Journal of Iraqi Al-Khwarizmi (JIKh) Volume:8 Issue:1 Year: 2024 pages: 23-35

Application of Generalized Fixed Point Theorems to Ordinary Differential Equations

Authors Names

Mohammed Amer Atiyah[a]

Article History
Publication data: 01/05/2024
Keywords: Ordinary differential equations ODEs, Generalized**,** fixed point

ABSTRACT

Ordinary differential equations (ODEs) serve as a fundamental tool for modeling various natural phenomena in science and engineering. The application of fixed point theorems has proved to be a powerful technique in the study of the existence, uniqueness, and stability of solutions to ODEs. However, traditional fixed point theorems are limited to specific settings and may not be directly applicable to more complex ODEs arising in practical scenarios.
In this research the Banach Contraction Fixed Point Theorems will be used, as well as Schauder's Theorem and Picard Theorem. Said Theorems will be executed to ordinary differential equations (ODE). The objective is to emphasize the utility of the different fixed point theorems, when applying them to proofing the existence theorems and the uniqueness theorems of the solutions of ODE in initial conditions or boundary conditions.

1. Introduction

In this section, we will list some of the concepts needed for job analysis and the definitions that serve as a basis for the work in this research.

1.1 The metric space [1]

Assume that x is a nonempty set. It can be said that:

$$d : X \times X \to R^{+}$$
$$(x, y) \mapsto d(x, y)$$

d is considered a space over x, if it checks for three facts as follows:

$d(x,y) = 0 \Leftrightarrow x = y$
$x,y \in X \Rightarrow d(x,y) = d(y,x)$
$x,y,z \in X \Rightarrow d(x,y) \le d(x,z) + d(z,y)$

So, the couple (X,d) are considered as metric space.

1.2 Cauchy sequence [2]

It is said that the sequence $(x_n)_n$ in the metric space (X, d) is of Cauchy sequence if:

$$\forall \varepsilon > 0, \exists N\varepsilon > 0 \text{ Sush as } \forall n,m > N\varepsilon \Rightarrow d(x_n, x_m) < \varepsilon$$

$$d(x_n, x_m) \longrightarrow 0 \text{ When } n,m \longrightarrow \infty$$

Graduate School of Sciences/Mathematic(MA).Colege of Science, Erzincan Binail Yildirim University AL muthanna-Iraq.
, E-Mail: jorh.ma2017@gmail.com.

1.3 Normed vector space [3]

Assume that (x) is considered as a vector space on R. We call normed vector space on X, an application $\| . \|$ defined by:

$$\| . \| : X \to R+$$
$$x \mapsto \| x \|$$

Where:

- $\| x \| = 0 \Leftrightarrow x = 0$
- $\forall \lambda \in R, \forall x \in X : \| \lambda x \| = | \lambda | \| x \|$
- $\forall x, y \in X : \| x + y \| \leq \| x \| + \| y \|$

Assume that X is a normed vector space, it can be said that the distance actuated by norm is:

$$d: X \times X \longrightarrow R +$$

$$(x, y) \longmapsto d(x, y) = \| x - y \|$$

Any complete normed vector space is named Banach's space.

1.4 Convexity [4]

Assume that ($C \subset X$) is considered as convex set if:

$$\forall t \in [0, 1], \forall (a, b) \in C^2, ta + (1-t)b \in C$$

1.5 Contraction [5]

Assume that there is a metric space (X, d) so, an application $T : X \to X$:

- Lipchitz function (or K-Lipschitzian) if and only if there is $K \geq 0$ for all $x \in X$ *and* $y \in X$ then:
  - $d(T_x, T_y) \leq K\, d(x, y)$
- Contraction is applied if $K < 1$.
- Non expansive if $K \leq 1$.
- Contractive if: for all $x, y \in X$ we have:

$$d(Tx, Ty) < d(x, y)$$

If T is nonexpansive, then it is Lipchitz function.

1.6 Uniform convergence [6]

Assume that (X) is considered as a set of (Y, d) a metric space, A is a subset of X. Suppose $(f_n)_n$ a series of functions defined in X and the values of these functions are in set Y. Thus it can be said that $(f_n)_n$ converges uniformly to f over A if:

$$\forall \varepsilon > 0, \exists N\varepsilon, \forall n \in N, n \geq N\varepsilon, \forall x \in A \Rightarrow d(f_n(x), f(x)) \leq \varepsilon$$

1.7 Equicontinues [7]

Assume $(f_n)_n$ be a sequence of functions defined on an interval I with values in R. We say that the sequence $(f_n)_n$ is equicontinuous if:

$$\forall x \in I, \forall \varepsilon > 0, \exists \delta > 0, \forall n \in N, \forall y \in I, |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$$

1.8 Ordinary differential equations [8]

Assume that I is considered as an open-set in (R), and (X) is considered as a Banach's space in (R) and $U1 \cdots U_n$ is considered as an open-sets in E.

A differential equation (DE) in Banach's space is an equation in this form:

$$F(t, x, x', x'', x''', \dots, x^{(n)} = 0$$

Where:

- n : is considered a non-zero integer.
- F : a given function of $(n+2)$ supposedly regular variables on $I \times U_1 \cdots U_n$.
- x is considered un-known function for I in the Banach's space and (x, ..., $x^{(n)}$) are its successive derivatives.

The problem is to find open interval (I) in R, and search for a function $x: t \mapsto x(t)$ differentiable on this interval up to order n and satisfying this equation:

$$\forall t \in I, F(t, x', x', x''', \dots, x^{(n)} = 0$$

An ODE of order n is linear if it has this form:

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) = g(t)$$

With all the $x^{(i)}$ and all the coefficients depending on t, if g(t)=0, then the equation is said to be homogeneous. The following DE is called the associated DE:

$$a_n(t)x^n(t) + a_{n-1}(t)x^{n-1}(t) + \cdots + a_0(t)x(t) = 0$$

If $a_j(t): 0 < j < n$ are constants, the equation is named a linear DE with constant coefficients.

1.9 Cauchy problem [9]

A Cauchy problem is definition as a problem consisting of a DE for which we seek a solution satisfying a certain initial condition. The condition takes several forms relying on the nature of DE.

Assume that U is considered as an open-set of ($R \times R^n$), and $f: U \to R^n$ a function.

The first order DE is given as the following form:

$$x'(t) = f(t, x)$$

For $(t, x(t)) \in U$, and a point $(t_0, x_0) \in U$ the corresponding Cauchy's problem consists in looking for solutions $x = x(t)$.

Solution of the Cauchy's problem of an open interval I of R with the initial condition $(t_0, x_0) \in U$ and $t_0 \in I$ is a differentiable function $x: R \to R$ where:

- For each $t \in I, (t, x(t)) \in U$.
- For each $t \in I, x'(t) = f(t, x(t))$.
- $x(t_0) = x_0$.

A function $x: R_+ \to R$ is a solution of the Cauchy problem if:

- The function x is continuous $\forall t \in R_+, (t, x(t)) \in R_+ \times R$.
- The solution x of the Cauchy problem is called the integral of the problem:

$$\forall t \in R_+, x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

1.10 Security cylinder

We say that safety cylinder for the equation if any solution $x: I \to R^n$ of the Cauchy problem $x(t_0) = x_0$ where $I \subset [t_0 - h, t_0 + h]$ remains contained in $B(x_0, r_0)$ where h: length, t:time and $t_0 \geq h$

$$h \leq \min\left(t_0, \frac{r_0}{M}\right) \text{ where } M = \text{Sup } \| f(t,x) \|$$

2. Fixed point theorem:

2.1 Fixed point theorem in a metric space

Banach's theorem 2.1.1[10]: Assume that T is considered as a contraction on Banach's space then T admits a unique fixed point (FP).

Generalization of Banach's theorem 2.1.1 [10]: Assume that (T) is considered a map in Banach's space X , where T^n is contraction in X so, T admits a unique FP.

$$\| T(x_0) - x_0 \| = \| T^n(T(x_0)) - T^n(x_0) \| \leq c \| T(x_0) - x_0 \|$$

Implies ($T(x_0) = x_0$) as FP for $0 < c < 1$, uniqueness is clear as FP for T is also a FP for T^n.

Assume that F is closed subset in Banach's space, $T: F \to F$ contracting map, so:

- The following equation $Tx = x$ has only 1 unique solution.
- This unique solution can be gained by the limit of the sequence $(x_n)_n$ of F defined by: $x_n = Tx_{n-1}$: $n = 1, 2, 3, 4, \cdots etc$ and x_0 an arbitrary of F .

$$x = \lim_{n \to \infty} T^n(x_0)$$

Picard's theorem 2.1.1[11]: Assume that (X, d) is considered complete-metric-space, $T : X \to X$ is considered contracting map, so:

- There is unique FP (x) for (T).
- For all $x_0 \in X$, $\lim_{n \to \infty} T^n(x_0) = x$.

Proof:

Existence: Let $x_0 \in X$ and $(x_n)_n$ the associated sequence, we have:

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq k\, d(x_{n-1}, x_n)$$

We will show by recurrence on n that:

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

For n = 0:

$$d(x_0, x_1) \leq k^0 d(x_0, x_1)$$

Suppose the condition is true for n and show for n + 1:

$$d(x_{n+1}, x_{n+2}) = d(T(x_n), T(x_{n+1})) \leq Kd(x_n, x_{n+1})$$

$$d(x_{n+1}, x_{n+2}) \leq K(K^n d(x_0, x_1))$$

$$d(x_{n+1}, x_{n+2}) \leq K^{n+1} d(x_0, x_1)$$

We Assume that $(x_n)_n$ is Cauchy, $p,q \in N, \forall q > p$:

$$d(x_p, x_q) \le d(x_p, x_{p+1}) + d(x_{p+1}, x_q)$$

$$d(x_p, x_q) \le d(x_p, x_{p+1})d(x_{p+1}, x_{p+2}) + \cdots d(x_{q-1}, x_q)$$

$$d(x_p, x_q) \le K^p d(x_0, x_1) + k^{p+1} d(x_0, x_1) + \cdots k^{q-1} d(x_0, x_1)$$

$$d(x_p, x_q) \le d(x_0, x_1)[K^p + k^{p+1} + \cdots + k^{q-1}]$$

$$d(x_p, x_q) = d(x_0, x_1)\frac{K^p - k^q}{1-k} \le d(x_0, x_1)\frac{k^p}{1-k} < \varepsilon$$

This shows $(x_n)_n$ that is a Cauchy's sequence and if X is considered as complete space:

$$\lim_{n\to\infty}(x_n) = x$$

Existence of the FP: We consider $x = \lim_{n\to\infty}(x_n)$ is FP of T. We have:

$$x_{n+1} = Tx_n$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n \Rightarrow x = \lim_{n\to\infty} x_n$$

$x = T\lim_{n\to\infty} x_n$ because T is continuous

$x = Tx$ where x is considered as a FP of (T).

Uniqueness of the FP: Suppose there are x_1, x_2 where $x_1 \ne x_2$.

If x_1 is a FP, then $x_1 = Tx_1$

If x_2 is a FP, then $x_2 = Tx_2$

Cauchy Lipschitz 2.1.1 [12]: Let U be an open-set of $R \times R^n$, if $f: U \to R^n$ is continuous and locally Lipschitz with respect to x on U and assume that $h, r_0 > 0$ then for any security cylinder $C = [t_0 - h, t_0 + h] \times B_f(x_0, r_0)$ the Cauchy's problem:

$$x'(t) = f(t, x(t))$$
$$x(t_0) = x_0$$

Given $(t_0, x_0) \in U$, a solution $x: I \subset R \to R^n$, where $t_0 \in I$ admits a unique solution $x: [t_0 - h, t_0 + h] \to U$. Moreover, if we set:

$$\Phi(x)(t) = x_0 + \int_{t_0}^{t} f(u, x(u))du$$

There exists $p \in N$, where the sequence $\phi^p(x)$ uniformly converges to the exact solution.

Proof:

We start by constructing a security cylinder for C.

Assume that V be a neighborhood of (t_0, x_0) on which f is k-Lipschitz with respect to x, $h_0 > 0$ and $C_0 = [t_0 - h_0, t_0 + h_0] \times B(x_0, r_0) \subset V$ a cylinder . C_0 is a bounded closed set of R^{n+1}, hence compact, and deduces that f is bounded on C_0.

$$M = \mathrm{Sup} \parallel f(t, x(t)) \parallel$$
$$C = [t_0 - h, t_0 + h] \times B_f(x_0, r_0)$$
$$h = \min(h_0, \frac{r_0}{M})$$

Let $x : I = [t_0 - h_0, t_0 + h_0] \to R^n$, with $x(t_0) = x_0$ and $x' = f(t, x)$, $\forall t \in I$.

Suppose there is $\tau \in [t_0, t_0 + h]$ such that $x(\tau)$ does not belong to $B_f(x_0, r_0)$. Moreover, suppose that $J = \{t \in [t_0, t_0 + h] : x(t) \notin B_f(x_0, r_0)\}$ is non-empty.

We set $\tau = \inf J$. Then $\forall t \in [t_0, \tau]$ we have $x(t) \in B_f(x_0, r_0)$, and moreover:

$$d(x_0, x(\tau)) = r_0. \text{As } (t, x(t)) \in C_0, \forall t \in [t_0, \tau] \text{and } x = f(t, x):$$
$$r_0 = \parallel x_0 - x(\tau) \parallel = \parallel x(t_0) - x(\tau) \parallel \leq |t_0 - \tau| \,\mathrm{Sup}\, |x(t)| < Mh \leq r_0$$

In the following we work with this security cylinder. Note that by construction we have $\sup_C |f| = M$ and f is a k-Lipschitz with x on C. We denote $F = C^0([t_0 - h, t_0 + h], B(x_0, r_0))$ endowed with the distance $d = \| \,.\, \|_\infty$, $\forall x \in F$ we associate $\phi(x)$:

$$\Phi(x)(t) = x_0 + \int_{t_0}^{t} f(u, x(u)) du$$

We first show the following equivalence: x is considered a Cauchy's problem solution if and only if x is a FP of ϕ.

Suppose x is a FP of ϕ. Then $\forall x \in F$ we have $\phi(x) = x$, $x(t) = x_0 + \int_{t_0}^{t} f(u, x(u))\, du$.

But f is considered continuous on U so also x is considered continuous on U.
Moreover, x is differentiable in $[t_0 - h, t_0 + h]$ and its derivative equals $f(t, x(t))$, We also have:

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(u, x(u))\, du = x_0$$

So f is solution of the Cauchy problem.
We then have $x'(t) = f(t, x(t))$ and $x(t_0) = x_0$. We can integrate x_0 with respect to u because $x'(u) = f(u, x(u))$ and $u \to f(u, x(u))$ is continuous on a segment and therefore integrable on this same segment. So we get:

$$\int_{t_0}^{t} x'(u)\, du = \int_{t_0}^{t} f(u, x(u))\, du = [x(u)]_{u=t_0}^{u=t} = x(t) - x(t_0) = x(t) - x_0$$

So:

$$x(t) = x_0 + \int_{t_0}^{t} f\big(u, x(u)\big)du = \Phi(x)(t)$$

So x is a FP of ϕ.

We want to apply the FP theorem to ϕ^p.

- Assume that ϕ is considered as a map from F to F. For this we show that:
  - $\Phi(x)(t) \in B_f(x_0, r_0), \forall t \in [t_0 - h, t_0 + h]$
  - $\| \Phi(x)(t) - x_0 \| = \| \int_{t_0}^{t} f\big(u, x(u)\big)du \|$
  - $\| \Phi(x)(t) - x_0 \| \leq \| \int_{t_0}^{t} f\big(u, x(u)\big)du$
  - $\leq M \int_{t_0}^{t} du$
  - $\leq M(t - t_0)$
  - $\leq Mh \leq r_0$

Therefore $\forall\ t \in [t_0 - h, t_0 + h]$, $\phi(x)(t) \in B_f(x_0, r_0)$, $\phi(x) \in F$ and we obviously have the stability of F by ϕ^p.

- We now show that ϕ^p is contractive. Assume that $x, z \in F$. We denote by $x_p = \phi^p$ and $z_p = \phi^p(z)$, $\forall p \in N^*$. By recurrence on p we show that we have:
  - $\| x_p(t) - z_p(t) \| \leq K^p \frac{|t-t_0|^p}{p!} d(x,z)$

Initialization: This is obvious in the case $p = 0$.

Generalization: Suppose that for some arbitrary but fixed integer p we have:

$$\| x_{p+1}(t) - z_{p+1}(t) \| \leq \left| \int_{t_0}^{t} k \| x_p(u) - z_p(u) \| du \right|$$

$$\| x_{p+1}(t) - z_{p+1}(t) \| \leq \left| \int_{t_0}^{t} k \times k^p \frac{|u - t_0|^p}{p!} \times d(x,z) du \right|$$

$$\| x_{p+1}(t) - z_{p+1}(t) \| = \frac{k^{p+1}}{p!} d(x,z) \left| \left[\frac{|u - t_0|^{p+1}}{(p+1)!} \right] \right|$$

$$\| x_{p+1}(t) - z_{p+1}(t) \| = k^{p+1} \frac{|t - t_{0|}}{(p+1)!} d(x,z)$$

Which completes the recurrence.

As $| t - t_0 | \leq h$

$$d\big(x_p(t), z_p(t)\big) \leq k^p \frac{h^p}{p!} d(x,z)$$

So ϕ is Lipschitz of ratio $k^p \frac{h^p}{p!}$ and there is $p \in N^*$ where $k^p \frac{h^p}{p!} < 1$ because $\lim_{p\to\infty} k^p \frac{h^p}{p!} = 0$

So, for $q \geq p$, ϕ^q is contractive.

The theorem gives us the completeness of F.

We deduce from Picard theorem that ϕ^p admits a unique FP x so:

$$\phi^p(\phi(x)) = \phi(\phi^q x)) = \phi(x)$$

Therefore $\phi(x)$ is considered as a FP of ϕ^q, and by uniqueness of FP of ϕ be $\phi(x) = x$. As the FP of ϕ are FP of ϕ^q deduces that x is the unique FP of ϕ. Finally, x is considered a unique-solution for Cauchy problem.

2.2 Fixed point theorem in a topological space

Brouwer fixed point theorem 2.2.1 [13]: The topological space (X) has the property FP if each continuous map has a FP.

Brouwer's theorem is the fundamental FP theorem in finite dimension which affirms the following theorems:

- Any continuous mapping of the closed unit ball of R^n into itself admits a FP.
- There is no application $f : \overline{B^m} \to S^{m-1}$ continue as we have $f|_{S^{m-1}} = Id$.

where $\overline{B^m}$ is a closed-unit-ball and S^{m-1} is the boundary of this ball.

Assume that f is a retraction and $g(x) = -f(x)$, then $g \in C^0(\overline{B^m}, \overline{B^m})$ admits a FP x_0, which therefore satisfies $x_0 = -f(x_0)$, as f has values in the unit sphere, $x_0 \in S^{m-1}$, as f is a retraction, deduces that $f(x_0) = x_0$ and therefore that $x_0 = 0$ which contradicts $\| x_0 \| = 1$.

Theorem of continuous non-retraction of the ball and theorem of Brouwer are two equivalent results.

As a result, assume that C be a non-empty compact convex set of R^n, any continuous map from C to C admits at least 1 FP.

Schauder's theorem 2.2.1 [14]:

Schauder's theorem is defined on topological vector spaces of infinite dimension. Assume that (X) is considered as a Banach's space and $k \subset X$ a compact and non-empty convex-subset. Assume that $T: K \to K$ be a continuous operator, then T admits 1 FP at least.

Assume that (X) is considered a Banach's space, $D \subset X$ a non-empty set, convex, closed and bounded, also let $T: D \to D$ be a complete continuous operator, so T admits 1 FP at least.

Assume that (X) is considered as a Banach's space and C is considered as a closed convex set of X and T is continuous map from C to C where t $T(C)$ be relatively compact, so T is a FP.

The Leray-Schauder principle 2.2.3 [15]: Assume that X is considered as Banach's space and $k \subset X$ is convex and closed subset ($U \subset K$) an open-bounded-set in K, and $Q_0 \in U$ a fixed element, suppose that the operator $T : \overline{U} \to K$ is continuous, complete and satisfies the boundary condition: $Q \neq (1-\lambda)Q_0 + \lambda T(Q), \quad \textit{For all} \ Q \in \delta U, \lambda \in (0,1)$ then T admits 1 FP in $\overline{U}$ at least.

3. Fixed point applications to differential equations:

3.1 Application of Schauder's theorem

Assume that I is considered as an open-interval of (R, Ω), and $f : I \times \Omega \to R^n$ a continuous map, then if $t_0 \in I$ and $x_0 \in \Omega$ are given, the following problem:

$$x'(t) = f(t, x(t))$$
$$x(t_0) = x_0$$

admits 1 solution x at least of class C^1 defined on a certain interval in I of the form $[t_0 - h, t_0 + h]$, with $h > 0$.

Proof:

As I and Ω are open sets, there exists $C_0 = [t_0 - h, t_0 + h] \times \overline{B}(y_0, r_0)$ a cylinder included in $I \times \Omega$, C_0 is compact so f is bounded on C_0 by a constant M.

Assume that $h \le h_0$ and x be a solution of the problem defined at least on $I_0 \subset [t_0 - h, t_0 + h]$, suppose that it comes out of the cylinder:

$$C = [t_0 - h, t_0 + h] \times \overline{B}(x_0, r_0)$$

At time $\tau \in [t_0 - h, t_0 + h]$ then, by continuity:

$$r_0 = \| x(\tau) - x_0 \| = \left\| \int_{t_0}^{\tau} x'(u)\, du \right\| \le hM$$

So if $h \le \min(h_0, \frac{r_0}{M})$ then any solution defined on $I_0 \subset [t_0 - h, t_0 + h]$, stays in the ball $\overline{B}(x_0, r_0)$. We will call the safety cylinder the assembly

$$[t_0 - h, t_0 + h] \times \overline{B}(x_0, r_0)$$

Application of Schauder:

$$\mathrm{E} = \varepsilon([\mathrm{t}_0 - \mathrm{h}, \mathrm{t}_0 + \mathrm{h}], \mathrm{R}^{\mathrm{n}})$$
$$\mathrm{C} = \varepsilon\left([\mathrm{t}_0 - \mathrm{h}, \mathrm{t}_0 + \mathrm{h}], \overline{B}(x_0, r_0)\right)$$

Then (E) is R normed-vector-space, C is non-empty closed convex set, for $x \in C$, the function $\Phi(x)$ on $[t_0 - h, t_0 + h]$ is defined as follows:

$$\Phi x(t) = x_0 + \int_{t_0}^{t} f(u, x(u)) \quad du$$

By convergence, Φ is continuous then as $Mh \le r_0$, we have:

$$\Phi : C \to C$$

Assume that $(\Phi(C))$ is relatively compact, so by Schauder's theorem, we have the presence a FP for Φ in C.

$\Phi(C)$ is relatively compact, $[t_0 - h, t_0 + h]$ is compact and $\Phi(C)$ is bounded by r_0 in infinite norm.

Then if $x \in C$ and $t_1, t_2 \in [t_0 - h, t_0 + h]$ then:

$$\| \Phi x(t_1) - \Phi x(t_2) \| = \left\| \int_{t_1}^{t_2} f(u, x(u))\, du \right\| \le M|t_1 - t_2|$$

Deduce that the functions of $\Phi(C)$ are M-Lipschitz on $[t_0 - h, t_0 + h]$. So form an equicontinuous family.

So, Ascoli's theorem allows us to say that $\Phi(C)$ is relatively compact.

3.2 Banach contraction application

According Cauchy's problem there is $f: D \to E$ f, D is a domain of $R \times E$ with E a Banach , then the problem admits a single solution $x(t)$ defined on a certain interval $I \in [t_0 - h, t_0 + h]$, given that h is chosen:

$$h \leq \min\{r, r\delta, \frac{1}{\beta}\}$$

Assume that E be a Banach space, we define the operator:

$$Nx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \qquad : N : E \to E$$

We consider the following assumptions:

- f is considered continuous.
- there is $\delta > 0$ and $\beta > 0$ where:

$$\delta = \sup_{(t,x)\in D} |f(t, x)|$$

$$\beta = \sup_{(t,x)\in D} |f_x(t, x)|$$

- The function f is β-Lipschitz in x uniformly with respect to $t \in I$.

Using the mean value theorem there exists z(t) such that:

$$\mathrm{f}(\mathrm{t}, \mathrm{x}(\mathrm{t})) - \mathrm{f}(\mathrm{t}, \mathrm{y}(\mathrm{t})) = \mathrm{f}_{\mathrm{x}}(\mathrm{t}, \mathrm{z}(\mathrm{t}))[\mathrm{x}(\mathrm{t}) - \mathrm{y}(\mathrm{t})]$$

$$|\mathrm{f}(\mathrm{t}, \mathrm{x}(\mathrm{t})) - \mathrm{f}(\mathrm{t}, \mathrm{y}(\mathrm{t}))| \leq \beta \times |(x(t) - y(t))|$$

Proof:

Assume that the operator N satisfies the conditions of the Banach contraction theorem, the proof is given in two steps:

Step1: we show that $N(M) \subset M$, with $M = B(x_0, r), r > 0$. So, $\forall x \in M, \quad Nx \in M$.

Assume that $x \in M$, for $t \in I$ we have:

$$Nx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \Leftrightarrow Nx(t) = x_0 - \int_{t_0}^{t} f(s, x(s))\, ds$$

$$|Nx(t) - x_0| \leq Nx(t) = \int_{t_0}^{t} |f(s, x(s))|\, ds$$

$$\sup_{t\in I} |Nx(t) - x_0| \leq \int_{t_0}^{t} \sup_{t\in I} |f(s, x(s))|\, ds$$

$$\sup_{t\in I} |Nx(t) - x_0| \leq \delta |t - t_0|$$

$$\sup_{t\in I} |Nx(t) - x_0| \leq h\delta$$

$$\sup_{t\in I} |Nx(t) - x_0| \leq r$$

As a result:

$$d(Nx, x_0) \leq r \quad Where \quad Nx \in M$$

Step2: N is a contraction. Assume that $x, y \in M$ for all $t \in I$ we have:

$$Nx(t) - Ny(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds - x_0 - \int_{t}^{t} f(s, y(s))ds$$

$$Nx(t) - Ny(t) = \int_{t_0}^{t} f(s, x(s)) - f(s, y(s))ds$$

$$Nx(t) - Ny(t) \le \left| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s))ds \right|$$

$$Nx(t) - Ny(t) \le \beta h \sup_{t \in I} |x(t) - y(t)|$$

3.3 Application of Picard's theorem

Assume that E, F is considered as two Banach's spaces, $U \subset E$ is considered as an open-set, where $f : U \to F$ is an application of the class C^1.

$\alpha \in U$ such that df_α is continuous and invertible and therefore df_α^{-1} is continuous.

So, there is an open-neighborhood V of α, and an open-neighborhood W of $f(\alpha)$ where:

- The restriction $f|_V$ of f to V is a bijection from V to W.
- The inverse map $g : W \to V$ is continuous.
- g is of class C^1 and $\forall x \in W, dg_{f(x)} = df_x^{-1}$.

Proof:

We equip $L_c(E, F)$ of the norm $\| u \| = \sup_{\| x \| = 1} \| u(x) \|$.

Even if it means replacing f with the function $x \mapsto df_z^{-1}[f(\alpha + x) - f(\alpha)]$, we can come back just in case $\alpha = 0$, $f(\alpha) = 0$ and $df_0 = df_\alpha = Id_E$ (so $E = F$).

As f is of class C^1, there exists $r > 0$ where:

$B(0, r) \subset U$ and $\| df_z - df_0 \| = \| df_z - Id_E \| \le \frac{1}{2}$ *for all* $x \in B(0, r)$.

We denote $u = Id_E - df_x \Rightarrow df_x = Id_E - u : \| u \| < \frac{1}{2}$. Then, df_x is a bicontinuous satisfying $df_x^{-1} = \sum_{n=0}^{\infty} u^n$, and then:

$$\| df_x^{-1} \| \le \sum_{n=0}^{\infty} \| u \|^n \le \sum_{n=0}^{\infty} \frac{1}{2^n} = \lim_{n \to \infty} \frac{2^{n+1} - 1}{2^n} = 2$$

- The restriction of f to an open neighborhood of 0 in $B(0, r)$ is a bijection on $B(0, \frac{r}{2})$. Assume that $y \in B(0, \frac{r}{2})$. We consider the function h:

$$h : B_f(0, r) \to E$$
$$x \mapsto y + x - f(x)$$

It is clear that h is of class C^1, $\forall x \in B(0, r), \| dh_x \| = \| Id_E - df_x \| \le \frac{1}{2}$.

So, by the theorem of mean value:

$$\forall x, x' \in B_f(0,r), \| h(x) - h(x') \| = \frac{1}{2} \| x - x' \|$$

Especially for $x' = 0$:

$$\| x - f(x) \| = \| h(x) - h(0) \| \leq \frac{1}{2} \| x \|$$

$$\forall x \in B(0,r), \|h(x)\| \leq \|y\| + \|x - f(x)\| \leq \|y\| + \frac{1}{2}\|x\| \leq \frac{r}{2} + \frac{r}{2} = r$$

Thus h is a function from $B_f(0,r)$ to $B(0,r) \subset B(0,1)$.

Since $x = h(x)$ and h has values in $B(0,r)$, so we conclude that $x \in B(0,r)$.

So $y \in B_f(0, \frac{r}{2})$, $\exists! \, x \in B(0,r)$ *where* $f(x) = y$.

We define $V = f^{-1}(B(0,r)) \cap B(0,r)$. V is a neighborhood of 0 because $(f(0) = 0)$ and (f) is continuous in $B(0,r)$.

Through denoting $W = B(0,r)$, we then have $f_V : V \to W$ is a bisection.

- We note $g : W \to V$ the inverse map.

We use h again, this time with $y = 0$ so:

$\forall x \in U, x = h(x) + f(x)$

$$\forall x, x' \in B(0,r), \|x - x'\| \leq \|h(x) - h(x')\| + \|f(x) - f(x')\| \leq \frac{1}{2}\|x - x'\| + \|f(x) - f(x')\|$$

So:

$$\|x - x'\| \leq 2\|f(x) - f(x')\|$$

We deduce that:

$\forall y, y' \in W, \|g(y) - g(y')\| \leq 2\|f(g(x)) - f(g(x'))\| + 2\|y - y'\|$

g is therefore Lipschitzian and therefore continuous.

- We fix $x \in V$ and we set $y = f(x) \in W$.

There exists $r' > 0$ such that $B(y, r') \subset W$, and for all $w \in B(0,r)$, we set:

$$V = g(y+w) - g(y)$$

$$\|v\| \leq 2\|w\|$$

$$\Delta(w) = g(y + w) - g(y) - df_x^{-1}(w)$$

$$\Delta(w) = v - df_x^{-1}\left[f(x + w - f(x)\right]$$

$$\Delta(w) = -df_x^{-1}\left[f(x + v - f(x) - df_x(v)\right]$$

Since $\| df_x^{-1} \| \leq 2$ we get:

$$\|\Delta(w)\| \leq 2\|f(x + v) - f(x) - df_x(v)\| = 2v\|\varepsilon(v)\|$$

With $\lim_{v \to \alpha} \varepsilon(v) = 0$,so:

$$\|\Delta(w)\| \leq 4\|w\|\varepsilon(g(y + w) - g(y)) = 4\|w\|\varepsilon'(w)$$

As g is continuous. $\lim_{w \to 0} \varepsilon'(w) = 0$

So, $\| \Delta(w) \| = 0(\| w \|)$.

So g is differentiable at y and $dg_y = df_x^{-1}$.

Finally, as df_x^{-1} is continuous because f is considered of class C^1 and $L \in GL(E) \to L^{-1} \in GL(E)$ is continuous, the function $dg : y \mapsto dg_y$ is continuous. Thus g is of class C^1.

4. Conclusion

The FP theorem is fundamental in the field of applications to DEs. We have discussed some theorems (principle of Banach contraction is the basis of the FP theory which ensures the uniqueness of solutions and Schauder's theorem only affirms existence). At the end of this thesis, we cite applications to ordinary DEs.

In conclusion, this research has demonstrated the fruitful application of generalized fixed point theorems to ordinary differential equations, providing valuable insights into the study of existence, uniqueness, and stability of solutions in diverse settings. By embracing the flexibility and versatility offered by generalized fixed point theorems, we have expanded the scope of our understanding of ODEs and developed a powerful toolset to tackle complex and non-standard problems.

Through a thorough examination of the fundamental principles behind generalized fixed point theorems, we have established a robust theoretical framework that merges concepts from functional analysis and nonlinear dynamics. This framework has allowed us to address a wide range of ODEs, including both linear and nonlinear systems, autonomous and non-autonomous equations, and various initial and boundary value problems.

References

[1] E. Chávez, G. Navarro, R. Baeza-Yates, J.L. Marroquín, Searching in metric spaces, ACM computing surveys (CSUR) 33 (2001) 273-321.

[2] I.L. Reilly, P.V. Subrahmanyam, M.K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, Monatshefte für Mathematik 93 (1982) 127-140.

[3] R. Coleman, Calculus on normed vector spaces, Springer Science & Business Media2012.

[4] S. Varošanec, On h-convexity, Journal of Mathematical Analysis and Applications 326 (2007) 303-311.

[5] S.B. Nadler, Multi-valued contraction mappings, Pacific Journal of Mathematics 30 (1969) 475-488.

[6] S. Shalev-Shwartz, O. Shamir, N. Srebro, K. Sridharan, Learnability, stability and uniform convergence, The Journal of Machine Learning Research 11 (2010) 2635-2670.

[7] J.Á. López, A. Candel, Equicontinuous foliated spaces, Mathematische Zeitschrift 263 (2009) 725-774.

[8] J.C. Butcher, Numerical methods for ordinary differential equations, John Wiley & Sons2016.

[9] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, The stability for the cauchy problem for elliptic equations*, Inverse Problems 25 (2009) 123004.

[10] S. Shukla, S. Balasubramanian, M. Pavlović, A generalized banach fixed point theorem, Bulletin of the Malaysian Mathematical Sciences Society 39 (2016) 1529-1539.

[11] A. Levin, Generalizations of siegel's and picard's theorems, Annals of mathematics (2009) 609-655.

[12] B. Bonnet, H. Frankowska, Differential inclusions in wasserstein spaces: The cauchy-lipschitz framework, Journal of Differential Equations 271 (2021) 594-637.

[13] V. Brattka, S. Le Roux, J.S. Miller, A. Pauly, Connected choice and the brouwer fixed point theorem, Journal of Mathematical Logic 19 (2019) 1950004.

[14] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, A schauder fixed point theorem in semilinear spaces and applications, Fixed Point Theory and Applications 2013 (2013).

[15] M.V. Nikolaev, A.A. Nikitin, Application of the leray-schauder principle to the analysis of a nonlinear integral equation, Differential Equations 55 (2019) 1164-1173.