Al-Qadisiyah Journal of Pure Science

Volume 29 | Number 2 Article 11

12-20-2024

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Recommended Citation
huniwi, Rand Raheem and Al-Mayahi, Noori F. (2024) "On Fuzzy Soft Modular space," *Al-Qadisiyah Journal of Pure Science*: Vol. 29 : No. 2 , Article 11.
Available at: https://doi.org/10.29350/2411-3514.1289

ORIGINAL STUDY

On fuzzy Soft Modular Space

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Abstract

In this paper, we will present a definition of fuzzy soft modular space and some properties of fuzzy soft modular (convergent, continuous, bounded) are explained Instead of the prevailing definition.

Keywords: Fuzzy set, Soft set, Fuzzy soft set, Fuzzy soft metric space, Soft modular space

1. Introduction

In 1999, Russian researcher Molodtsov [1] originated the idea of soft set theory as a Mathematical tool for dealing with uncertainty and decision making problems. There are Many practical applications of soft set theory in various fields of sciences including social Sciences, physics, engineerin economics, computer science and medical sciences. Maji et al. [4] applied soft set theory in decision making problems and defended many operations on soft sets. Yildirim et al. [3] presented the notion of soft ideal for a soft topological space and defined soft I-Baire spaces for a soft ideal topological spaces as well. Soft topology and soft metric spaces has studied by many researchers in the last decade [6], [7].

2. Basic concepts about soft sets

2.1. Definition (2.1) [2]

A pair (F, E) is said to be a soft set over X, where F is a function given by $F{:}E \rightarrow P(X)$

2.2. Definition (2.2) [5]

Let $\mathcal{R}$ be the set of real numbers, B($\mathcal{R}$) be the collection of all non-empty bounded subsets of $\mathcal{R}$ and E taken as a set of parameters. Then a function $F : E \rightarrow B(\mathcal{R})$ is called a soft real set. If a soft real set is a singleton soft set, it will be said a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\check{0}$ and $\check{1}$ are the soft real numbers where $\check{0}(e) = 0$, $\check{1}(e) = 1$ for all $e \in E$.

2.3. Remark (2.3) [3]

The set of all soft real numbers is denoted by $\mathcal{R}(E)$ and the set of all non-negative soft real numbers by $\mathcal{R}^*$ (E).

2.4. Remark (2.4) [7]

Let $\tilde{r}, \tilde{s}, \tilde{t} \tilde{\in} \mathcal{R}(E)$. Then the soft addition $\tilde{r} \tilde{+} \tilde{s}$ of $\tilde{r}, \tilde{s}$ and soft scaler multiplication $\tilde{t} \tilde{\cdot} \tilde{r}$ of $\tilde{t}$ and $\tilde{r}$ are defined by:

1. $(\tilde{r} + \tilde{s})(e) = \tilde{r}(e) + \tilde{s}(e)$, for all $e \in E$.
2. $(\tilde{r} - \tilde{s})(e) = \tilde{r}(e) - \tilde{s}(e)$, for all $e \in E$.
3. $(\tilde{r} \cdot \tilde{s})(e) = \tilde{r}(e) \cdot \tilde{s}(e)$, for all $e \in E$.
4. $(\tilde{r} \tilde{/} \tilde{s})(e) = \tilde{r}(e) / \tilde{s}(e)$, and $\tilde{s}(e) \neq 0$ for all $e \in E$.

2.5. Remark (2.5) [3]

For two soft real numbers $\tilde{r}, \tilde{s}$, we have:

1. If $\tilde{r} \tilde{\leq} \tilde{s}$, then $\tilde{r} + \tilde{t} \tilde{\leq} \tilde{s} + \tilde{t}$; for all $\tilde{t} \tilde{\in} \mathcal{R}(A)$.
2. If $\tilde{r} \tilde{\leq} \tilde{s}$, then $\tilde{r} \cdot \tilde{t} \tilde{\leq} \tilde{s} \cdot \tilde{t}$; for all $\tilde{t} \tilde{\in} \mathcal{R}(A)^*$.

Received 7 November 2022; accepted 30 December 2022.
Available online 18 April 2025

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https://doi.org/10.29350/2411-3514.1289

2.6. Theorem (2.6) [5]

Let F_E, G_E, H_E are soft sets in $S(X)$, $\breve{x}_e \neq \breve{\varnothing}$. Then the following hold:

(i) $\forall e \in E, (e, \varnothing) \tilde{\in} F_E$.
(ii) $\breve{x}_e \tilde{\in} [F_E \breve{\cup} G_E]$ iff $\breve{x}_e \tilde{\in} F_E \breve{\vee} \breve{x}_e \tilde{\in} G_E$.
(iii) $\breve{x}_e \tilde{\in} [F_E \tilde{\cap} G_E]$ iff $\breve{x}_e \tilde{\in} F_E \tilde{\wedge} \breve{x}_e \tilde{\in} G_E$.
(iv) $\breve{x}_e \tilde{\in} [F_E \breve{\setminus} G_E]$ iff $\breve{x}_e \tilde{\in} F_E \tilde{\wedge} \breve{x}_e \widetilde{\notin} G_E$.

2.7. Definition (2.7) [7]

A soft point (F,E) over X is said to be a soft point if there is exactly one e ∈ E, such that

F(e) = {x} for some x ∈ X and F(e) = ∅, $\forall e \in E/\{e\}$. It will be denoted by $\breve{x}_e$.

2.8. Definition (2.8) [7]

Let (F, E) be a soft set over $\breve{X}$. The set soft (F, E) is said to be a soft vector and denoted by $\breve{x}_e$ if there is exactly one e $\in E$. Such that F(e) = $\{x\}$ for some x ∈ X and F(e) = $\breve{\varnothing}$,

$\forall\ e \in E \setminus \{e\}$ set. The set of all soft vector over $\breve{X}$ will be denoted by SV($\breve{X}$).

2.9. Definition (2.9) [7]

The set SV($\breve{X}$) is called soft vector space.

2.10. Theorem (2.10) [6]

Every soft set can be expressed as union of all soft points belonging to it. Conversely, any set of soft points can be considering as a soft set.

2.11. Definition (2.11): [4]

A pair (F, E) is called a fuzzy soft set over (X, E), if F: $E \rightarrow I^X$ is a function from E into I^X. The collection of all fuzzy soft sets over (X, E) is denoted by $F(X, E)$.

2.12. Definition (2.12) [4]

A Fuzzy soft set (F,A) over (X, E) is said to be an absolute fuzzy soft set, if for all e ∈ E, F(e) is a fuzzy universal set $\tilde{1}$ over X and is denoted by $\tilde{E}$.

2.13. Definition (2.13) [4]

A fuzzy soft set (F, E) over (X, E) is said to be a null fuzzy soft set, if for all e ∈ E, F(e) is the null fuzzy set $\tilde{0}$ over X. It is denoted by $\breve{\varnothing}$.

2.14. Definition (2.15) [4]

For two fuzzy soft sets (F, A) and (G, B) in $F(X, E)$ we say that $(F, A) \breve{\subseteq} (G, B)$ if $A \breve{\subseteq} B$ *and* $F(e)(x) \breve{\leq} G(e)(x)$.

2.15. Definition (2.16) [4]

Two fuzzy soft sets (F, A) and (G, B) *in* $F(X, E)$ are equal if $F \breve{\subseteq} G$ *and* $G \breve{\subseteq} F$.

2.16. Definition (2.17) [4]

The different between two fuzzy soft sets (F,A) and (G,B) in F (X, E) is a fuzzy soft set ($F/G, E$) (say) defined by (F/G) (e) = $F(e)/G(e)$ for each $e \in E$.

$$(F/G)(e) : X \rightarrow I, (F/G)(e)(x) = F(e)(x) \breve{\cap} (G(e)(x))^c \\ = \min\{F(e)(x), G^c(e)(x)\} \forall x \in X.$$

2.17. Definition (2.18) [4]

The complement of a fuzzy soft set (F,E) is a fuzzy soft set (F^c, E) defined by F^c (e) = $1/F(e)$ for each $e \in E$,

$$(F^c(e))(x) = 1 - F(e)(x)\ \forall x \in X.$$

2.18. Definition (2.19) [1]

Let (F, A) and (G, B) be two fuzzy soft sets in F (X, E) with $A \breve{\cap} B \neq \Phi$, then:

a) their intersection ($F \breve{\cap} G, C$) is a fuzzy soft set, where $C = A \breve{\cap} B$ and $(F \breve{\cap} G)(e) = F(e) \breve{\cap} G(e)$ for each $e \in C$, $(F \breve{\cap} G)(e)(x) = \min\{(e, F_e(x)), (e, G_e(x))$
b) their union $(F \breve{\cup} G, C)$ is a fuzzy soft set, where $C = A \breve{\cup} B$ *and* $(F \breve{\cup} G)e = F(e) \breve{\cup} G(e)$ for each $e \in C$ $(F \breve{\cup} G)(e)(x) = \max\{F(e)(x), G(e)(x)\}$

3. Fuzzy soft modular

In this section We will define a fuzzy soft modular space and some properties.

3.1. Definition (3.1)

Let SV($\breve{X}$) be a soft vector space *over a field F*, A function $\breve{\mathcal{M}}$: SV($\breve{X}$) $\rightarrow \mathcal{R}^*$ (I) is said fuzzy soft modular on SV($\breve{X}$) if satisfies the following condition:

1. $\breve{\mathcal{M}}(\breve{x}_{e1}) = \overline{0} \Leftrightarrow \breve{x}_{e1} = \breve{0},\ \breve{x}_{e1} \breve{\in}$ SV $(\breve{X})$.

2. $\check{\mathcal{M}}(\alpha\check{x}_{e1}) = \check{\mathcal{M}}(\check{x}_{e1})$ for $\check{\alpha} \check{\in} F$ with $|\check{\alpha}| = \check{1}$
3. $\check{\mathcal{M}}(\alpha\check{x}_{e1} + \beta\check{y}_{e2}) \check{\leq} \check{\mathcal{M}}(\check{x}_{e1}) \bigoplus \check{\mathcal{M}}(\check{y}_{e2})$ iff $\check{\alpha}, \check{\beta} \check{\geq} \check{0}$, for all $\check{x}_{e1}, \check{y}_{e2} \check{\in} \mathrm{SV}(\check{X})$.

The soft vector space $\mathrm{SV}(\check{X})$ with the fuzzy soft modular $\check{\mathcal{M}}$ on $\check{X}$ is said to be a fuzzy soft modular space and denoted by $(\check{X}, \check{\mathcal{M}})$.

3.2. Example (3.2)

Let $\mathrm{SV}(\check{X}) = \mathbb{R}^2$ with $\check{\mathcal{M}}(\check{x}_{e1}, \check{y}_{e2}) = \check{x}_{e1} \bigoplus \check{y}_{e2}$, for any pair $(\check{x}_{e1}, \check{y}_{e2})$ in $\mathrm{SV}(\check{X})$, then $(\check{X}, \check{\mathcal{M}})$ *is* fuzzy soft modular space.

3.2.1. Solution

Let $(\check{x}_{e1}, \check{y}_{e2}), \in \mathbb{R}^2$ and $\gamma, \beta, \lambda \in F$ with $\gamma + \beta = 1$

1. Since $\mathcal{M}(\check{x}_{e1}, \check{y}_{e2}) = \overline{0}$ if and only if $\check{x}_{e1} \bigoplus \check{y}_{e2} = \overline{0}$ and since

$\check{x}_{e1} \bigoplus \check{y}_{e2} = \overline{0}$ if and only if $\check{x}_{e1} = \check{y}_{e2} = \overline{0}$, then $\check{\mathcal{M}}(\check{x}_{e1}, \check{y}_{e2}) = \overline{0}$ if and only if the pair $(\check{x}_{e1}, \check{y}_{e2})$ the zero in $\mathbb{R}^2$

2. $\check{\mathcal{M}}(\gamma(\check{x}_{e1}, \check{y}_{e2})) = \mathcal{M}((\gamma\check{x}_{e1}, \gamma\check{y}_{e2})) = \gamma\check{x}_{e1} \bigoplus \gamma\check{y}_{e2}$.

Since $|\gamma| = 1$, then $\check{\mathcal{M}}((\check{x}_{e1}, \check{y}_{e2})) = \check{x}_{e1} \bigoplus \check{y}_{e2} = \check{\mathcal{M}}((\check{x}_{e1}, \check{y}_{e2}))$.

3. $\check{\mathcal{M}}(\alpha(\check{x}_{e1}, \check{y}_{e2}) + \beta(\check{z}_{e3}, \check{d}_{e_4})) = \alpha(\check{x}_{e1}, \check{y}_{e2}) \bigoplus \beta(\check{z}_{e3}, \check{d}_{e_4})$

$$\leq \left(\alpha\left(\check{x}_{e1} \bigoplus \check{y}_{e2}\right)\right) \bigoplus \left(\beta\left(\check{z}_{e3} \bigoplus \check{y}_{e2}\right)\right),$$

$$\text{Let } \mathrm{N} = \check{x}_{e1} \bigoplus \check{y}_{e2}, \mathrm{M} = \check{z}_{e3} \bigoplus \check{d}_{e_4}$$

$= \alpha N \bigoplus \beta \bigodot M$, where g $: X \to X$ (by definition $\gamma \bigodot A$)
$= \mathrm{g(N)} \bigoplus \mathrm{g(M)}$
Thus $\mathcal{R}^2$ is fuzzy soft modular space

3.3. Definition (3.3)

A function $\check{\mathfrak{H}}$: $\mathrm{SP}(\check{X}) \times \mathrm{SP}(\check{X}) \to \mathcal{R}^*$ (I)is said to be fuzzy soft metric on $\mathrm{SP}(\check{X})$ if $\check{\mathfrak{H}}$ satisfies the following conditions:

1. $\check{\mathfrak{H}}$ $(\check{x}_{e1}, \check{y}_{e2}) \check{\geq} \overline{0}$ for all $\check{x}_{e1}, \check{y}_{e2} \check{\in} \mathrm{SP}(\check{X})$.
2. $\check{\mathfrak{H}}$ $(\check{x}_{e1}, \check{y}_{e2}) = \overline{0}$ if and only if $\check{x}_{e1} = \check{y}_{e2} \check{\in} \mathrm{SP}(\check{X})$.
3. $\check{\mathfrak{H}}$ $(\check{x}_{e1}, \check{y}_{e2}) = \check{\mathfrak{H}}$ $(\check{y}_{e2}, \check{x}_{e1})$ for all $\check{x}_{e1}, \check{y}_{e2} \check{\in} \mathrm{SP}(\check{X})$.
4. $\check{\mathfrak{H}}$ $(\check{y}_{e2}, \check{z}_{e3}) \check{\leq} \check{\mathfrak{H}}$ $(\check{x}_{e1}, \check{y}_{e2}) \bigoplus \check{\mathfrak{H}}$ $(\check{y}_{e2}, \check{z}_{e3})$ for all $\check{x}_{e1}, \check{y}_{e2}, \check{z}_{e3} \check{\in} \mathrm{SP}(\check{X})$.

The soft vector space $\mathrm{SP}(\check{X})$ with the fuzzy soft metric $\check{\mathfrak{H}}$ on $\check{X}$ is said to be a fuzzy soft metric space and denoted by $(\check{X}, \check{\mathfrak{H}})$.

3.4. Definition (3.4)

A sequence of soft vectors $\{\check{x}_{e_n}\}$ in $(\check{X}, \check{\mathcal{M}})$ is said to be convergent to $\check{x}_0$if $\forall \overline{\varepsilon} > \overline{0}$,
$\exists k \in Z$ *such that* $\check{\mathcal{M}}(\check{x}_{e_n} - \check{x}_{e_0}) \check{<} \overline{\varepsilon}, \forall n \geq k$ and is denoted by $\check{x}_{e_n} \to \check{x}_{e_0}$*as* $n \to \infty$ or $\lim_{n\to\infty} \check{x}_{e_n} = \check{x}_{e_0}$, $\check{x}_{e_0}$*is said to be* the limit of sequence $\check{x}_{e_n}$ as $n \to \infty$.

3.5. Definition (3.5)

A sequence $\{\check{x}_{e_n}\}$ in $(\check{X}, \check{\mathcal{M}})$ is said to be a Cauchy sequence if corresponding to every $\overline{\varepsilon} \check{>} \overline{0}$, $\exists$ $m \in N$ *such that* $\check{\mathcal{M}}(\check{x}_{e_n} - \check{x}_{e_j}) \check{<} \overline{\varepsilon}$, $\forall$ $n, j \geq m$, *i.e*

$$\check{\mathcal{M}}\left(\check{x}_{e_n} - \check{x}_{e_j}\right) \to \overline{0}, as\ n, j \to \infty.$$

3.6. Definition (3.6)

Let $(\check{X}, \check{M})$ be a fuzzy soft modular space. Then $(\check{X}, \check{M})$ is said to be complete if every Cauchy sequence in $\check{X}$ convergent to a soft vector of $\check{X}$

3.7. Theorem (3.7)

Every fuzzy soft modular space $(\check{X}, \check{M})$ is soft metric space $(\check{X}, \check{\mathfrak{H}})$.

3.7.1. Proof

Let $(\check{X}, \check{M})$ be a soft modular space, defined $\check{\mathfrak{H}} : \mathrm{SV}(\check{X}) \times \mathrm{SV}(\check{X}) \to \mathcal{R}^*$ (I) by

$$\check{\mathfrak{H}}\left(\check{x}_{e_1}, \check{y}_{e_2}\right) = \check{\mathcal{M}}\left(\check{x}_{e_1} - \check{y}_{e_2}\right) \text{ for all } \check{x}_{e_1}, \check{y}_{e_2} \check{\in} \mathrm{SV}(\check{X})$$

1. Let all $\check{x}_{e_1}, \check{y}_{e_2} \check{\in} \mathrm{SV}(\check{X})$

$$\check{x}_{e_1} \ominus \check{y}_{e_2} \to \check{\mathcal{M}}\left(\check{x}_{e_1} - \check{y}_{e_2}\right) \check{\geq} \overline{0} \to \check{\mathfrak{H}}\left(\check{x}_{e_1}, \check{y}_{e_2}\right) \check{\geq} \overline{0}.$$

2. Let all $\check{x}_{e_1}, \check{y}_{e_2} \check{\in} \mathrm{SV}(\check{X})$

$$\check{\mathfrak{H}}\left(\check{x}_{e_1}, \check{y}_{e_2}\right) = \overline{0} \Leftrightarrow \check{\mathcal{M}}\left(\check{x}_{e_1} - \check{y}_{e_2}\right) = \overline{0} \Leftrightarrow \check{x}_{e_1} - \check{y}_{e_2} = \overline{0} \Leftrightarrow \check{x}_{e_1} = \check{y}_{e_2}.$$

3. Let $\check{x}_{e_1}, \check{y}_{e_2} \check{\in} \mathrm{SV}(\check{X})$

$$\check{\mathfrak{H}}\left(\check{x}_{e_1},\check{y}_{e_2}\right)=\check{\mathcal{M}}\left(\check{x}_{e_1}-\check{y}_{e_2}\right)=\check{\mathcal{M}}\left(-\left(\check{y}_{e_2}-\check{x}_{e_1}\right)\right)$$
$$=\check{\mathcal{M}}\left(\check{y}_{e_2}-\check{x}_{e_1}\right)$$

$$=\check{\mathfrak{H}}\left(\check{y}_{e_2},\check{x}_{e_1}\right).$$

4. Let $\check{x}_{e_1},\check{y}_{e_2},\check{z}_{e3}\check{\in}\mathrm{SV}(\check{X})$

$$\check{\mathcal{M}}\left(\check{x}_{e_1}-\check{y}_{e_2}\right)=\check{\mathcal{M}}\left(\left(\check{x}_{e_1}-\check{z}_{e3}\right)+\left(\check{z}_{e3}-\check{y}_{e_2}\right)\right)$$

$$\check{\le}\check{\mathcal{M}}\left(\check{x}_{e_1}-\check{z}_{e3}\right)\bigoplus\check{\mathcal{M}}\left(\check{z}_{e3}-\check{y}_{e_2}\right)$$

$$\Rightarrow\check{\mathfrak{H}}\left(\check{x}_{e_1},\check{y}_{e_2}\right)\check{\le}\check{\mathfrak{H}}\left(\check{x}_{e_1}-\check{z}_{e3}\right)\bigoplus\check{\mathfrak{H}}\left(\check{z}_{e3}-\check{y}_{e_2}\right).$$

It follow that $(\check{X},\check{\mathfrak{H}})$ is fuzzy soft metric on $\check{X}$ and this fuzzy soft metric is called the fuzzy soft metric induced by fuzzy soft modular.

3.8. Definition (3.8)

Let $(\check{X},\check{\mathcal{M}})$ be a fuzzy soft modular space. The fuzzy soft open ball with center $\check{x}_{e_1}\check{\in}SV(\check{X})$ and radius $\overline{r}\check{>}\overline{0}$ is denoted and defined by

$$\mathrm{B}\left(\check{x}_{e_1},\overline{r}\right)=\left\{\check{y}_{e_2}\check{\in}\check{X}_M:\check{\mathcal{M}}\left(\check{x}_{e_1}-\check{y}_{e_2}\right)\check{<}\overline{r}\right\}$$

Similarly, the fuzzy soft closed ball with center $\check{x}_{e_1}\check{\in}SV(\check{X})$ and radius $\overline{r}\check{>}\overline{0}$ is denoted and defined by

$$\overline{\mathrm{B}\left(\check{x}_{e_1},\overline{r}\right)}=\left\{\check{y}_{e_2}\check{\in}\check{X}_M:\check{\mathcal{M}}\left(\check{x}_{e_1}-\check{y}_{e_2}\right)\check{\le}\overline{r}\right\}.$$

3.9. Definition (3.9)

Let $(\check{X},\check{M})$ be a fuzzy soft modular space and $\check{A}\check{\subseteq}\check{X}$ we say that $\check{A}$ is fuzzy soft open set if for every $\check{x}_{e_1}\check{\in}\check{A}$ there exist $\overline{r}\check{>}\overline{0}\ni\mathrm{B}(\check{x}_{e_1},\check{r})\check{\subset}\check{A}$. A subset $\check{A}$ of $\check{X}$ is said to be fuzzy soft closed if its complement is fuzzy soft open, that is, $\check{A}^c=\check{X}-\check{A}$ is fuzzy soft closed.

3.10. Definition (3.10)

Let $(\check{X},\check{\mathcal{M}}),(\check{Y},\check{\mathcal{M}})$ be two fuzzy soft modular spaces. The linear function $f:SV(\check{X})\to Y$ is said bounded if $f(\check{A})$ is bounded set in Y for all $\check{A}$ bounded set in $\check{X}$.
i-e : $\forall\{\check{A}$ bounded set in $SV(\check{X})\,f(\check{A})$ bounded set in $Y\}$.

3.11. Theorem (3.11)

Let $(\check{X},\check{M})$ *and* $(\check{Y},\check{M})$ be a two fuzzy soft modular spaces and let $\check{x}_{e_n}\to\check{x}_{e_1},\check{y}_{e_n}\to\check{y}_{e_1}$, such that $\{\check{x}_{e_n}\}$ and $\{\check{y}_{e_n}\}$ are two sequences in $SV(\check{X})$ and $\alpha,\beta\in F/\{0\}$ then $\alpha f(\check{x}_{e_n})+\beta g(\check{y}_{e_n})\to\alpha f(\check{x}_{e_1})\bigoplus\beta g(\check{y}_{e_1})$ whenever f and g are two identity functions.

3.11.1. Proof

Let $\check{x}_{e_n}\to\check{x}_{e_1}$ and $\check{y}_{e_n}\to\check{y}_{e_1}$

$$\check{\mathcal{M}}\left(\left(\alpha f\left(\check{x}_{e_n}\right)+\beta g\left(\check{y}_{e_n}\right)\right)-\left(\alpha f\left(\check{x}_{e_1}\right)+\beta g\left(\check{y}_{e_1}\right)\right)\right)$$

$$=\check{\mathcal{M}}\left(\alpha\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)\right)+\beta\left(g\left(\check{y}_{e_n}\right)-g\left(\check{y}_{e_1}\right)\right)\right)$$

$$\le\check{M}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)\right)\bigoplus\check{M}\left(g\left(\check{y}_{e_n}\right)-g\left(\check{y}_{e_1}\right)\right)$$
$$=\check{M}(\check{x}_{e_n}-\check{x}_{e_1})\bigoplus\check{M}(\check{y}_{e_n}-\check{y}_{e_1})$$

Since $\check{\mathcal{M}}(\check{x}_{e_n}-\check{x}_{e1})\to\overline{0}$ *and* $\check{\mathcal{M}}(\check{y}_{e_n}-\check{y}_{e_1})\to\overline{0}$
Then
$\check{\mathcal{M}}((\alpha(f(\check{x}_{e_n})+\beta g(\check{y}_{e_n})-(\alpha f(\check{x}_{e_1})+\beta g(\check{y}_{e_1})))\to\overline{0}$ as $n\to\infty$
Then $\alpha f(\check{x}_{e_n})+\beta g(\check{y}_{e_n})\to\alpha f(\check{x}_{e_1})\bigoplus\beta g(\check{y}_{e_1})$.

3.12. Definition (3.12)

Let $(\check{X},\check{\mathcal{M}}),(\check{Y},\check{\mathcal{M}})$ be two fuzzy soft Modular spaces. The function
$f:\check{X}\to\check{X}$ is said to be continuous at $\check{x}_{e_0}\check{\in}\mathrm{SV}(\check{X})$ if for all $\overline{\varepsilon}\check{>}\overline{0}$ and there exists $\overline{\delta}\check{>}\overline{0}$ such that for all $\check{x}_{e_1}\check{\in}\mathrm{SV}(\check{X})$

$$\check{M}\left(\check{x}_{e_1}-\check{x}_{e_0}\right)<\overline{\delta}\Rightarrow\check{M}\left(f\left(\check{x}_{e_1}\right)-f\left(\check{x}_{e_0}\right)\right)<\overline{\varepsilon}.$$

The function f is called continuous function, if it continuous at every point of $\mathrm{SV}(\check{X})$.

3.13. Remark (3.13)

Every identity function in fuzzy soft modular space $(\breve{X}, \breve{M})$ is continuous functions in fuzzy soft modular space.

3.13.1. Proof

For all $\bar{\varepsilon} > \bar{0}$ *take* $\bar{\varepsilon} = \bar{\delta} . \bar{\delta} > \bar{0}$ $\breve{\mathcal{M}}(\breve{x}_{e_n} - \breve{x}_{e_1}) < \bar{\delta}\breve{\mathcal{M}}(f(\breve{x}_{e_n}) - f(\breve{x}_{e_1})) = \breve{\mathcal{M}}(\breve{x}_{e_n} - \breve{x}_{e_1}) < \bar{\delta} < \bar{\varepsilon}$ then f is continuous at $\breve{x}_{e_1}$ Since $\breve{x}_{e_1}$ is an arbitrary point then f is continuous function

3.14. Theorem (3.14)

Let $\breve{X}$ be fuzzy soft modular space over F. Then the function $f : \breve{X} \to X$, $f(\breve{x}_{e_1}, \breve{y}_{e_2}) = \breve{x}_{e_1} \bigoplus \breve{y}_{e_2}$ is continuous functions.

3.14.1. Proof

Let $\breve{x}_{e_0}, \breve{y}_{e_0} \,\breve{\in}\, \mathrm{SV}(\breve{X})$ and $\{\breve{x}e_n\}, \breve{y}e_n\} \,\breve{\in}\, \mathrm{SV}(\breve{X})$ *such that* $\breve{x}_{e_n} \to \breve{x}_{e_0}$ *and* $\breve{y}_{e_n} \to \breve{y}_{e_0}$ *as* $n \to \infty$

$$\breve{\mathcal{M}}\left(f\left(\breve{x}_{e_n}, \breve{y}_{e_n}\right) - f\left(\breve{x}_{e_0}, \breve{y}_{e_0}\right)\right) = \breve{\mathcal{M}}\left(\left(\breve{x}_{e_n} + \breve{y}_{e_n}\right) - \left(\breve{x}_{e_0} + \breve{y}_{e_0}\right)\right) = \breve{\mathcal{M}}\left(\left(\breve{x}_{e_n} - \breve{x}_{e_0}\right) + \left(\breve{y}_{e_n} - \breve{y}_{e_0}\right)\right) \le \breve{\mathcal{M}}\left(\breve{x}_{e_n} - \breve{x}_{e_0}\right) \bigoplus \breve{\mathcal{M}}\left(\breve{y}_{e_n} - \breve{y}_{e_0}\right)$$

Since $\breve{\mathcal{M}}(\breve{x}_{e_n} - \breve{x}_{e_0}) \to \bar{0}$ *and* $\breve{\mathcal{M}}(\breve{y}_{e_n} - \breve{y}_{e_0}) \to \bar{0}$ *as* $n \to \infty$, *we have*

$\breve{\mathcal{M}}(f(\breve{x}_{e_n}, \breve{y}_{e_n}) - f(\breve{x}_{e_0}, \breve{y}_{e_0})) \to \bar{0}$ *as* $n \to \infty$

Then $f(\breve{x}_{e_n}, \breve{y}_{e_n}) \to f(\breve{x}_{e_1}, \breve{y}_{e_1})$ *as* $n \to \infty$ *is continuous function at* $(\breve{x}_{e_1}, \breve{y}_{e_2})$ *and* $(\breve{x}_{e_1}, \breve{y}_{e_2})$ *is any point in* $\breve{X} \times \breve{X}$ *, therefore f is continuous function.*

3.15. Theorem (3.15)

Let $(X, \mathcal{M}), (Y, \mathcal{M})$ be a fuzzy soft modular spaces, then the function $f : \mathrm{SV}(\breve{X}) \to Y$ is continuous at $\breve{x}_{e_0} \,\breve{\in}\, \mathrm{SV}(\breve{X})$ if and only if for all sequence $\{\breve{x}_{e_n}\}$ convergent to $\breve{x}_{e_0} \,\breve{\in}\, \mathrm{SV}(\breve{X})$ then the sequence $\{f(\breve{x}_{e_n})\}$ is convergent to $f(\breve{x}_{e_0})$ in Y

3.15.1. Proof

Suppose the function f is continuous in x_0 and let $\{\breve{x}_{e_n}\}$ is a sequence in $\mathrm{SV}(\breve{X})$ such that $\breve{x}_{e_n} \to \breve{x}_{e_0}$.

Let $\bar{\varepsilon} \in (0, 1)$, since f is continuous in $\breve{x}_{e_0} \Rightarrow$ there exist $\bar{\delta} > \bar{0}$, such that for all $\breve{x}_{e_1} \in SV(\breve{X}) : \breve{\mathcal{M}}(\breve{x}_{e_1} - \breve{x}_{e_0}) < \bar{\delta} \Rightarrow \breve{\mathcal{M}}(f(\breve{x}_{e_1}) - f(\breve{x}_{e_0})) < \bar{\varepsilon}$ Since

$\breve{x}_{e_n} \to x_0, \bar{\delta} > \bar{0}$, there exist $k \in Z^+$ such that $\breve{\mathcal{M}}(\breve{x}_{e_n} - \breve{x}_{e_0}) < \bar{\varepsilon}$ for all $n \ge k$ hence $\breve{\mathcal{M}}(f(\breve{x}_{e_n}) - f(\breve{x}_{e_0})) < \bar{\varepsilon}$ for all $n \ge k$

Then $f\left(\breve{x}_{e_n}\right) \to f\left(\breve{x}_{e_0}\right)$.

Conversely suppose the condition in the theorem is true.

Suppose f is not continuous at $\breve{x}_{e_0}$.

There exist $\bar{\varepsilon} > \bar{0}$ such that for all $\bar{\delta} > \bar{0}$, there exist $\breve{x}_{e_0} \,\breve{\in}\, \mathrm{SV}(\breve{X})$ and

$$\breve{\mathcal{M}}\left(\breve{x}_{e_1} - \breve{x}_{e_0}\right) < \bar{\delta} \Rightarrow \breve{\mathcal{M}}\left(f\left(\breve{x}_{e_1}\right) - f\left(\breve{x}_{e_0}\right)\right) \ge \bar{\varepsilon}$$

That is mean $\breve{x}_{e_n} \to \breve{x}_{e_0}$ in $\mathrm{SV}(\breve{X})$ but $f(\breve{x}_{e_n}) \nrightarrow f(\breve{x}_{e_0})$ in $\breve{Y}$ this contradiction, f is continuous at $\breve{x}_{e_0}$.

3.16. Theorem (3.16)

Let $(\breve{X}, \breve{\mathcal{M}}), (\breve{Y}, \breve{\mathcal{M}})$ be fuzzy soft modular spaces and let $f : \mathrm{SV}(\breve{X}) \to Y$ be a linear function. Then f is continuous either at every point of $\mathrm{SV}(\breve{X})$ or at no point of $\mathrm{SV}(\breve{X})$.

3.16.1. Proof

Let $\breve{x}_{e_1}$ and $\breve{x}_{e_2}$ be any two point of $\mathrm{SV}(\breve{X})$ and suppose f is continuous at $\breve{x}_{e_1} \,\breve{\in}\, \mathrm{SV}(\breve{X})$, Then for each $\bar{\varepsilon} > \bar{0}$ there exist $\bar{\delta} > \bar{0}$ such that $\breve{x}_{e_0} \,\breve{\in}\, \mathrm{SV}(\breve{X})$.

$$\breve{\mathcal{M}}\left(\breve{x}_{e_0} - \breve{x}_{e_1}\right) < \bar{\delta} \Rightarrow \breve{\mathcal{M}}f\left(\breve{x}_{e_0}\right) -_f \breve{x}e_1\big)\big) < \bar{\varepsilon}$$

$$\textit{Now } \breve{\mathcal{M}}\left(\breve{x}_{e_0} - \breve{x}_{e_2}\right) < \bar{\delta} . \breve{\mathcal{M}}\left(\breve{x}_{e_0} + \breve{x}_{e_1} - \breve{x}_{e2}\right) - \breve{x}e_1\big) < \bar{\delta}$$

$$\Rightarrow \breve{\mathcal{M}}\left(f\left(\breve{x}_{e_0}\right) +_f \breve{x}e_1\big) -_f \breve{x}e_2\big) -_f \breve{x}e_1\big)\right) < \bar{\varepsilon}$$

$$\Rightarrow \breve{\mathcal{M}}\left(f\left(\breve{x}_{e_0}\right) -_f \breve{x}e2\big)\right) < \bar{\varepsilon}$$

then f is continuous at $\breve{x}_{e_2} \,\breve{\in}\, \mathrm{SV}(\breve{X})$, since $\breve{x}_{e_2}$ is an arbitrary point, then f is continuous.

3.17. Corollary (3.17)

Let $(\breve{X}, \breve{\mathcal{M}}), (\breve{Y}, \breve{\mathcal{M}})$ be two fuzzy soft modular spaces and let $f : \mathrm{SV}(\breve{X}) \to Y$ be a linear function. If f is continuous at 0 then it is continuous at every point.

3.17.1. Proof

Let $\{\breve{x}_{e_n}\}$ *be a sequence in* $\mathrm{SV}(\breve{X})$ *such that* $\breve{x}_{e_n} \to \breve{x}_{e_0}$, *Since f is continuous at* 0, *then:*

For all $\bar{\varepsilon} > \bar{0}$, *there exist* $\bar{\delta} > \bar{0}, : (\breve{x}_{e_n} - \breve{x}_{e_1})\mathrm{SV}(\breve{X})$

$$\check{\mathcal{M}}\left(\left(\check{x}_{e_n}-\check{x}_{e_0}\right)-0\right)<\overline{\delta}\Rightarrow\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}-\check{x}_{e_0}\right)-f(0)\right)<\overline{\varepsilon},$$

$$\check{\mathcal{M}}\left(\check{x}_{e_n}-\check{x}_{e_0}\right)<\overline{\delta}\Rightarrow\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_0}\right)-f(0)\right)<\overline{\varepsilon},$$

$$\check{\mathcal{M}}\left(\check{x}_{e_n}-\check{x}_{e_0}\right)<\overline{\delta}\Rightarrow\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_0}\right)-(0)\right)<\overline{\varepsilon},$$

$$\check{\mathcal{M}}\left(\check{x}_{e_n}-\check{x}_{e_0}\right)<\overline{\delta}\Rightarrow\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_0}\right)\right)<\overline{\varepsilon},$$

$\check{x}_{e_n}\rightarrow\check{x}_{e_n}\Rightarrow f(\check{x}_{e_n})\rightarrow f(\check{x}_{e_0})$ *Then f is continuous at* $\check{x}_{e_0}$ *Since* $\check{x}_{e_0}$ *is arbitrary point, therefore f is continuous function.*

3.18. Theorem (3.18)

Let $(\check{X},\check{\mathcal{M}}),(\check{Y},\check{\mathcal{M}})$ be two fuzzy soft modular spaces. If the function $f:\check{X}\rightarrow Y, g:\check{X}\rightarrow Y$ are two continuous functions then:

1. $f+g$ is continuous function.
2. kf where $k\in F/\{0\}$ is continuous function.

3.18.1. Proof

Let $\{\check{x}_{e_n}\}$ be a sequence in $\mathrm{SV}(\check{X})$ such that $\check{x}_{e_n}\rightarrow\check{x}_{e_1}$ Since f and g are two continuous functions at $\check{x}_{e_1}$, Then for all $\overline{\varepsilon}>\overline{0}$ there exist $\overline{\delta}>\overline{0}$ such that for all $\check{x}_{e_1}\in\mathrm{SV}(\check{X}):\check{\mathcal{M}}(\check{x}_{e_n}-\check{x}_{e_1})<\overline{\delta}\Rightarrow\check{\mathcal{M}}(f(\check{x}_{e_n})-f(\check{x}_{e_1}))<\overline{\varepsilon}$,

And $\check{\mathcal{M}}\left(\check{x}_{e_n}-\check{x}_{e_1}\right)<\overline{\delta}\Rightarrow\check{\mathcal{M}}\left(g\left(\check{x}_{e_n}\right)-g\left(\check{x}_{e_1}\right)\right)<\overline{\varepsilon},$

Now $\check{\mathcal{M}}\left((f+g)\left(\check{x}_{e_n}\right)-(f+g)\left(\check{x}_{e_1}\right)\right),$

$$=\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}\right)+g\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)-g\left(\check{x}_{e_1}\right)\right),$$

$$\leq\check{\mathcal{M}}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)\right)\bigoplus\check{\mathcal{M}}\left(g\left(\check{x}_{e_n}\right)-g\left(\check{x}_{e_1}\right)\right),$$

$$<(\varepsilon\bigoplus\varepsilon)(x)=sup\{min\{\varepsilon(y),\varepsilon(z)\}:y\bigoplus z=x\},$$

Therefore $f+g$ is continuous function.

2. Let $\{\check{x}_{e_n}\}$ be a sequence in X such that $\check{x}_{e_n}\rightarrow\check{x}_{e_1}$, then for all $\overline{\varepsilon}>\overline{0}$ there exist $\overline{\delta}>\overline{0}$ such that $\check{M}(\check{x}_{e_n}-\check{x}_{e_1})<\overline{\delta}$ implies $\check{M}(f(\check{x}_{e_n})-f(\check{x}_{e_0}))<\overline{\varepsilon}$

Then for all $\overline{\varepsilon}>\overline{0}$ there exist $\overline{\delta}>\overline{0}$ such that $\check{M}(x_n\check{-}x)<\overline{\delta}$ implies

$$\check{M}\left((kf)\left(\check{x}_{e_n}\right)-(kf)\left(\check{x}_{e_1}\right)\right)=\check{M}\left(k\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)\right)\right)=\check{M}\left(f\left(\check{x}_{e_n}\right)-f\left(\check{x}_{e_1}\right)\right)<\overline{\varepsilon}$$

Therefore kf is continuous function.

Funding

Self-funding.

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