Fractional Differential Equations with Algebraic Structures for Concentration-Dependent Transport

Authors Names

Soran Noori Saleh[a,b], Asrin Farman Shekh[c], Srwa Mansoor Hasan[d], Kamaran Jamal Hamad[e] Hunar Jihangir Azeez [f]

Publication data: 30/ 6 /2024

Keywords: *Algebraic Structures - Fractional Differential Equations - Concentration-Dependent Transport - Transport Equations - Algebraic Modeling - Fractional Calculus*

ABSTRACT

In this paper, a new method is adopted which provides an easy and convenient way to separate partial differential equations involving fractional derivatives and delays. These fractional derivatives also refer to derivatives of an arbitrary real system. The method is based on the matrix approach developed by Podlubny, as presented in the article "Calculating Fractional Calculus and Applied Analysis" (Vol. 3, No. 4, 2000, 359-386).

In order to implement this method, a set of MATLAB procedures was developed, and sample code was provided to solve the examples discussed in the paper. In the end, we can say that this paper highlights the importance of algebraic structures in fractional differential equations, especially in the context of concentration-dependent transport, as the proposed method provides a clear and flexible way to characterize these equations and demonstrate their effectiveness through numerical examples.

1. Introduction and literature review:

After the development of science, kinetic equations involving fractional derivatives were recognized as a useful way to describe transport dynamics in complex systems. In most cases, these systems exhibit rapidly vanishing correlations and a time history that differs from conventional rules. A few examples are Gaussian or exponential laws. Systems with Hamiltonian chaos, turbulent media, complex system interactions, fluid and plasma turbulence, protein molecule dynamics, groundwater contamination, and motions under optical tweezers. [1-2-3]

Derived from random walk models, fractional kinematic equations have the advantage of being able to include external force terms and solve boundary value problems as well as being able to take into account transport in the phase space extended by position and velocity data. Despite all this, the development of numerical methods is now crucial as the complex integral-differential nature of these equations limits analytical solutions to relatively basic scenarios. The similar diffusion equation for particle density, given by u(x, t), in a one-dimensional space is a good example to illustrate how fractionation occurs in kinetic equations with integer partial derivatives.[4-5]

$$\frac{\partial u}{\partial t} = \chi \frac{\partial 2u}{\partial x2} \text{ , } (t > 0, a < x < b) \text{ ……. (1)}$$

Where the constant χ is the diffusion coefficient.

[a]*Sulaimani Polytechnic University, Chamchamal technical ,Iraq,* [b]*Ministry of Education, General Directorate of Education of Sulaimani, Directorate of Education Chamchamal,Sheikh Reza Talabani High School – Science,* [c]*ministry of higher education and scientific research, Erbil polytechnic university, soran technical college.* [d]*Computer Science, College of Sciences, Charmo University, Kurdistan Region, Iraq.* [e]*Department of Mathematic, Faculty of Science, Soran University, Kurdistan Regional Government, Iraq.* [f]*Ministry of Higher Education and Scientific Research Mathematics Department-college of Science Education-Salahaddin University -Kurdistan Region -Iraq*

By replacing a fractional derivative of order α smaller than 1 with a time derivative of first order, the fractional time diffusion equation is produced by the first type of partitioning.

$$ {}_0^C D\ {}_t^\alpha u = \chi \frac{\partial 2u}{\partial x2} , (t > 0, a < x < b) \ldots\ldots\ldots (2) $$

Here, ${}_0^C D\ {}_t^\alpha$ is the formula for the Caputo fractional derivative, which is [6]

$$ {}_a^C D_x^\mu \phi(x) = \frac{1}{\Gamma(m-\mu)} \int_a^x \frac{\phi^{(m)}(\xi) d\xi}{(x-\xi)^{\mu-m+1}}, \qquad (m-1 < \mu \leq m) \qquad (3) $$

When α = 1 is used in (2), the classical diffusion equation is obtained. (1).

There are two versions of the time fractional diffusion equation in the literature that replace the fractional Riemann-Lovell derivative with the Caputo derivative [2]. But it is more practical to apply it in physical situations since Caputo's derivative makes it possible to state the initial conditions in terms of the unknown function and its integer-order derivatives, and he also suggested that the initial conditions have a physical interpretation of the fractional Riemann-Liouville derivative. [7].

We will focus on the form of the equation using the Caputo derivative in this discussion of algebraic structures in fractional differential equations for concentration-dependent transport, as the ease of expressing nonzero initial conditions motivates this choice [8]. In the second form of segmentation, a fractional derivative of order β between 1 and 2 will likewise replace the second-order spatial derivative, so that this transformation leads to the spatial fractional diffusion equation [50].

$$ \frac{\partial u}{\partial t} = \chi \frac{\partial^\beta u}{\partial |x|^\beta}, \quad (t > 0, \ \ a < x < b) \qquad (4) $$

where (with respect to the spatial component) ∂β /∂|x|β is the partial symmetric Riesz derivative, which is half the sum of the left- and right-side Derivatives of Riemann-Liouville [9-10]:

$$ \frac{d^\beta \phi(x)}{d|x|^\beta} = D_R^\beta \phi(x) = \frac{1}{2} \Big({}_a D_x^\beta \phi(x) + {}_x D_b^\beta \phi(x) \Big), \qquad (5) $$

where the definitions of the Riemann-Liouville derivatives on the left and right sides are

$$ {}_a D_x^\mu \phi(x) = \frac{1}{\Gamma(m-\mu)} \left(\frac{d}{dx} \right)^m \int_a^x \frac{\phi(\xi) d\xi}{(x-\xi)^{\mu-m+1}}, \quad (m-1 < \mu \leq m), \quad (6) $$

$$ {}_x D_b^\mu \phi(x) = \frac{1}{\Gamma(m-\mu)} \left(-\frac{d}{dx} \right)^m \int_x^b \frac{\phi(\xi) d\xi}{(\xi-x)^{\mu-m+1}}, \quad (m-1 < \mu \leq m), \quad (7) $$

The classical diffusion equation (1) can be obtained by reducing equation (4) to $\beta = 2$. Different kinds of derivatives are used in other asymmetric fractional extensions, though. Several alternatives exist, such as employing the left-sided Riemann-Liouville derivative in place of the symmetric Riesz derivative [11,12] or an asymmetric derivative in conjunction with asymmetric operators. distinct for derivatives on the left and right sides [13].

There are various generalizations of space-time partial diffusion equations, such as multi-dimensional partial diffusion and kinetic equations [5-17], and space-time fractional generalizations. In addition to the presence of different formulas that include different systematic forces in the fractional Fokker-Planck equations of space and time [16- 17], fractional derivatives of variable and dispersed orders in equations, and transport coefficients variable, and others. There is a need for an efficient numerical solution method that is relatively simple and capable of dealing with different forms of fractional kinematic equations due to the expansion of the fractional kinematic field.

While a basic framework for the numerical solution of many existing numerical tools for ordinary rational equations has been established, the number of numerical methods available for solving rational equations with partial derivatives is relatively limited. Therefore, the currently active field of research is developing effective numerical schemes for such equations. Let's take a quick look at the various strategies employed in the literature [18].

The main distinction between numerical approaches is how they handle regular and fractional derivatives. For instance, the L2 differentiation approach is used in [19] to solve the diffusion reaction equation with the left Riemann-Liouville derivative between 1 and 2. The fractional integral that arises in the definition of the fractional Riemann-Liouville derivative is about the same for both the L2 technique and its variant, L2C. Within [20]. It has been demonstrated that the former is more accurate for orders greater than 1.5, whereas the latter is most accurate for orders smaller than 1.5. The semi-implicit scheme and the explicit forward Euler formula were applied to the first-order time derivative.

To describe the fractional time derivative of Riemann-Liouville The L1 scheme from [20] was employed by Langlands and Henry [21] with an order ranging from 1 to 2. Yusti [22] took into account the Grünewald-Letnikov approximation of the Riemann-Liouville time derivative and employed the weighted average of second-order space derivatives. A refinement of the Grünwald-Letnikov approximation was recently presented by Scherer et al. [23] expressly for the function's Caputo derivative. With this modification, the fractional diffusion equations were numerically solved with the use of non-zero beginning conditions and the Caputo time derivative.

Liu and associates [24] By employing a different approach to solve the fractional space Fokker-Planck equation with fixed coefficients on the fractional derivative term, they were able to transform the partial differential equation into a system of ordinary differential equations. This was accomplished by applying the method of lines.

A theoretical framework for the Galerkin finite element approach of the steady-state partial advection diffusion problem was developed by Irvin and Robb [25–26]. They

expanded their method to include fractional derivative terms in multidimensional partial differential equations with constant coefficients.

Falco and Abate [27] tackled the temporal fractional diffusion equation on a semi-infinite field by numerically inverting the two-dimensional Laplace transform. However, to solve the time fractional diffusion equation in a finite domain, Lin and Xu [28] suggested a method based on Legendre's spectral method in space and the finite difference scheme in time.

Liang and Chen [29] used a combination of symbolic computations and numerical inversion of the Laplace transform to solve the time partial propagation wave equation with a time derivative of order between 1 and 2. The derivative of the transformed Caputo time in hydrodynamic equations for heterogeneous porous media has been approximated by means of a modification of Yuan and Agrawal's [29] method, which entails converting the fractional derivative to an infinite integral over the auxiliary internal variables [30].

Monte Carlo methods can be used to solve fractional kinematic equations, including specifically stochastic walking-based methods [31]. Gorenflo, Mainardi, and associates created a number of random walk schemes that were applied to fractional diffusion equations based on the Grunwald-Letnikov approximation. These schemes have been applied to solve various types of partial diffusion equations, including asymmetric spatial partial diffusion equations in the Lévy-Filler formula, symmetric spatial partial diffusion equations, temporal partial diffusion equations with the Caputo derivative, and spatiotemporal fractionation. Diffusion equations [32].

Gorenflo and Abdel-Rahim proposed discrete approximations of temporal fractional diffusion processes with heterogeneous drift toward the origin by generalizing the Ehrenfest jar model. Liu and associates [33]. They modeled the Lévy-Feller diffusion and advection process with constant drift using random walk and finite difference methods. In order to solve one-dimensional space-based partial diffusion and advection equations with space-dependent coefficients, Meerschaert and coauthors employed a stochastic particle tracking technique [34].

To tackle the nonlinear evolution problem involving the fractional Laplace operator, a method based on the numerical solution of coupled stochastic differential equations driven by symmetric stable Lévy processes is also proposed [35]. From the above, we find that the above-mentioned works highlight the increasingly crucial role of numerical solutions of fractional differential equations in applications of non-integer order methods and models. These techniques can be specifically applied to study algebraic structures in fractional differential equations for concentration-dependent transport

We describe a novel approach to numerically solve partial differential equations in this study. Our method is based on the discrete rational operators' matrix form representation, which was first presented in [10]. Using trigonometric matrices, this method offers a coherent framework for n-fold integration as well as numerical differentiation of any order, including integer order.

By applying our approach to the numerical solution of differential equations, we are able to unify the solution of both partial differential equations and integer-order partial differential equations, leading to a significant simplification in the numerical solutions

of partial differential equations. Additionally, because of its generality, algebraic structures in fractional differential equations of concentration-dependent transport can be treated.

2. Data and Methods:

Unlike widely used numerical methods, which obtain the solution step by step by moving from the previous time layer to the next layer to differentiate the differentiation and integration operators in the arbitrary real system, our method is based on the triple bar matrix approach [10]. As a result, the entire time interval of interest is taken into consideration. As a whole it also allows us to create a network of discriminating nodes. This step results in a two-dimensional network of nodes in the case of one spatial dimension. An example of this distinction is shown in Figure 1. Unknown function values should be identified at the internal nodes (shaded area in Figure 1). Later on, the system of algebraic equations will be constructed using the known values at the borders.

The left side of the resulting system is formed by an algebraic equation system that is obtained at all internal nodes concurrently by approximating the equation.

The right side of the system is then finished by applying the beginning and boundary conditions. The temporal levels in Figure 1 are numbered from bottom to top, and within each temporal level, the highlight nodes are numbered from right to left. Although standard numbering works well, we have used this numbering convention for the sake of clarity in presenting our approach as well. An outline of the fundamental instruments required for this technique, including transformers, demodulators, Kronecker products, and triangle strip matrices, is given below. We then demonstrate how these methods can be applied to obtain an algebraic system of equations by approximating the partial derivatives of any real system and the problem itself.

2.1 Triangular strip matrices:

For the matrices with a specific structure, we used in our research called triangular bar matrices [10, 36], which were also mentioned in [37-38].

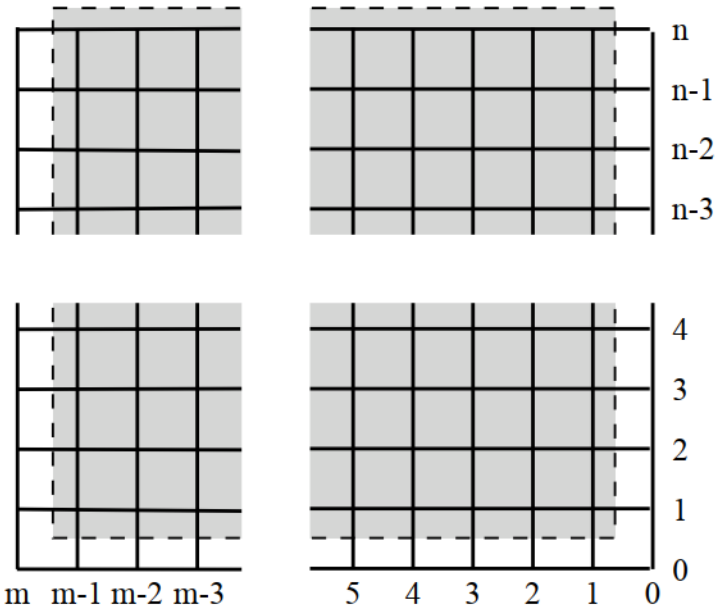

Fig. 1 Nodes and their numbering from bottom to top and from right to left

We will need lower triangular strip matrices,

$$L_N = \begin{bmatrix} \omega_0 & 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & \omega_0 & 0 & 0 & \cdots & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 & 0 \\ \omega_N & \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 \end{bmatrix}, \qquad (8)$$

and upper triangular strip matrices,

$$U_N = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \ddots & \omega_{N-1} & \omega_N \\ 0 & \omega_0 & \omega_1 & \ddots & \ddots & \omega_{N-1} \\ 0 & 0 & \omega_0 & \ddots & \omega_2 & \ddots \\ 0 & 0 & 0 & \ddots & \omega_1 & \omega_2 \\ \cdots & \cdots & \cdots & \cdots & \omega_0 & \omega_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_0 \end{bmatrix}, \qquad (9)$$

Through the first column (row) the lower (upper) triangular bar matrix can be completely represented. Hence, if we introduce the truncation operation, truncN(·), which truncates the energy series ϱ(z) in a general case

$$\varrho(z) = \sum_{k=0}^{\infty} \omega_k z^k \qquad (10)$$

to the polynomial ϱ *N*(*z*),

$$\operatorname{trunc}_N\left(\varrho(z)\right) \stackrel{\text{def}}{=} \sum_{k=0}^{N} \omega_k z^k = \varrho_N(z), \qquad (11)$$

A lower (or upper) triangular bar matrix can be represented using the generative series ϱ (z) In the context of fractional differential equations with algebraic structures for concentration-dependent transport, The generating series for the set of lower (or upper) triangular matrices LN (or UN), where N = 1, 2,..., is the function ϱ (z). As seen in [10], the generation chain of triangular bar matrices can be used to describe a variety of operations on them, including addition, subtraction, multiplication, and inverse (10). The commutativity of triangular bar matrices is a crucial characteristic. We can observe that matrices C and D move if they are both lower (or upper) triangular bar matrices

$$CD = D\,C \qquad (12)$$

2.2 Kronecker matrix product:

The n×m matrix A and the p×q matrix B's Kronecker product A⊗B

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{bmatrix}, \qquad (13)$$

is the np × mq matrix having the following block structure:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix}. \qquad (14)$$

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad (15)$$

Then

$$A \otimes B = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 0 & 0 & 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & -12 & -15 & -18 \end{bmatrix}. \qquad (16)$$

The Kronecker product has several properties that are important in the context of fractional differential equations with algebraic structures for concentration-dependent transport that are relevant for later sections. These characteristics, outlined in [39], include the following:

1. The Kronecker product A ⊗ B is also a range matrix if A and B are range matrices.

2. The Kronecker product of A ⊗ B is a lower (or upper) triangular matrix if A and B are lower (or upper) triangular matrices.
3. En ⊗ A and A ⊗ Em are the two particular Kronecker products that need to be employed; En is a n × n identity matrix.

For example, if A is a 2 x 3 matrix: En ⊗ A and A ⊗ Em will be used in the relevant calculations and formulas**.**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \tag{17}$$

Then

$$E_2 \otimes A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \end{bmatrix} \tag{18}$$

$$A \otimes E_3 = \begin{bmatrix} a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} \end{bmatrix} \tag{19}$$

This demonstrates that the left multiplication of An×m by En results in a n × n diagonal matrix when the matrix A is iterated on the diagonal, while the right multiplication of An×m by Em results in a sparse matrix composed of n × m diagonal blocks.

2.3 Eliminators:

The suggested approach makes use of aliquots, a particular class of matrices [10]. These matrices are obtained by picking only specific rows and eliminating all others from the N × N unit matrix E. For instance, by eliminating the first row of E, S1 is obtained; similarly, by eliminating the second row of E, S2 is acquired; and finally, by eliminating both the first and second rows of E, S1,2 is achieved. In other words, rk is found by eliminating rows that correspond to r1, r2,..., rk.

In the situation of infinite matrices, similar matrices have been investigated [40]. Only rows of A whose numbers differ from those of rows1,2,..., rk are present in the product Sr1, r2,..., rkA. The product ASrT1, r2,..., rk only comprises columns A whose numbers differ from r1, r2,..., rk if A is a N × N square matrix. The following straightforward example serves as an illustration of this removers' property: This demonstrates removers' primary feature.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \qquad S_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad S_1 A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

$$A S_1^T = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}; \qquad S_1 A S_1^T = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

2.4 Shifters:

For some types of differential operator approximations (e.g., one of the approximations to the symmetric Riesz derivative below in this article), and especially for the numerical solution of differential equations of arbitrary order, it is convenient to introduce another special type of matrix {shifters}, which will represent discrete transformations, such as delays. (fractional or integer) with a delay, for example.

In [10], transformers—though this name is not used—were utilized to create basic triangular bar matrices. We differentiate between two categories of variables:

(N + 1) × (N + 1) matrices $E_{N,p}^{+}$, p = 1, …..N

1. With zeros everywhere else and numbers on the p-th diagonal above the main diagonal, the matrix $E_{N,p}^{-}$, p = 1, …..N.

2. with zeros everywhere else and numbers on the p-th line beneath the main diagonal. Additionally, we indicate $E_{N,0}^{\pm}$, ≡ EN as the unit matrix.

If we start with UN+1 and then use transformers and demodulators, we can write the transformation of all the coefficients in the triangular bar matrix UN in the southwest direction (bottom left) easily:

$$-1UN = S1\ E_N^{-}+1;1\ UN+1\ E_N^{-}+1;1\ S_N^{\mathrm{T}}+1 \qquad (20)$$

Similarly, it is simple to determine how all of the coefficients in the triangular strip matrix UN have shifted in the north-east (top-right) direction:

$$+1UN = S_{N+1}\ E_N^{+}+1;1\ UN+1\ E_N^{+}+1;1\ S_1^{\mathrm{T}} \qquad (21)$$

3. Discretization of ordinary fractional derivatives:

According to [10], at all nodes of the equidistant discriminant network it is possible to approximate the fractional Riemann-Liouville or Caputo derivative on the left side

$$\mathrm{v}\ (\alpha)\ (\mathrm{t}) = 0\ D_{\mathrm{t}}^{\alpha}\ \mathrm{v}\ (\mathrm{t}). \quad \mathrm{t} = \mathrm{j}\tau\ (\mathrm{j} = 0, 1, .. ., \mathrm{n})$$

simultaneously using the upper triple band matrix B_{n}^{α}as 1:

$$\left[v_n^{(\alpha)}\ \ v_{n-1}^{(\alpha)}\ \ \dots\ v_1^{(\alpha)}\ \ v_0^{(\alpha)}\right]^T = B_n^{(\alpha)} \left[v_n\ \ v_{n-1}\ \ \dots\ v_1\ \ v_0\right]^T \qquad (22)$$

Where

$$B_n^{(\alpha)} = \frac{1}{\tau^\alpha}\begin{bmatrix} \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots & \omega_{n-1}^{(\alpha)} & \omega_n^{(\alpha)} \\ 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots & \omega_{n-1}^{(\alpha)} \\ 0 & 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} \\ 0 & 0 & \cdots & 0 & 0 & \omega_0^{(\alpha)} \end{bmatrix} \qquad (23)$$

$$\boldsymbol{\omega_j^{(\alpha)}} = (-1)^j \binom{\alpha}{j}, \quad j = 0, 1, \ldots, n. \qquad (24)$$

The Riemann-Liouville or Caputo fractional derivative can also be approximated on the right-hand side

$$v(\alpha)(t) = t\, \boldsymbol{D}_{\boldsymbol{b}}^{\alpha}\, v(t)$$

simultaneously at all nodes of the equidistant net differentiation $t = j\,\tau$ ($j = 0, 1, \ldots, n$) using the lower triangular strip matrix $\boldsymbol{F_n^{(\alpha)}}$:

$$\begin{bmatrix} v_n^{(\alpha)} & v_{n-1}^{(\alpha)} & \ldots & v_1^{(\alpha)} & v_0^{(\alpha)} \end{bmatrix}^T = F_n^{(\alpha)} \begin{bmatrix} v_n & v_{n-1} & \ldots & v_1 & v_0 \end{bmatrix}^T \qquad (25)$$

$$F_n^{(\alpha)} = \frac{1}{\tau^\alpha}\begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{n-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_n^{(\alpha)} & \omega_{n-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix} \qquad (26)$$

By applying its definition (5), one can approximate the symmetric Riesz derivative of order β as a combination of approximations (22) and (25) of the left- and right-side Riemann-Liouville derivatives. Another approach is to use the centered fractional differences approximation recently proposed by Ortigueira [41-42]. The general formula remains the same:

$$\begin{bmatrix} v_m^{(\beta)} & v_{m-1}^{(\beta)} & \ldots & v_1^{(\beta)} & v_0^{(\beta)} \end{bmatrix}^T = R_m^{(\beta)} \begin{bmatrix} v_m & v_{m-1} & \ldots & v_1 & v_0 \end{bmatrix}^T \qquad (27)$$

1. The Kabuto derivative is approximated from the left side by taking one step forward, while by taking one step backward the Kabuto derivative is approximated from the right side. This leads to the formation of a matrix.

$$R_m^{(\beta)} = \frac{h^{-\alpha}}{2}\left[{}_{-1}U_m + {}_{+1}U_m \right] \qquad (28)$$

2. The following is present in the second scenario (Ortigueira's definition [41]). symmetric matrix:

$$R_m^{(\beta)} = h^{-\beta} \begin{bmatrix} \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_m^{(\beta)} \\ \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \cdots & \omega_{m-1}^{(\beta)} \\ \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \cdots & \omega_{m-2}^{(\beta)} \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{m-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} \\ \omega_m^{(\beta)} & \omega_{m-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} \end{bmatrix} \tag{29}$$

$$\omega_k^{(\beta)} = \frac{(-1)^k \Gamma(\beta+1)\cos(\beta\pi/2)}{\Gamma(\beta/2-k+1)\,\Gamma(\beta/2+k+1)}, \qquad k = 0, 1, \ldots, m \tag{30}$$

Both of these approximations of the symmetric Riesz derivatives yield similar numerical results. These approximations also lead to a well-formed matrix of the algebraic system resulting when solving fractional differential equations numerically.

4. Partial derivatives discretization in space and time:

In Figure 2 we find the simplest scheme for the implicit estimation of the classical diffusion equation. Two nodes are located in the temporal direction for the approximation of the time derivative, and three nodes are located in the spatial direction for the symmetric approximation of the spatial derivative. This stencil covers two-time layers specifically. But when considering a fractional order time derivative, all-time levels must be involved from the beginning, with five-time layers as shown in Figure 3. In addition, when the fractional-order temporal derivative with symmetrical fractional-order spatial derivatives is taken into account all nodes in the current temporal layer must be used, from the leftmost to the rightmost spatial discrimination node as shown in Figure 4.

The nodes (ih; jτ), where j is a real number between 0 and n, are the representation of all time layers at the i-th spatial discretization node. It has been shown in [10] that all values of the α-th order time derivative of u(x; t) at these nodes may be approximated using the discrete analogue of differentiation of arbitrary order.

$$\left[u_{i,n}^{(\alpha)}\ u_{i,n-1}^{(\alpha)}\ \ldots\ u_{i,2}^{(\alpha)}\ u_{i,1}^{(\alpha)}\ u_{i,0}^{(\alpha)}\right] = B_n^{(\alpha)} \left[u_{i,n}\ u_{i,n-1}\ \ldots\ u_{i,2}\ u_{i,1}\ u_{i,0}\right]^T \tag{31}$$

To get a simultaneous approximation of the α-th order time derivative of u(x, t) at all nodes in Figure 1, we need to organize all function values uij at the discriminant nodes in the form of a column vector. That is, a column vector U must be generated, where each element of U corresponds to a certain node (ij) and contains a value uij. By arranging the function values into this column vector, we can effectively represent the entire system of equations for the discrete problem. This column vector allows us to express the simultaneous approximation of the order time derivative α of u(x, t) at all nodes, which is necessary for solving fractional differential equations with algebraic structures for concentration-dependent transport

$$u_{nm} = \Big[u_{m,n}\ u_{m-1,n}\ \cdots\ u_{1,n}\ u_{0,n} \\ u_{m,n-1}\ u_{m-1,n-1}\ \cdots\ u_{1,n-1}\ u_{0,n-1} \\ \cdots\cdots\cdots \\ u_{m,1}\ u_{m-1,1}\ \cdots\ u_{1,1}\ u_{0,1} \\ u_{m,0}\ u_{m-1,0}\ \cdots\ u_{1,0}\ u_{0,0} \Big]^T \qquad (32)$$

Visually, Figure 1 represents the discriminative nodes of the problem. To arrange the nodes in the desired order, we stack these nodes vertically to create a column vector, starting with n time layer nodes, followed by (n-1) time layer nodes, and so on. The matrix that converts the vector Unm into the vector Ut(α), which denotes the partial partial derivative of order α with respect to the time variable, can be obtained using the Kronecker product. This involves taking the matrix $\boldsymbol{B_n^\alpha}$, which is equivalent to the unit matrix Em and the fractional normal derivative of order α. The Kronecker product of these two matrices gives us the required transformation matrix.

(where n denotes the number of time steps)

(where m denotes the number of spatial discrimination nodes)

This matrix is considered to have an important role in solving fractional differential equations with algebraic structures for concentration-dependent transport.

$$T_{mn}^{(\alpha)} = B_n^{(\alpha)} \otimes E_m \qquad (33)$$

This concept is illustrated in Figure 5, where the fractional order time derivative at the gray node is approximated using the white and gray nodes. The matrix that converts the vector U into the vector Ux(β), which denotes the partial partial derivative of the order β with respect to the spatial variable, can alternatively be obtained by using the Kronecker product. Taking the unit matrix En and the matrix Rm(β) is necessary for this. The symmetric normal Riesz derivative of order β is represented by the matrix Rm(β), as described in references [41-42].

We obtain the required transformation matrix by performing the Kronecker product between the unit matrix En and the matrix Rm(β). This matrix is crucial in solving fractional differential equations with algebraic structures for concentration-dependent transport.

$$S_{mn}^{(\alpha)} = E_n \otimes R_n^{(\beta)} \qquad (34)$$

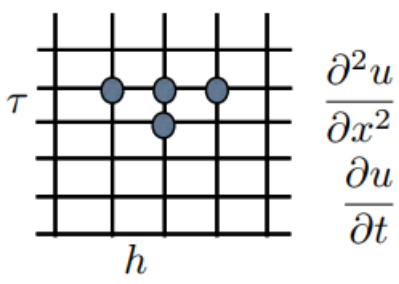

Fig. 2 An integer-order derivative stencil

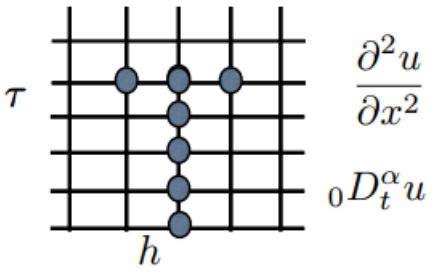

Fig. 3 A template when dealing with fractional time derivatives

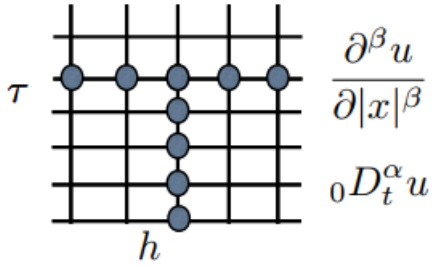

Fig. 3 In the event of fractional temporal and spatial derivatives, a stencil

This is further illustrated in Figure 5, where the black and gray nodes (which represent all discriminative nodes from leftmost to rightmost) in the same gray node are utilized to approximate the symmetric fractional order Riesz derivative. Once we obtain these approximations of the partial fractional derivatives with respect to both variables, we can proceed to discretize the general form of the fractional diffusion equation by substituting the derivatives with their discrete counterparts, as shown in the figure

$$ {}_0^C D_t^\alpha u - \chi \frac{\partial^\beta u}{\partial |x|^\beta} = f(x,t) \qquad (35)$$

is discretized as

$$ \left\{ B_n^{(\alpha)} \otimes E_m - \chi\, E_n \otimes R_m^{(\beta)} \right\} u_{nm} = f_{nm}, \qquad (36)$$

5. Utilization inside MATLAB:

To facilitate the implementation of the proposed method, we used a set of MATLAB procedures [43].

Using fractional difference methods, the BCRECUR function returns the coefficients needed to approximate fractional derivatives.

The matrix approximating the back difference of the normal fractional derivative from the left side is returned by the BAN function.

The matrix approximating the normal fractional derivative is returned by the FAN function from the right side.

Using formulas (28) and (29), respectively, the RANSYM function and the RANORT function return matrices to the symmetric Riesz approximation.

The ELEMINATOR function returns the eliminator array.

Finally, the SHIFT function performs operations (20) and (21).

These MATLAB procedures are designed to facilitate the implementation and calculation of fractional differential equations with algebraic structures for concentration-dependent transport.

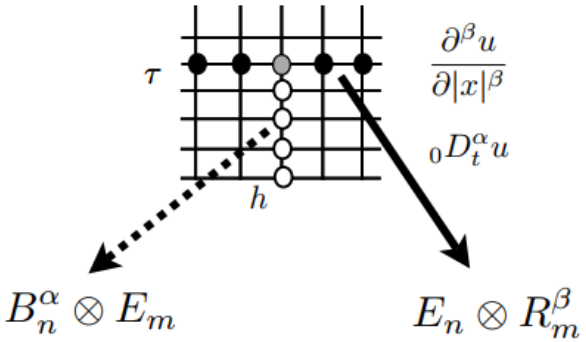

Fig. 5 Partial derivatives are dematerialized

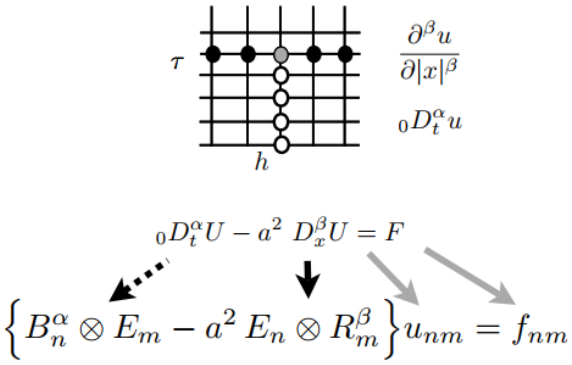

Fig. 6 Partial derivatives and the equation's discretization

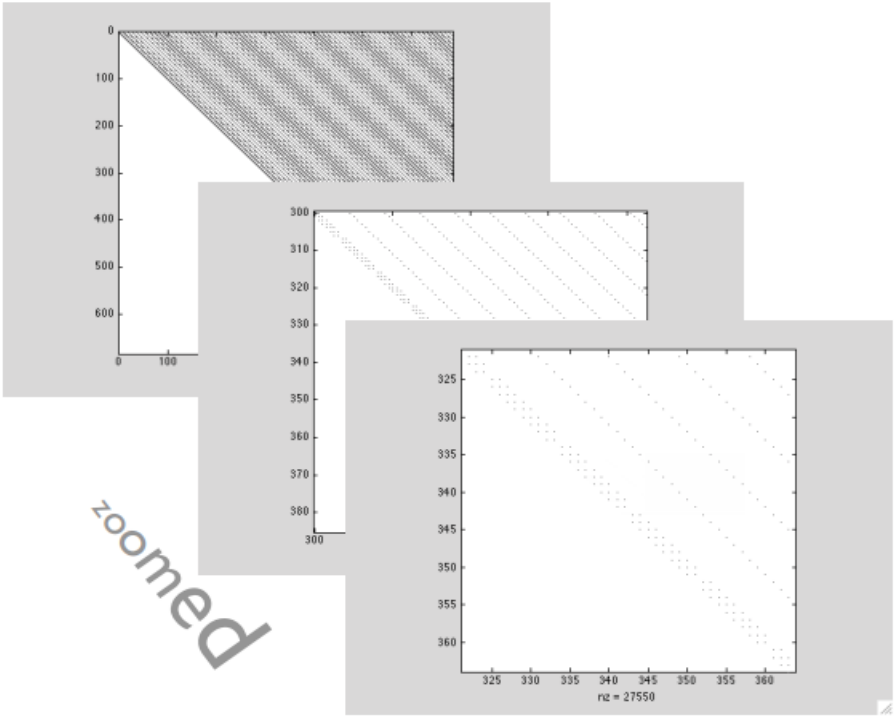

Fig. 7 The resulting algebraic system's matrix structure

Conclusion:

In this research paper, a novel method employing algebraic structures for solving partial differential equations involving fractional derivatives and delays in concentration-dependent transport dynamics has been presented. This method, based on the matrix approach by Podlubny, provides a clear and flexible way to handle complex systems where fractional derivatives play a crucial role.

The development of algebraic structures in fractional differential equations is essential for accurately modeling systems exhibiting anomalous transport phenomena, such as chaotic systems, turbulent media, protein molecule dynamics, and groundwater contamination. The adoption of fractional derivatives allows for a more thorough

description of these systems' dynamics compared to traditional integer-order approaches.

The numerical implementation of the proposed method has been facilitated through a set of MATLAB procedures, enabling researchers to efficiently solve fractional differential equations and demonstrate the effectiveness of the approach through practical examples. By discretizing fractional derivatives in both time and space domains and utilizing algebraic tools like triangular strip matrices, Kronecker products, and shifters, the method provides a systematic and rigorous framework for solving complex concentration-dependent transport problems.

This research underscores the importance of leveraging algebraic structures in fractional calculus to advance our understanding and modeling capabilities in various scientific and engineering fields. By providing a comprehensive methodology to tackle concentration-dependent transport dynamics, this work contributes to the growing body of knowledge in fractional differential equations and their applications.

In conclusion, the proposed method offers a promising avenue for researchers and practitioners to address challenging problems in concentration-dependent transport systems with fractional derivatives, paving the way for a deeper exploration of complex transport phenomena with improved numerical techniques and analytical tools.

References:

1. A. Chechkin, V. Y. Gonchar, J. Klafter, R. Metzler, Fundamentals of L´evy flight processes., Advances in Chemical Physics 133 (2006) 439- 496.
2. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Reports. 339 (1) (2000) 1-77.
3. I. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, Physics Today 55 (2002) 48-54.
4. G. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, Phys. Rep. 371 (1) (2002) 461-580.
5. R. Klages, G. Radons, I. M. Sokolov (eds.), Anomalous Transport, Wiley-VCH, Berlin, 2008.
6. M. Caputo, Elasticit`a e dissipazione, Zanichelli, Bologna, 1969.
7. I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, Fractional Calculus and Applied Analysis 5 (4) (2002) 367-386.
8. N. Heymans, I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheologica Acta 45 (5) (2006) 765-771.
9. I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
10. I. Podlubny, Matrix approach to discrete fractional calculus, Fractional Calculus and Applied Analysis 3 (4) (2000) 359-386.
11. D. del Castillo-Negrete, B. A. Carreras, V. E. Lynch, Front dynamics in reaction-diffusion systems with L´evy flights: A fractional diffusion approach, Phys. Rev. Lett. 91 (1) (2003) Article 018302.

12. M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion equations, Journal of Computational and Applied Mathematics 172 (1) (2004) 65-77.
13. M. M. Meerschaert, H.-P. Scheffler, C. Tadjeran, Finite difference methods for two-dimensional fractional dispersion equation, Journal of Computational Physics 211 (1) (2006) 249-261.
14. M. M. Meerschaert, H.-P. Scheffler, C. Tadjeran, Finite difference methods for two-dimensional fractional dispersion equation, Journal of Computational Physics 211 (1) (2006) 249-261.
15. R. Friedrich, Statistics of Lagrangian velocities in turbulent flows, Physical Review Letters 90 (8) (2003) Article 084501.
16. R. Metzler, E. Barkai, J. Klafter, Deriving fractional Fokker-Planck equations from a generalized master equation, Europhysics Letters 46 (4) (1999) 431-436.
17. I. M. Sokolov, J. Klafter, Field-induced dispersion in subdiffusion, Physical Review Letters 97 (2006) Article 140602.
18. I. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, Physics Today 55 (2002) 48-54.
19. V. E. Lynch, B. A. Carreras, D. del Castillo-Negrete, K. Ferreira-Mejias, Numerical methods for the solution of partial differential equations of fractional order, Journal of Computational Physics 192 (2003) 406-421.
20. K. B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
21. T. Langlands, B. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, Journal of Computational Physics 205 (2) (2005) 719-736.
22. S. Yuste, Weighted average finite difference methods for fractional diffusion equations, Journal of Computational Physics 216 (1) (2006) 264-274.
23. R. Scherer, S. L. Kalla, L. Boyadjiev, B. Al-Saqabi, Numerical treatment of fractional heat equations, Applied Numerical Mathematics 58 (8) (2008) 1212-1223.
24. Q. Liu, F. Liu, I. Turner, V. Anh, Approximation of the Levy- Feller advection-dispersion process by random walk and finite difference method, Journal of Computational Physics 222 (1) (2007) 57-70.
25. V. J. Ervin, J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numerical Methods for Partial Differential Equations 22 (3) (2005) 558-576.
26. V. J. Ervin, J. P. Roop, Variational solution of fractional advection dispersion equations on bounded domains in *rd*, Numerical Methods for Partial Differential Equations 23 (2) (2006) 256-281.
27. P. P. Valko, J. Abate, Numerical inversion of 2-d Laplace transforms applied to fractional diffusion equation, Applied Numerical Mathematics 53 (2005) 73-88.
28. Y. Lin, C. Xu, Finite difference/spectral approximations for the time fractional diffusion equation, Journal of Computational Physics 225 (2) (2007) 1533-1552.
29. L. Yuan, O. P. Agrawal, A numerical scheme for dynamic systems containing fractional derivatives, Journal of Vibration and Acoustics 124 (2) (2002) 321-324.

30. J.-F. Lu, A. Hanyga, Wave field simulation for heterogeneous porous media with singular memory drag force, Journal of Computational Physics 208 (2) (2005) 651-674.
31. R. Gorenflo, G. De Fabritiis, F. Mainardi, Discrete random walk models for symmetric L´evy{Feller diffusion processes, Physica A 269 (1999) 79-89.
32. R. Gorenflo, F. Mainardi, D. Moretti, P. Paradisi, Time fractional diffusion: A discrete random walk approach., Nonlinear Dynamics 29 (2002) 129-143.
33. Q. Liu, F. Liu, I. Turner, V. Anh, Approximation of the Levy-Feller advection-dispersion process by random walk and finite difference method, Journal of Computational Physics 222 (1) (2007) 57-70.
34. Z. Yong, D. A. Benson, M. M. Meerschaert, H.-P. Scheffler, on using random walks to solve the space-fractional advection-dispersion equations, Journal of Statistical Physics 123 No.1 (2006) 89-110.
35. D. Stanescu, D. Kim, W. A. Woyczynski, Numerical study of interacting particles approximation for integro-differential equations, Journal of Computational Physics 206 (2) (2005) 706-726.
36. D. A. Suprunenko, R. I. Tyshkevich, Commutative Matrices, Nauka I Tekhnika, Minsk, 1966.
37. B. V. Bulgakov, Kolebaniya (Vibrations), Gostekhizdat, Moscow, 1954.
38. F. R. Gantmakher, Theory of Matrices, Nauka, Moscow, 1988.
39. C. F. van Loan, The ubiquitous Kronecker product, Journal of Computational and Applied Mathematics 123 (2000) 85-100.
40. R. G. Cooke, Infinite Matrices and Sequence Spaces, Fizmatgiz, Moscow, 1960.
41. M. D. Ortigueira, Riesz potential operators and inverses via fractional centred derivatives, International Journal of Mathematics and Mathematical Sciences Article ID 48391 (2006) 1-12.
42. M. D. Ortigueira, A. G. Batista, On the relation between the fractional Brownian motion and the fractional derivatives, Physics Letters A 372 (2008) 958-968.
43. I. Podlubny, A. Chechkin, T. Skovranek, Y. Chen, B. Vinagre Jara, Matrix approach to discretization of ODEs and PDEs of arbitrary real order, http://www.mathworks.com/matlabcentral/fileexchange/22071 (November 12, 2008).