with the following properties:

E = 21.575 × 10^4 N / mm^2; v = 0.3 and $\rho = 78 \times 10^{-6}$ N / mm^3

The results are given in Table 1 in terms of the non-dimensional factor c defined by

$$c = \frac{4a^2 f}{p\ t}\sqrt{\frac{3r\left(1-v^2\right)}{E}}$$

in which f is the frequency in cps and a is the plate side. Thus for the dimensions and material used c is given by $c_i = 0.01631 f_i$. It is clear from Table 2 that the results obtained, using the beam-column analogy for the present study are always converging towards the experimental values. Even when a beam-column analog with a coarse mesh size 4 × 4 is used, acceptable accuracy is obtained.

6.4 Example 4

In order to check the accuracy of the beam-column method in studying the vibration of plates in the presence of in-plane static loads, this example has been carried out (Fig. 6).

The plate dimensions are taken 1000 × 1000 × 1 mm, and material properties are E = 21 × 10^4 N / mm^2; v = 0.3 and $\rho = 78 \times 10^{-6}$ N / mm^3.

The intensity of the in-plane hydrostatic tension N is assumed to be given by $Na^2/p^2 D = 10$ which corresponds to N = 1.896 N / mm. The grillage used is of 8 × 8 mesh size and thus the statically equivalent nodal forces are 237 N / mesh. Table 2 gives comparison of the results obtained using the grillage method and those obtained by Mohsin and Sadek[13] and the exact solution given in the same reference from a research done by Leissa[16], for the first five natural frequencies (cps) of the plate. It is clear from Table 3 that the results obtained using the grillage method are very close to the exact values obtained by using the series solution.

7. Summary and Conclusion

A grid-based method used to deal with plated structures including the effect of both stability and vibration on behavior. It was seen that the method is efficient and accurate enough to use in analyzing plates with simple or complex support conditions. It is obvious from applications and examples solved that the method is competitive compared with finite element method.

Refernces

1. Timoshenko, S.P., and Gere, J.M. *Theory of Elastic Stability*, McGraw-Hill, 1961.
2. Cook, R.D., Malkus, D.S., and Plesha, M.E. *Concepts and Applications of Finite Element Analysis*. John Wiley and Sons, New York, 1989.
3. Rao, G.V. “Fundamental Frequency of a Square Panel

with Multiple Point Supports on Edges.", *J. Sound & Vib.*, Vol. 38, 271, 1975.
4. Cheung, Y.K. *Finite Strip Method in Structural Analysis*. Pergamon Press, Oxford ,1976.
5. MacNeal, R.H. "The Solution of Elastic Plate Problems by Electrical Analogues.", *J. Appl. Mech.*, Vol. 18, 59-67, 1951.
6. Yettram, A.L., and Husain, H.M. "The Representation of a Plate in Flexure by a Grid of Orthogonally Connected Beams." *Int. J. Mech. Sci.*, Vol. 7, 243-251, 1965a.
7. Yettram, A.L., and Husain, H.M. "Grid-Framework Method for Plates in Flexure.", *J. Eng. Mech. Div.*, Vol. 91, No. EM3, 243-251 (1965b).
8. Mohsin, M.E., and Sadek, E.A. "Beam-Analog for Plate Elements." *J. Eng. Struct. Div.*, Vol. 102, No. ST1, 125-145, 1976.
9. Ovunc, B.A. "Dynamics of Frameworks by Continuous Mass Method." *Comp. & Struct.*, Vol. 4, 1061-1089, 1974.
10. Wittrick, W.H., and Williams, F.W. "A General Algorithm for Computing Natural Frequencies of Elastic Structures.", *Quart. J. Mech. Appl. Math.*, Vol. 24, 263-284 (1971).
11. Lee, I.W., Jung, H.J., Kim, M.C., and Robinson, A.R. "Eigensolution Method for Structures with Multiple or Close Eigenvalues.", *J. Eng. Mech. Div.*, Vol. 125, No. 11, 1286-1292, 1999.
12. Chan, S.L., and Gu, J.X. "Exact Tangent Stiffnesses for Imperfect Beam-Column Members.", *J. Struct. Eng. Div.*, Vol. 126, No. 9, 1094-1102, 2000.
13. Mohsin, M.E., and Sadek, E.A. "On the Dynamics of Plates using a Beam-Analog.", *Comp. & Struct.*, Vol. 12, 267-272, 1980.
14. Hurty, W.C., and Rubinstein, M.F. *Dynamics of Structures*. Prentice-Hall, New Jersey, 1964.
15. Ali, A.A. "Vibration and Stability Analysis of Frame-Type Structures and Plates using Beam-Column Analogy.", *Ph.D. Thesis*, The University of Technology, Baghdad, 2004.
16. Leissa, A.W. "Vibration of Plates.", NASA SP-160, 1969.

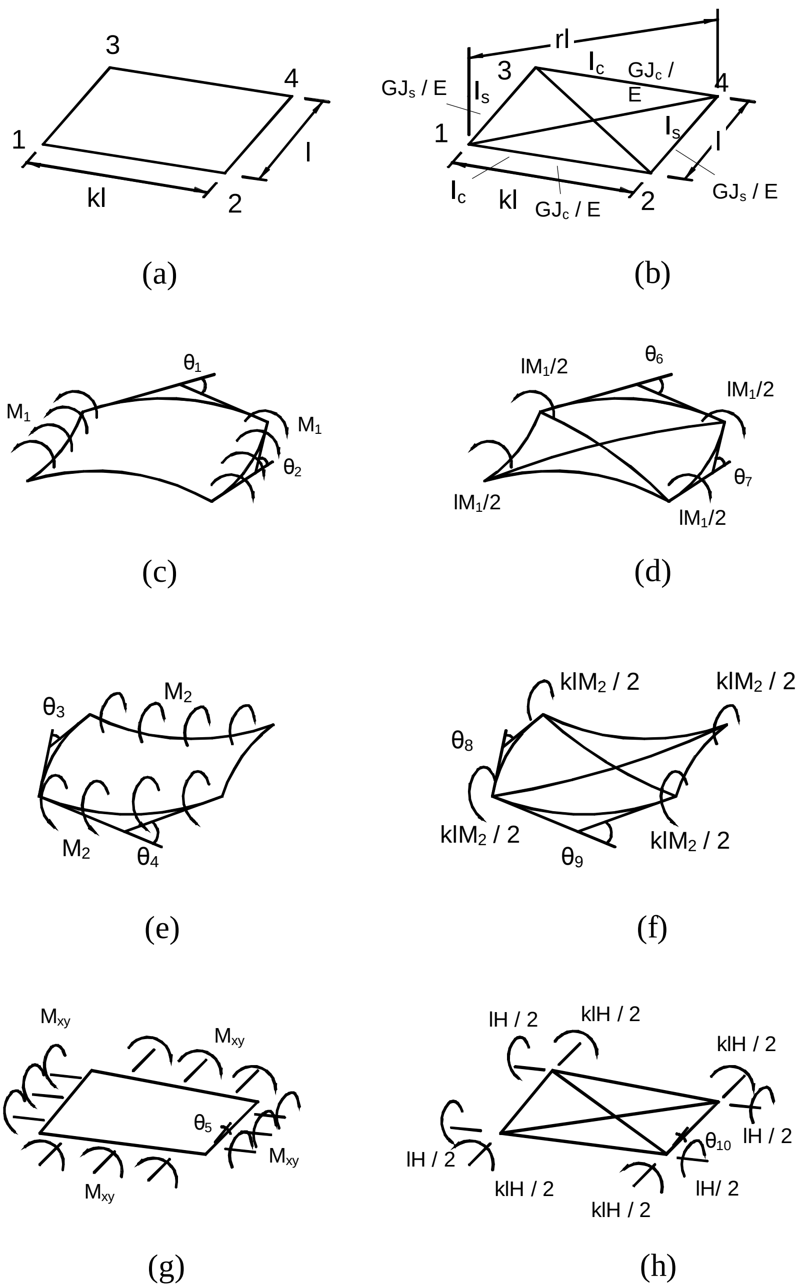
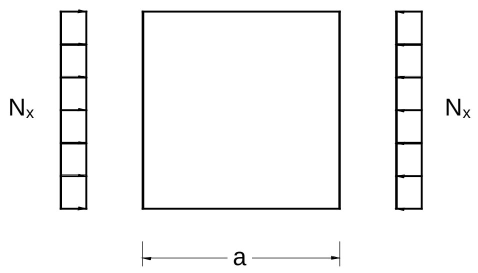
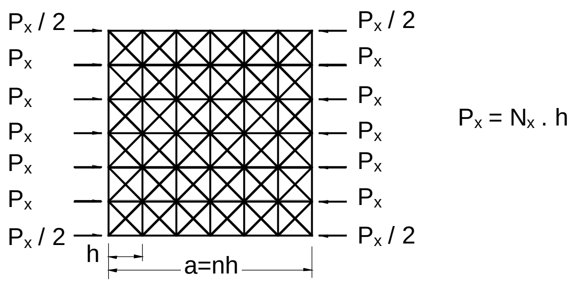


Fig. 1: Plate element and equivalent framework model [7].



(a)



(b)

Fig. 2: (a) Square element plate under action of uniform in-plane load. (b) Beam-column analogy used.

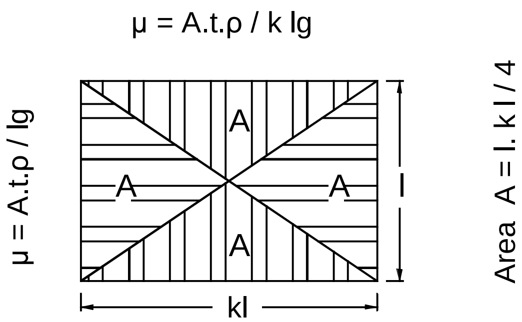


Fig. 3: Lumping of the distributed mass of the plate element into concentrated mass along the side beams of the analog.

Table 1: Simply supported square plate of side *a*, central load *P* using 6×6 grid (Example 1).

$\delta / (Pa^2 / D)$	Present	0.01186
	Mohsin and Sadek[8]	0.01190
	Exact[1]	0.0116
M / P	Present	0.3112
	Mohsin and Sadek[8]	0.3117
	Exact[1]	0.2980
R / P	Present	0.118
	Mohsin and Sadek[8]	0.110
	Exact[1]	0.122

Note: Poisson's ratio, v = 0.3, δ is central deflection, *M* is central moment, and *R* is concentrated corner reaction.

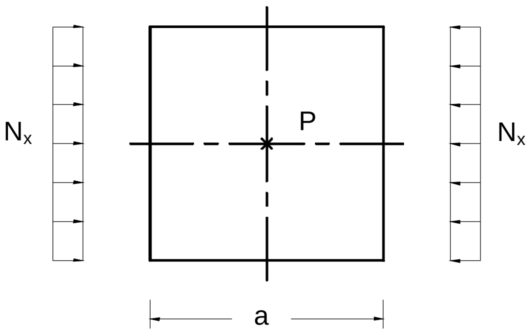


Fig. 4: Simply supported plate under action of uniform in-plane pressure together with out-of-plane central load (Example 2).

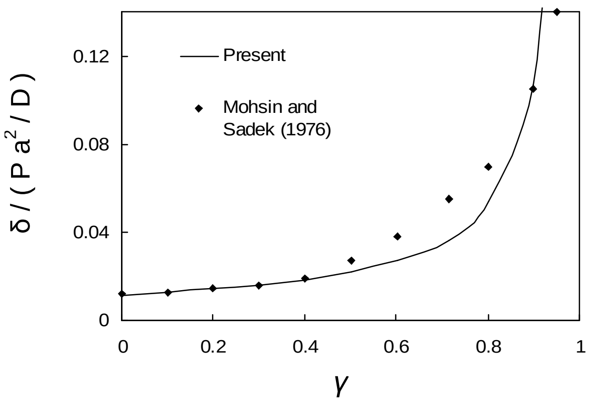


Fig. 5: Effect of in-plane compressive load on central deflection (Example 2).

Table 2: The nondimensional factors for natural frequencies of a square cantilever plate (Example 3).

c_i	Finite Element		Mohsin and Sadek [13]		Present		Experimental
	4×4	6×6	4×4	6×6	4×4	6×6	
c_1	0.352	0.352	0.347	0.351	0.347	0.350	0.351
c_2	0.863	0.863	0.835	0.850	0.829	0.846	0.858
c_3	2.182	2.169	2.039	2.107	2.038	2.104	2.138
c_4	2.735	2.746	2.528	2.649	2.524	2.644	2.712
c_5	3.133	3.138	2.955	3.055	2.922	3.038	3.089

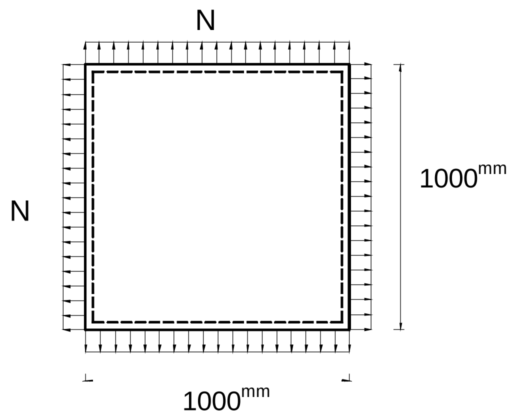


Fig. 6: Simply supported plate under action of uniform in-plane tension (Example 4).

Table 3: The natural frequencies of a square simply supported plate under tension $N\, a^2 / \pi^2 D = 10$ (Example 4).

f_i	Mohsin and Sadek (13) 8×8	Exact	Present 8×8
f_1	11.95	11.96	11.94
f_2	21.05	21.15	21.01
f_3	29.10	29.30	29.24
f_4	41.80	42.20	41.72
f_5	53.90	54.80	53.50

APPENDIX: General Stiffness Matrix

The stiffness matrix for any member of the analog shown in Fig. A.1 is

$$\{f\}=[k]\{d\} \qquad \text{(A.1)}$$

where $\{f\}$ and $\{d\}$ are force and displacement vectors, and

$$[k]=\begin{bmatrix} \frac{GJ}{L}a_1\cot a_1 & 0 & 0 & -\frac{GJ}{L}a_1\csc a_1 & 0 & 0 \\ & \frac{EI_z}{L}F_2 & -\frac{EI_z}{L^2}F_4 & 0 & \frac{EI_z}{L^2}F_1 & -\frac{EI_z}{L^2}F_3 \\ & & \frac{EI_z}{L^3}F_6 & 0 & \frac{EI_z}{L^2}F_4 & \frac{EI_z}{L^3}F_5 \\ & & & \frac{GJ}{L}a_1\cot a_1 & 0 & 0 \\ & \text{Symmetric} & & & \frac{EI_z}{L}F_2 & \frac{EI_z}{L^2}F_4 \\ & & & & & \frac{EI_z}{L^3}F_6 \end{bmatrix} \qquad \text{…(A.2)}$$

where

$$\alpha_1 = \omega L \left(\frac{\rho}{Gg} \right)^{\frac{1}{2}} \qquad \text{(A.3)}$$

in which ω is the angular frequency, ρ is the specific weight of element, G is the shear modulus of the material and g is the gravitational acceleration.

The dynamic-stability functions, F_1 to F_6, for a beam subjected to harmonic loads, while under the action of an axial load, P, are given in Ali [(15)], and are given here briefly.

Dynamic-Stability Functions

1. *General case of dynamic excitation in the presence of axial static load:*

$$F_1(a,d) = -\frac{(a^2 + d^2)(a \sinh d - d \sin a)}{f_1(a,d)}$$

$$F_2(a,d) = -\frac{(a^2 + d^2)(d \cosh d \sin a - a \sinh d \cos a)}{f_1(a,d)}$$

$$F_3(a,d) = -\frac{ad(a^2 + d^2)(\cosh d - \cos a)}{f_1(a,d)}$$

$$F_4(a,d) = \frac{ad(d^2 - a^2)(\cosh d \cos a - 1) + 2ad^2 \sinh d \sin a}{f_1(a,d)}$$

$$F_5(a,d) = \frac{ad(a^2 + d^2)(d \sinh d + a \sin a)}{f_1(a,d)}$$

$$F_6(a,d) = -\frac{ad(a^2 + d^2)(d \cosh d \sin a + a \sinh d \cos a)}{f_1(a,d)}$$

where

$$a = \left[\frac{b^2}{2} + \left(\frac{b^4}{4} + l^4 \right)^{1/2} \right]^{1/2} \qquad \text{(A.4)}$$

$$d = \left[-\frac{b^2}{2} + \left(\frac{b^4}{4} + l^4 \right)^{1/2} \right]^{1/2} \qquad \text{(A.5)}$$

$$b = L \left(\frac{Q}{EI} \right)^{1/2} \qquad \text{(A.6)}$$

$$l = L \left(\frac{Arw^2}{EIg} \right)^{\frac{1}{4}} \qquad \text{(A.7)}$$

and

$$f_1(a,d) = 2ad(\cosh d \cos a - 1) + (a^2 - d^2) \sinh d \sin a \qquad \text{…(A.8)}$$

2. *Case of dynamic excitation with zero axial static load:*

$$F_1(l) = -l \frac{\sinh\lambda - \sin\lambda}{\cosh\lambda \cos\lambda - 1}$$

$$F_2(l) = -l \frac{\cosh l \sin l - \sinh l \cos l}{\cosh l \cos l - 1}$$

$$F_3(\lambda) = -\lambda^2 \frac{\cosh\lambda - \cos\lambda}{\cosh\lambda \cos\lambda - 1}$$

$$F_4(\lambda) = \lambda^2 \frac{\sinh\lambda \sin\lambda}{\cosh\lambda \cos\lambda - 1}$$

$$F_5(\lambda) = \lambda^3 \frac{\sinh\lambda + \sin\lambda}{\cosh\lambda \cos\lambda - 1}$$

$$F_6(\lambda) = -\lambda^3 \frac{\cosh\lambda \sin\lambda + \sinh\lambda \cos\lambda}{\cosh\lambda \cos\lambda - 1}$$

3. *Case of static loading:*

a. Axial compressive load;

$$F_1 = S_1(b) = \frac{b(\sin b - b)}{b \sin b + 2(\cos b - 1)}$$

$$F_2 = S_2(b) = -\frac{b(\sin b - b \cos b)}{b \sin b + 2(\cos b - 1)}$$

$$F_3 = S_3(b) = \frac{b^2(\cos b - 1)}{b \sin b + 2(\cos b - 1)}$$

$$F_4 = -S_3(b)$$

$$F_5 = S_4(b) = \frac{b^3 \sin b}{b \sin b + 2(\cos b - 1)}$$

$$F_6 = -S_4(b)$$

b. Axial tensile load;

$$F_1 = S_1(b) = \frac{b(\sinh b - b)}{b \sinh b - 2(\cosh b - 1)}$$

$$F_2 = S_2(b) = -\frac{b(\sinh b - b \cosh b)}{b \sinh b - 2(\cosh b - 1)}$$

$$F_3 = S_3(b) = \frac{b^2(\cosh b - 1)}{b \sinh b - 2(\cosh b - 1)}$$

$$F_4 = -S_3(b)$$

$$F_5 = S_4(b) = -\frac{b^3 \sinh b}{b \sinh b - 2(\cosh b - 1)}$$

$$F_6 = -S_4(b)$$

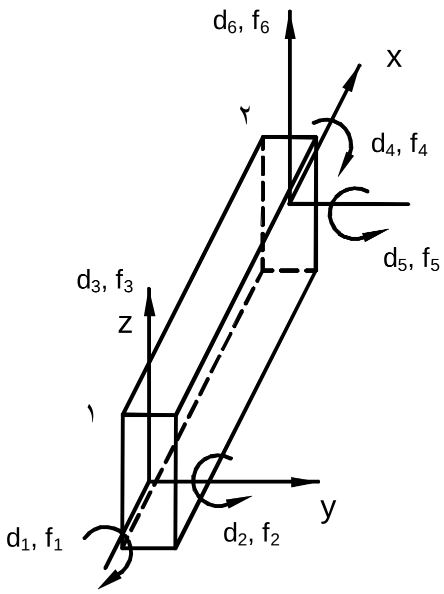


Fig. A.1: Deformations d_i and reactions f_i in member coordinate axes system, *xyz*.