Eng. & Technology, Vol. 25, No.2, 2007

The Boundary Element Method Versus The Finite Element Method For Solving Two-Dimensional Continuum Problems

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Received on: 28 /4 / 2004
Accepted on: 2/ 12 / 2004

Abstract

Stress analysis problems in geomechanics are ideally suited to the method of boundary elements, as this technique usually requires a very small number of nodes by comparison to finite elements. As only the surface of the continuum needs to be discretized, problems extending to infinity can be described by a very small number of elements on the soil surface or around a tunnel or excavation . In addition, the boundary conditions of the infinite domain can be properly defined using boundary elements, as the technique is based on fundamental solutions valid for unbounded domains.

Herein, a comparison is made between the finite element method and the boundary element method in solving two-dimensional stress analysis problems. It is concluded that the results of the boundary element method are greatly improved when increasing the number of elements, especially at the regions of stress concentration. A good agreement can be obtained between the results of the two methods. One must keep in mind that in the boundary element method, errors due to discretization are restricted to the boundaries compared to the finite element method where the entire domain needs to be discretized. This advantage makes the use of the boundary element method easier and faster.

Keywords: Boundary element method, Finite element method, Two-dimensional stresses and strains.

طريقة العناصر الحدودية مقابل طريقة العناصر المحددة في حل مسائل الأوساط المستمرة ثنائية الأبعاد

الخلاصة

إن مسائل تحليل الاجهادات في ميكانيك التربة تعتبر ملائمة لطريقة العناصر الحدودية من حيث أن هذه التقنية تتطلب عادة عددا صغيرا جدا من العقد مقارنة مع العناصر المحددة. و بسبب كون تمثيل سطح المجال هو المطلوب لوحده، فان المسائل الممتدة إلى المـا لانهاية يمكن وصفها بعدد محدود جدا من العناصر على سطح التربة أو حول الأنفاق أو الحفريات . بالإضافة إلى ذلك يتم تعريف الشروط الحدودية للمجالات اللامنتهية بصورة صحيحة باستعمال العناصر الحدودية بسبب كون هذه التقنية مبنية على الحلول الأساسية الملائمة للمجالات غير المحدودة. في هذا البحث أجريت مقارنة بين طريقة العناصر المحددة و طريقة العناصر الحدودية في حل مسائل تحليل الاجهادات ثنائية الأبعاد. تم التوصل إلى أن نتائج طريقة العناصر الحدودية تتحسن بشكل كبير عند زيادة عدد العناصر و خاصة في مناطق تمركز الاجهادات. و يمكن الحصول على توافق جيد بين نتائج الطريقتين. كماأن الأخطاء الناجمة عن التقسيم في طريقة العناصر الحدودية تكون محدودة مقارنة مع طريقة العناصر المحددة حيث تحتاج إلى تقسيم المجال كله. هذه الفائدة تجعل طريقة العناصر الحدودية أسهل و أسرع لكثير من المسائل.

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https://doi.org/10.30684/etj.25.2.2
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1. Introduction:

The most popular techniques for the numerical analysis of engineering problems are the finite difference method (FDM), the finite element method (FEM) and the boundary element method (BEM). A brief description of the basic concepts of the last two methods will be discussed, next (Gudehus, 1977).

2. The Finite Element Method

A governing integral equation for a boundary value problem may be obtained in terms of a variational principle, and the solution is defined as the one which extremizes the integral expression, or it may be formulated from a weighted - residual principle (Zienkiewicz, 1977). The FEM is based upon the solution of such domain integral equations by means of piecewise discretization. The problem domain is divided into a number of smaller and simpler subdomains, known as finite elements, for which it is easier to apply the relevant variational principle so as to obtain elemental equations in terms of unknown values at specified nodes in each element. The equations of the elements are then assembled together, and the matrix equations involving the nodal values within the whole domain is obtained, and it can be solved in terms of the given boundary conditions. The FEM is considered the most popular numerical technique ever used for engineering analysis, and it has a very wide range of applications for different aspects of science and technology, (EL-Zafrany, 1992).

3. The Boundary Element Method

There are many engineering problems for which it is possible to represent the governing equations by a system of boundary integral equations (BIEs); that is, the integrated unknown parameters, in such equations, appear only in integrals over the boundary of the problem domain. There are many numerical approaches for the solution of such equations, and each approach gives the solution of such equations, and each one of them may be called a boundary integral equation method (BIEM).

3.1 Characteristics of the Boundary Element Method

The boundary element method is a most popular numerical technique for the direct solution of BIEM. It is based upon piecewise discretization of the problem boundary in terms of sub–boundaries, known as boundary elements, in a way similar to that employed for the finite element method. The main advantages of the BEM compared with domain numerical techniques can be summarized in the following statements (EL-Zafrany, 1992):

1. For many applications, the dimensionality of the problem is reduced by one, resulting in a considerable reduction in the data

and computer central processing unit (CPU) time required for the analysis.
2. The BEM is ideal for problems with infinite domains, such as problems of soil mechanics, fluid mechanics and acoustics.
3. No interpolation errors exist inside the domain.
4. Boundaries at infinity can be modeled conveniently without truncating the outer at some arbitrary distance from the region of interest.
5. Surface problems, such as those of elastic fracture mechanics, or elastic contact, is dealt with more efficiently and economically with the BEM.
6. Valuable representation can be obtained for stress concentration problems.
7. The BEM offers a fully continuous solution inside the domain, and the problem parameters can be evaluated directly at any point.

The boundary element method has also disadvantages and they can be outlined as follows:

1. The derivation of the governing BIEs may require a level of mathematics higher than that with other methods, but the procedure of the BEM itself is not different from that of the FEM.
2. It leads to fully populated matrices for the equations to be solved, thus it is not possible to employ the elegant FEM solvers such as the banded or frontal solvers with the BEM.
3. The BIEs of nonlinear problems may have domain integrals which require the use of domain elements for their evaluation, thus losing the main advantage of the dimensionality reduction mentioned earlier.
4. The method is not accurate for problems within narrow strips or curved shell structures.

3.2 Range of Application

In principle, this method can be applied to any problem for which the governing differential equation is either linear or incrementally linear. In problems involving elliptic differential equations, the solutions are direct, whereas for parabolic and hyperbolic systems of equations, marching processes in time have to be introduced. Thus, a very wide range of physical problems is encompassed, e.g. those of steady state and transient potential flow, elastostatics, elastodynamics, elastoplasticity, acoustics, … etc., can all be solved by either the direct or the indirect formulations of BEM.

The BEM can also be used in conjunction with other numerical techniques, such as the finite element or finite difference methods, in a hybrid formulation. Such composite solutions extend the range of application almost indefinitely since the BEMs have very distinct advantages for problems of large physical dimensions whereas the finite element methods are attractive procedures of incorporating finite size bodies into such systems or fine details in regions with rapidly varying properties (Banerjee and Butterfield, 1981).

3.3 Equations of Equilibrium

In the elastic stress analysis of a plane-stress, or a plane strain engineering component, there are eight basic parameters to be determined, namely: the displacements u and v, strains ε_x, ε_y and γ_{xy} and stresses σ_x, σ_y and τ_{xy}. They are governed, at any point inside the component, by eight partial differential equations.

Strain-displacement relationships:

$$\left.\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\end{aligned}\right. \qquad \textbf{(1)}$$

Stress-strain relationships (assuming orthotropic materials):

$$\left.\begin{aligned}\sigma_x &= d_{11}\varepsilon_x + d_{12}\varepsilon_y \\ \sigma_y &= d_{21}\varepsilon_x + d_{22}\varepsilon_y \\ \tau_{xy} &= d_{33}\gamma_{xy}\end{aligned}\right\} \qquad \textbf{(2)}$$

where (isotropic materials):

$$\left.\begin{aligned}d_{11} &= d_{22} = 2G(1-p)/(1-2p) \\ d_{12} &= d_{21} = 2Gp/(1-2p) \\ d_{33} &= G\end{aligned}\right\} \qquad \textbf{(3)}$$

G = shear modulus

p = ν (Poisson's ratio) for plane strain problems

$= \frac{\nu}{1+\nu}$ for plane stress problems.

Equations of equilibrium:

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0\end{aligned}\right\} \qquad \textbf{(4)}$$

with the following equations, at any point on the boundary:

$$\left.\begin{aligned}T_x &= l\sigma_x + m\tau_{xy} \\ T_y &= l\tau_{xy} + m\sigma_y\end{aligned}\right\} \qquad \textbf{(5)}$$

where: T_x and T_y are the traction components in x- and y- directions.

l and m are directional cosines in x- and y-directions, respectively of the normal on the boundary.

3.3.1. Two-Dimensional Equations in Terms of Displacement:

Substituting Equations (1) into (2), then the stress components may be expressed in terms of displacement components. Substituting the resulting equations into the equations of equilibrium (Equations 4), then the governing equations are reduced to the following elliptic partial differential equations in terms of displacement components u and v:

$$\left.\begin{aligned} \nabla^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}(\nabla.\vec{q}) + f_x/G = 0 \\ \nabla^2 v + \frac{1}{1-2\nu}\frac{\partial}{\partial y}(\nabla.\vec{q}) + f_y/G = 0 \end{aligned}\right\} \quad (6)$$

where $\vec{q} = u\hat{i} + v\hat{j}$, which is the displacement vector.

Biharmonic representation:

Gelerkin introduced strain functions G_x and G_y which may be expressed in terms of a vector known as the Gelerkin vector, i.e., (EL-Zafrany, 1992):

$$\vec{G} = G_x\hat{i} + G_y\hat{j} \quad (7)$$

such that (Little, 1973):

$$\vec{q} = \nabla^2\vec{G} - \frac{1}{2(1-\nu)}\nabla(\nabla.\vec{G}) \quad (8)$$

Writing the partial differential Equations (6) in the following vectorial form:

$$\nabla^2\vec{q} + \frac{1}{2(1-\nu)}\nabla(\nabla.\vec{q}) + \vec{f}/\mu = 0 \quad (9)$$

Then from the definition of the Gelerkin vector, the previous equation can be modified as follows:

$$\nabla^2(\nabla^2\vec{G}) + \vec{f}/\mu = 0$$

which can be rewritten explicitly in terms of the following biharmonic equations:

$$\left.\begin{aligned} \nabla^4 G_x + f_x/\mu = 0 \\ \nabla^4 G_y + f_y/\mu = 0 \end{aligned}\right\} \quad (10)$$

4. Fundamental Solution of Solid Continuum Problems:

4.1 Fundamental Displacements

A two-dimensional solid continuum problem is considered in a semi-infinite domain, with the x-y plane in a state of loading defined by a concentrated force acting at point (x_i,y_i) with a uniform distribution, in the z direction, over a thickness t, which has a constant value for the whole domain. The applied force is represented by the following vector (Fung, 1965):

$$\vec{F} = t(e_x\hat{i} + e_y\hat{j}) \quad (11)$$

where e_x and e_y are the x and y-components of the applied force per unit thickness and i^ and j^

From the definition of the two-dimensional Dirac delta function, a domain distribution of the load intensity equivalent to the applied force, may be expressed as follows (Fung, 1965):

$$\left.\begin{aligned} f_x^* &= e_x \delta(x - x_i, y - y_i) \\ f_y^* &= e_y \delta(x - x_i, y - y_i) \end{aligned}\right\} \quad \textbf{(12)}$$

Using Equations (6) and (7), the governing partial diferential equations for the above case may be written in the following displacement form:

$$\left.\begin{aligned} \nabla^2 u^* + \frac{1}{1-2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) + \frac{f_x^*}{G} = 0 \\ \nabla^2 v^* + \frac{1}{1-2\nu}\frac{\partial}{\partial y}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) + \frac{f_y^*}{G} = 0 \end{aligned}\right\} \quad \textbf{(13)}$$

and the solution to such expressions is known as the **fundamental solution**. Here u* and v* are displacements related to the forces fx* and fy*.

If the displacement components (u*, v*) are expressed in terms of the componants (G_x^*, G_y^*) of Galerkin's vector, such that:

$$\left.\begin{aligned} u^* = \nabla^2 G_x^* - \frac{1}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial G_x^*}{\partial x} + \frac{\partial G_y^*}{\partial y}\right) \\ v^* = \nabla^2 G_y^* - \frac{1}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial G_x^*}{\partial x} + \frac{\partial G_y^*}{\partial y}\right) \end{aligned}\right\} \quad \textbf{(14)}$$

then, Equations (13) can be reduced to the following biharmonic equations:

$$\left.\begin{aligned} \nabla^4 G_x^* + e_x \delta(x - x_i, y - y_i)/G = 0 \\ \nabla^4 G_y^* + e_y \delta(x - x_i, y - y_i)/G = 0 \end{aligned}\right\} \quad \textbf{(15)}$$

The previous equations lead to the conclusion that the parameters G_x^* ***and*** G_y^* can be defined in terms of the functions:

$$G_x^* = g^* e_x, G_x^* = g^* e_x \quad \textbf{(16)}$$

Hence, Equations(15) may be reduced to the following equation:

$$\nabla^4\gamma^* + \delta(x - x_i, y - y_i)/G = 0 \quad \textbf{(17)}$$

Defining another function ϖ^* such that:

$$\nabla^2\gamma^* = \varpi^*/G \quad \textbf{(18)}$$

Then Equation (17) can be rewritten in terms of the following Poisson's partial differential equation:

$$\nabla^4\varpi^* + \delta(x - x_i, y - y_i) = 0 \quad \textbf{(19)}$$

which has the following solution:

$$\varpi^* = \frac{1}{2\pi}\left[\log(1/r) + C_1\right] \quad \textbf{(20)}$$

where $r = (x^2 + y^2)^{1/2}$ and C_1 is a constant.

Substituting the above expression into equation (18), and using direct integration, it can be shown that:

$$\gamma^* = \frac{r^2}{8\pi\mu}\left[\log(1/r) + C_1 + 1\right] + C_2 \quad \textbf{(21)}$$

where C_1 and C_2 are arbitrary integration constants. Then, equations (14) become as:

$$u^*_\alpha(x - x_i, y - y_i) = G_{\alpha 1}(x - x_i, y - y_i)e_x + G_{\alpha 2}(x - x_i, y - y_i)e_y \quad \textbf{(22)}$$

where the fundamental solution parameter $G_{\alpha\beta}$ is expressed as follows:

$$G_{\alpha\beta}(x - x_i, y - y_i) = \nabla^2\gamma^*\delta_{\alpha\beta} - \frac{1}{2(1-\pi)}\left(\frac{\partial^2\gamma^*}{\partial x_\alpha \partial x_\beta}\right) \quad \textbf{(23)}$$

All explicit expressions for the fundamental solution parameters given here are found in (Al-Adthami, 2003).

4.2 Fundamental Strain:

The components of Cauchy's strain tensor can be defined for the previous case, as follows (Desai and Siriwardane, 1984):

$$\varepsilon^*_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u^*_\beta}{\partial x_\alpha} + \frac{\partial u^*_\alpha}{\partial x_\beta}\right) \quad \textbf{(24)}$$

and using Equation (22), the previous equation may be written in the following form:

$$\varepsilon^*_{\alpha\beta} = A^*_{\alpha\beta 1}e_x + A^*_{\alpha\beta 2}e_y \quad \textbf{(25)}$$

where

$$A^*_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial G_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial G_{\alpha\gamma}}{\partial x_\beta}\right) \quad \textbf{(26)}$$

All fundamental solutions given here are functions of the source point $(x-x_i, y-y_i)$.

4.3 Fundamental Stress:

Substituting the fundamental strain tensor defined by Equation (25) into the stress-strain relationships, then it can be proved that:

$$\sigma^*_{\alpha\beta} = D^*_{\alpha\beta 1}e_x + D^*_{\alpha\beta 2}e_y \quad \textbf{(27)}$$

4.4 Fundamental Traction:

If the fundamental stress components defined above are employed in Equations (5), then the corresponding components of fundamental tractions can be expressed in the following form:

$$\left.\begin{aligned} T_x^* &= F_{11}e_X + F_{12}e_y \\ T_y^* &= F_{21}e_X + F_{22}e_y \end{aligned}\right\} \quad \textbf{(28)}$$

4.5 Boundary Integral Equations:

The governing boundary integral equations are usually obtained by employing fundamental solutions as weighting functions in inverse weighted - residual expressions. For linear elastic problems, the Maxwell-Betti reciprocal theorem may also be used for direct derivation of boundary integral equations.

4.6 Boundary Integral Equations of Displacement:

Substituting the fundamental loading parameters defined by Equations (12) into the inverse expression, and using Dirac delta properties, it can be deduced that:

$$\left.\begin{aligned} &C_i u_i e_x + C_i v_i e_y + \oint_\Gamma (T_x^* u + T_y^* v)\,d\Gamma = \\ &\oint_\Gamma (T_x u^* + T_y v^*)\,d\Gamma + \iint_\Omega (f_x u^* + f_y v^*)\,dxdy \end{aligned}\right\} \quad \textbf{(29)}$$

where:

$u_i = u(x_i, y_i), \quad v_i = v(x_i, y_i)$

Employing fundamental displacements (Equation 22), and fundamental tractions (Equation 28), for arbitrary values of e_x, e_y, then Equation (19) can be split into the following boundary integral equations which are defined with respect to the source point (x_i, y_i):

$$C_i u_i + \oint_\Gamma (F_{11}u + F_{21}v)\,d\Gamma = \oint_\Gamma (G_{11}T_x + G_{21}T_y)\,d\Gamma + U(x_i, y_i) \quad \textbf{(30)}$$

$$C_i v_i + \oint_\Gamma (F_{12}u + F_{22}v)\,d\Gamma = \oint_\Gamma (G_{12}T_x + G_{22}T_y)\,d\Gamma + V(x_i, y_i) \quad \textbf{(31)}$$

where: $$\mathbf{U(x_i, y_i) = \iint_\Omega \left(G_{11}f_x + G_{21}fy\right)dxdy} \quad \textbf{(32)}$$

$$\mathbf{V(x_i, y_i) = \iint_\Omega \left(G_{12}f_x + G_{22}fy\right)dxdy} \quad \textbf{(33)}$$

which represent domain loading terms. If the source point (x_i, y_i) is inside the domain, then C_i=1, and Equations (30) and (31) may be modified as follows:

$$u(x_i, y_i) = U(x_i, y_i) + \int_{\Gamma} (G_{11}T_x + G_{21}T_y)d\Gamma - \int_{\Gamma} (F_{11}u + F_{21}v)d\Gamma \qquad (34)$$

$$v(x_i, y_i) = V(x_i, y_i) + \int_{\Gamma} (G_{12}T_x + G_{22}T_y)d\Gamma - \int_{\Gamma} (F_{12}u + F_{22}v)d\Gamma \qquad (35)$$

The analysis given in the remaining subsections will be limited to cases with source points being inside the domain.

4.7 Boundary Integral Equations of Strain:

Equations (34) and (35) can be differentiated partially with respect to x_i and y_i; that is, Cauchy's strain components may be defined at an internal point (x_i,y_i) as follows (Banerjee, 1994):

$$\left.\begin{aligned} \varepsilon_{xx}(x_i, y_i) &= \frac{\partial u_i}{\partial x_i}, \\ \varepsilon_{yy}(x_i, y_i) &= \frac{\partial v_i}{\partial y_i}, \\ \varepsilon_{xy} &= \frac{1}{2}\left[\frac{\partial u_i}{\partial y_i} + \frac{\partial v_i}{\partial x_i}\right] = \frac{1}{2}\gamma_{xy} \end{aligned}\right\} \qquad (36)$$

When employing displacement equations (Equations 34 and 35) in the previous expressions of strain components, integral terms are to be differentiated with respect to x_i and y_i. Then, the boundary integral equation for Cauchy's strain tensor may be expressed in the following form:

$$\varepsilon_{\alpha\beta}(x_i, y_i) = \iint_{\Omega} (A_{\alpha\beta 1} f_x + A_{\alpha\beta 2} f_y)dxdy + \int_{\Gamma} (A_{\alpha\beta 1}T_x + A_{\alpha\beta 2}T_y)d\Gamma - \int_{\Gamma} (B_{\alpha\beta 1}u + B_{\alpha\beta 2}v)d\Gamma \qquad (37)$$

where: $\mathbf{A}_{\alpha\beta\gamma} = -\mathbf{A}_{\alpha\beta\gamma}$, and

$$B_{\alpha\beta\gamma} = -\frac{1}{2}\left[\frac{\partial F_{\gamma\beta}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta}\right] \qquad (38)$$

4.8 Boundary Integral Equations of Stress:

Substituting the strain tensor defined by the boundary integral Equation (37) into the stress-strain relationships, then a boundary integral equation for the stress tensor at the internal source point (x_i, y_i) can be described, and expressed in the following form (Banerjee, 1994):

$$\sigma_{\alpha\beta}(x_i, y_i) = \iint_{\Omega} (D_{\alpha\beta 1} f_x + D_{\alpha\beta 2} f_y)\,dxdy + \int_{\Gamma} (D_{\alpha\beta 1} T_x + D_{\alpha\beta 2} T_y)\,d\Gamma - \int_{\Gamma} (E_{\alpha\beta 1} u + E_{\alpha\beta 2} v)\,d\Gamma \qquad (39)$$

where: $\Delta_{\alpha\beta\gamma} = -\Delta_{\alpha\beta\gamma}$ **(40)**

4.9 Numerical Treatment of the Boundary Integral Equations:

The boundary element method, as described in the previous sections, is based upon dividing the boundary into a suitable number of boundary elements, and approximating the boundary distributions of field function parameters such as displacements and tractions by interpolating them in terms of their nodal values within each element. Discretizing the boundary Γ of a two-dimensional elasticity problem into n_e boundary elements, the boundary integral equations (Equations 30 and 31) with respect to the source point may be rewritten as follows:

$$\left. \begin{aligned} C_i u_i + \sum_{e=1}^{n_e} \left[\int_{\Gamma_e} \{F_{11} u(\Gamma_e) + F_{21} v(\Gamma_e)\}\, d\Gamma \right] = \\ \sum_{e=1}^{n_e} \left[\int_{\Gamma_e} \{G_{11} T_x(\Gamma_e) + G_{21} T_y(\Gamma_e)\}\, d\Gamma \right] + U(x_i, y_i) \end{aligned} \right\} \qquad (41)$$

$$C_i v_i + \sum_{e=1}^{n_e} \left[\int_{\Gamma_e} \{ F_{12} u(\Gamma_e) + F_{22} v(\Gamma_e) \} d\Gamma \right] =$$

$$\sum_{e=1}^{n_e} \left[\int_{\Gamma_e} \{ G_{12} T_x(\Gamma_e) + G_{22} T_y(\Gamma_e) \} d\Gamma \right] + V(x_i, y_i) \qquad (42)$$

where each parameter in the form of $f(\Gamma_e)$ represents a field function parameter approximated over the boundary Γ_e of the eth element.

5. A Computer Program for Two-Dimensional Solid Continuum Problems:

A computer program based upon the theory of the two-dimensional solid continuum mechanics problems of the boundary element method with constant elements is coded in FORTRAN 77 and introduced herein. The program can deal with plane-stress and plane strain problems with surface and domain loading.

In the design of tunnels to be constructed in urban areas, it is necessary to estimate the magnitude and distribution of the stresses and settlements that are likely to occur due to a particular design and construction technique. Also, the effect of these stresses and movements upon existing surface and buried structures has to be studied.

The computer program is used for the determination of the stress and deformation fields around one cavity. The soil is assumed to be homogeneous, isotropic and a linearly elastic medium containing one opening representing the cavity dimensions and positions.

6. Applications:

Two expositions illustrating the application of the boundary element method for two-dimensional solid continuum problems are presented herein. These expositions show also, the accuracy of the boundary element method by comparing its results with those of finite element or analytical solutions.

6.1 Uniform Cylinder under Internal Pressure

This problem has been selected to give an indication of the accuracy of the boundary element method. In order to do that, the results obtained using the BEM are compared with the exact solution of the problem given by (Fenner, 1986) as follows:

$$u_r = (1+\nu) r p \left[(r_e / r)^2 + 1 - 2\nu \right] / \left[E(\lambda^2 - 1) \right] \qquad (43)$$

$$\sigma_r = -p \left[(r_e / r)^2 - 1 \right] / (\lambda^2 - 1) \qquad (44)$$

$$\sigma_\theta = p \left[(r_e / r)^2 + 1 \right] / (\lambda^2 - 1) \qquad (45)$$

where $l = \mathbf{r_e} / \mathbf{r_i}$

The cylinder shown in Figure (1) is subjected to a uniform internal pressure p, with no pressure applied on the outer surface. Only a quarter of the cylinder is considered for the analysis. The boundary element discretization using 20-constant elements is shown in Figure (2). Figure (3) shows the boundary element discretization using 40-constant boundary elements.

The boundary conditions are specified to avoid rigid body motion, i.e., zero displacements are prescribed in the x-direction along line AB and in the y-direction along line CD, as shown in Figure (2). The cylinder has the following properties:

Radius of internal surface r_i = 0.05 m,
Radius of external surface r_e = 0.1 m
Young's modulus E = 10.0×10^9 N/m^2,
Poisson's ratio $\nu = 0.3$,
Internal pressure p = 1.0×10^6 N/m^2.

The results for radial displacements and radial stresses are computed at 5 internal points. The results for these points computed from the exact solution and from the boundary element method (20-constant elements) are presented in Table (1).

The results for radial displacements and radial stresses are computed at 8 internal points lying on the radial line in the middle of the quarter section. The results for these points computed by both the exact analytical solution and the boundary element method (40-constant elements) are presented in Table (2).

The distribution of the radial displacements u_r along a radial line in the middle of the quarter section is plotted against the analytical solution, as shown in Figure (4). The radial stress distributions on the same radial line is plotted in Figure (5).

Table (1) - Radial displacements and stresses by the BEM and exact solutions.

		Exact solution		20-Constant elements	
X (m)	Y (m)	u_r (m)	σ_r (N/m^2)	u_r (m)	σ_r (N/m^2)
0.0389	0.0389	8.83E-06	-768595	9.38E-06	-601320
0.0459	0.0459	7.79E-06	-455621	8.28E-06	-495894
0.053	0.053	7.08E-06	-259259	7.60E-06	-275529
0.0601	0.0601	6.57E-06	-128028	7.12E-06	-271457
0.0672	0.0672	6.21E-06	-36011	6.79E-06	-67775

Table (2)- Radial displacements and stresses by the BEM and exact solutions.

		Exact solution		40-Constant elements	
X (m)	Y (m)	u_r (m)	σ_r (N/m^2)	u_r (m)	σ_r (N/m^2)
0.0371	0.0371	9.16E-06	-876039	9.52E-06	-882000
0.0406	0.0406	8.53E-06	-674858	8.71E-06	-739000
0.0442	0.0442	8.02E-06	-520000	8.21E-06	-552000
0.0477	0.0477	7.59E-06	-398262	7.75E-06	-424000
0.0513	0.0513	7.23E-06	-300832	7.52E-06	-302000
0.0548	0.0548	6.94E-06	-221644	7.28E-06	-284563
0.0583	0.0583	6.68E-06	-156412	7.01E-06	-165325
0.0619	0.0619	6.47E-06	-102040	6.70E-06	-134225
0.0654	0.0654	6.29E-06	-56245	6.51E-06	-79953
0.0689	0.0689	6.13E-06	-17313	6.46E-06	-43847

It is clear from these figures that the accuracy of the BEM is well established in relation to the closed form analytical solution. The figures indicate that more accurate results can be obtained by using finer meshes.

6.2 Semi-Infinite Medium Problem

In the field of geotechnical engineering, the semi-infinite nature of the soil domain may be efficiently modeled using boundary element techniques. An example of this is the problem of a strip foundation under a constant pressure.

The details of the problem are shown in Figure (6), the constants and parameters used in the analysis are as follows:

P = 2 psi (0.0137931 MPa),

B = 42.5 inch (1.0795 m)

E = 30×106 psi (206897 MPa)

ν = 0.25

Table (3) includes the comparison of horizontal stresses obtained by the analytical and the boundary element methods for eight internal points.

Table (4) includes the comparison of the vertical stresses obtained by the analytical and the boundary element methods for eight internal points.

Table (5) gives the comparison of shear stresses obtained by the analytical and the boundary element methods for eight internal points.

It is noticed from the previous tables that the agreement of results between the boundary element and the analytical solution is satisfactory.

Table (3)-Horizontal stresses by the BEM and exacts solutions.

Points	X (cm)	Y (cm)	Analytical solution	Boundary element solution
			σ_x (MPa)	σ_x (MPa)
1	0	95	0.00127	0.000321
2	25	95	0.00235	0.00146
3	50	95	0.00336	0.001168
4	75	95	0.00058	0.000262
5	0	50	0.000124	0.000401
6	25	50	0.000462	0.000015
7	50	50	0.001007	0.000659
8	75	50	0.001207	0.000953

The same problem is solved by non-linear elastic finite element method. The finite element mesh for half of the domain is drawn in Figure (7). Eight-node isoparametric elements are used. The side boundaries are assumed to be free to move vertically, while the bottom boundary is restrained against both horizontal and vertical movements. Figure (8) shows the vertical displacements on the surface obtained from the finite element method and the boundary element method. As can be noticed, the agreement is very high between them.

Table (4) - Vertical stresses by the BEM and exacts solutions.

Points	X (cm)	Y (cm)	Analytical solution	Boundary element solution
			σ_y (MPa)	σ_y (MPa)
1	0	95	0.00955	0.01048
2	25	95	0.004855	0.08515
3	50	95	0.001169	0.000855
4	75	95	0.0000724	0.00017
5	0	50	0.00473	0.00529
6	25	50	0.003952	0.00432
7	50	50	0.002483	0.002483
8	75	50	0.0013103	0.0011

Table (5) - Shear stresses by the exact solution and the BEM.

Points	X (cm)	Y (cm)	Analytical solution	Boundary element solution
			τ_{xy} (MPa)	τ_{xy} (MPa)
1	0	95	0.0	0.0
2	25	95	0.00295	0.00317
3	50	95	0.00163	0.00129
4	75	95	0.0002	0.000455
5	0	50	0.0	0.0
6	25	50	0.0012	0.0011
7	50	50	0.00152	0.00123
8	75	50	0.00124	0.00084

7. Comparison between FEM and BEM

1. In general, since only the boundaries are discretized, a much smaller system of equations is developed than when the finite element method is used.
2. In the case of the boundary element method, errors due to discretization are usually confined to the boundaries, as for the finite element method, the entire domain needs to be discretized. Hence, for the latter, discretization errors are present in each element of the domain whereas those are found only on the boundaries for the former.
3. Values of the solution variables need only be obtained where required at any specified internal points while in the finite element method, the variables are calculated at every node.
4. An advantage of the boundary element method is that the boundary at infinity can be modeled without truncating the domain at some arbitrary distance from the region of interest.

Figure (9) shows a comparison between the steps required to reach a solution from both FEM and BEM programs.

8. Conclusions:

1) The results of the boundary element method will be greatly improved when increasing the number of elements at boundaries, especially at the regions of concentrated stresses.

2) A good agreement can be obtained between the results of the finite element method and the boundary element method. Keeping in mind that in the boundary element method, errors due to

discretization are restricted to the boundaries compared to the finite element method where the entire domain needs to be discretized. This advantage makes the use of the boundary element method easier and faster.

3) In the boundary element method, the boundary at infinity can be modeled without truncating the domain at some arbitrary distance from the region of interest. This can also be done by the finite element method by using infinite elements.

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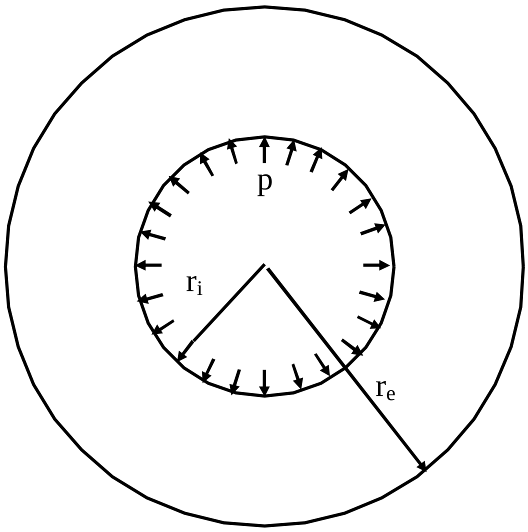

Fig. (1) – Uniform cylinder under constant internal pressure.

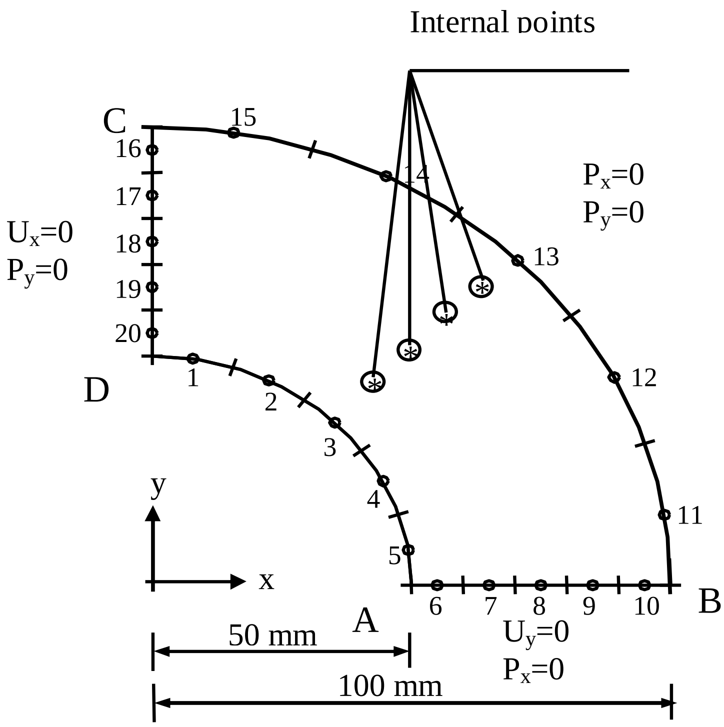

Fig. (2) – Pressurized cylinder mesh with 20-constant boundary elements.

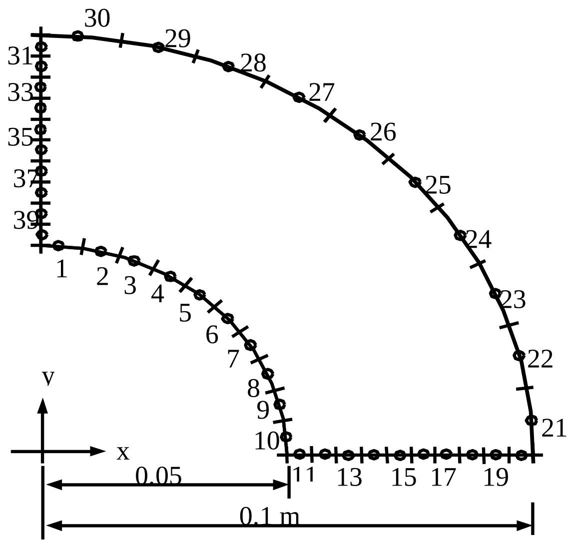

Fig. (3) – Pressurized cylinder mesh with 40-constant boundary elements.

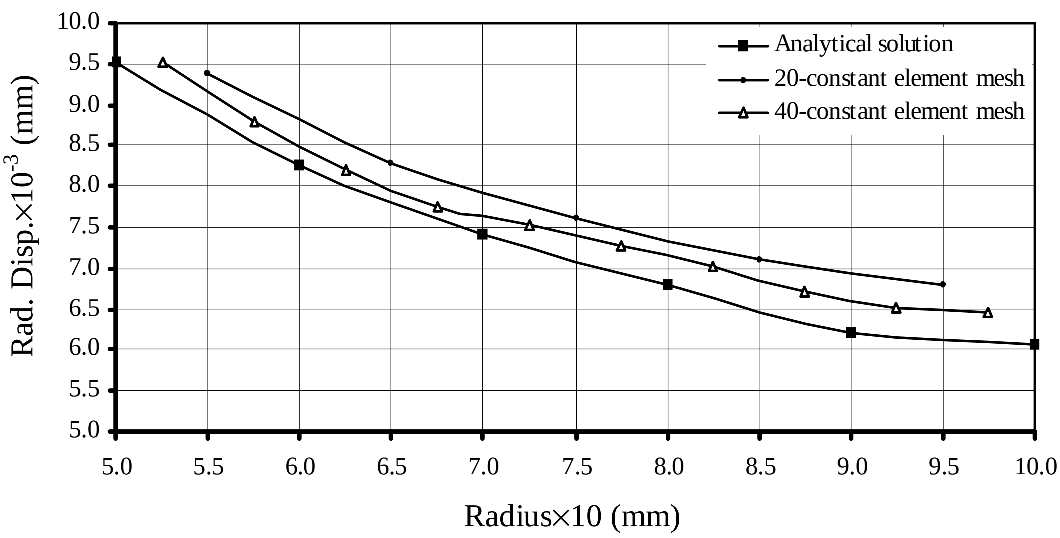

Fig. (4) – Radial displacement distribution for pressurized cylinder from EM and analytical solutions.

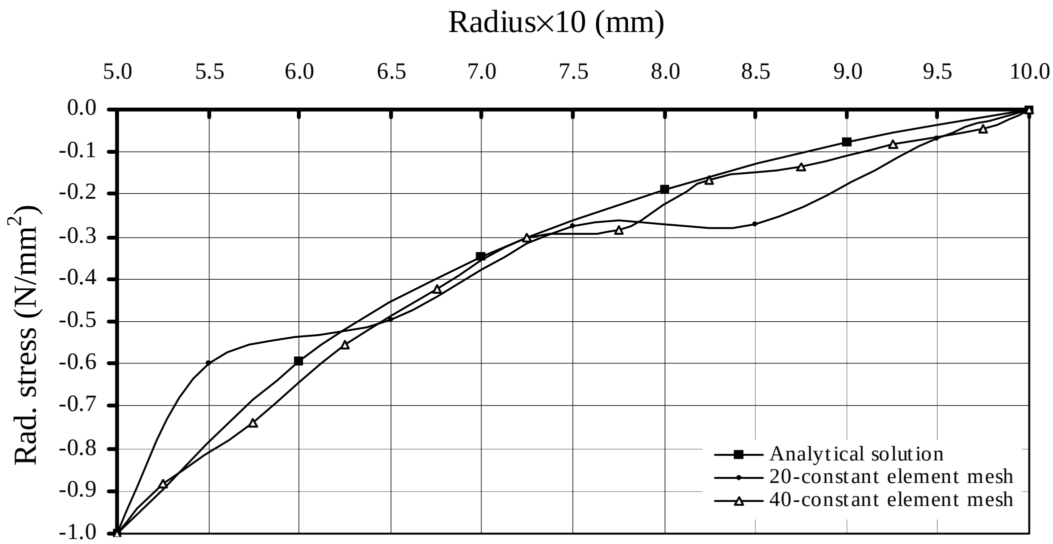

Fig. (5) – Radial stress distribution for pressurized cylinder from BEM and analytical solutions.

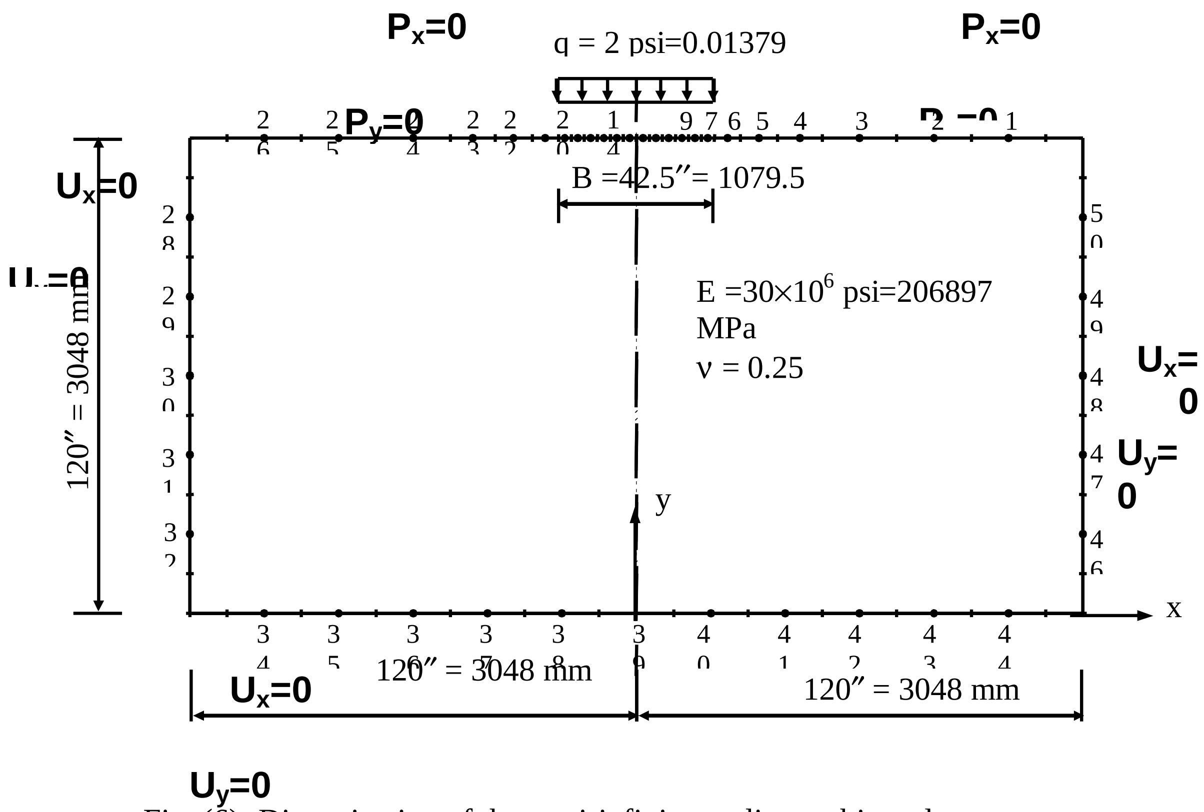

Fig. (6)- Discretization of the semi-infinite medium subjected to a constant pressure on a strip foundation.

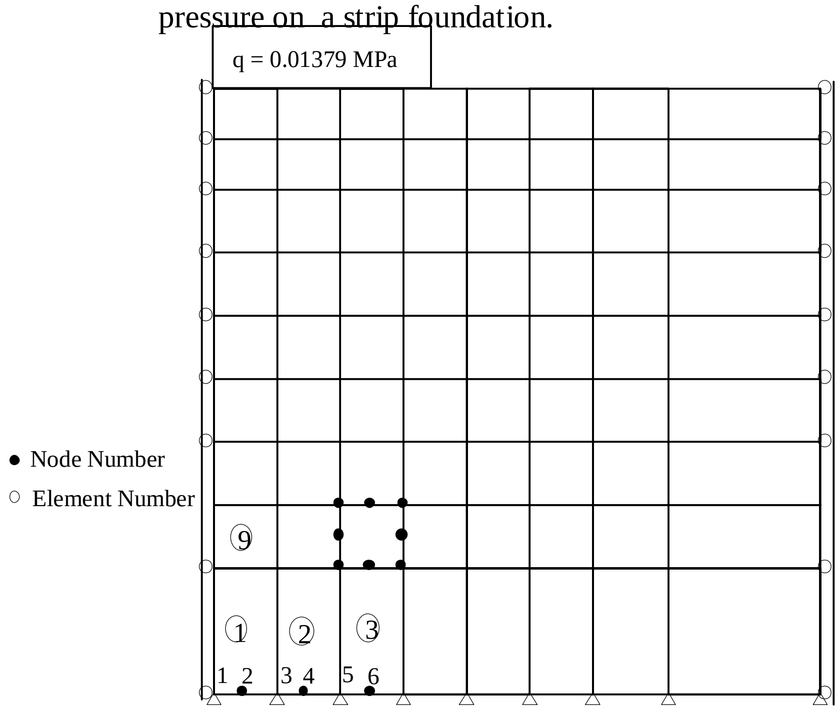

Fig. (7) – The finite element mesh.

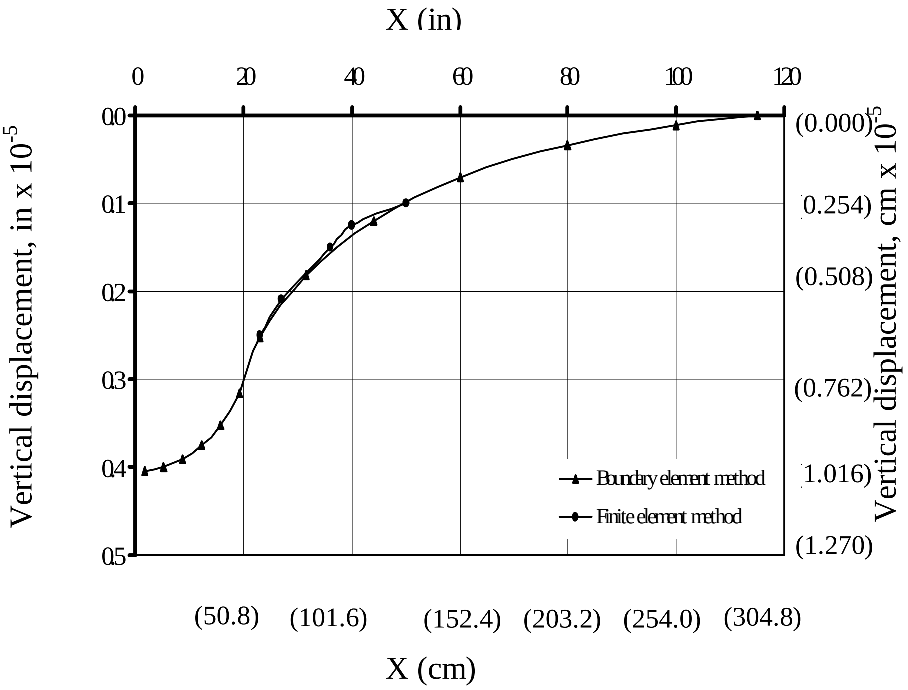

Fig. (8) -Vertical displacement on the surface by the FEM and BEM as a comparison

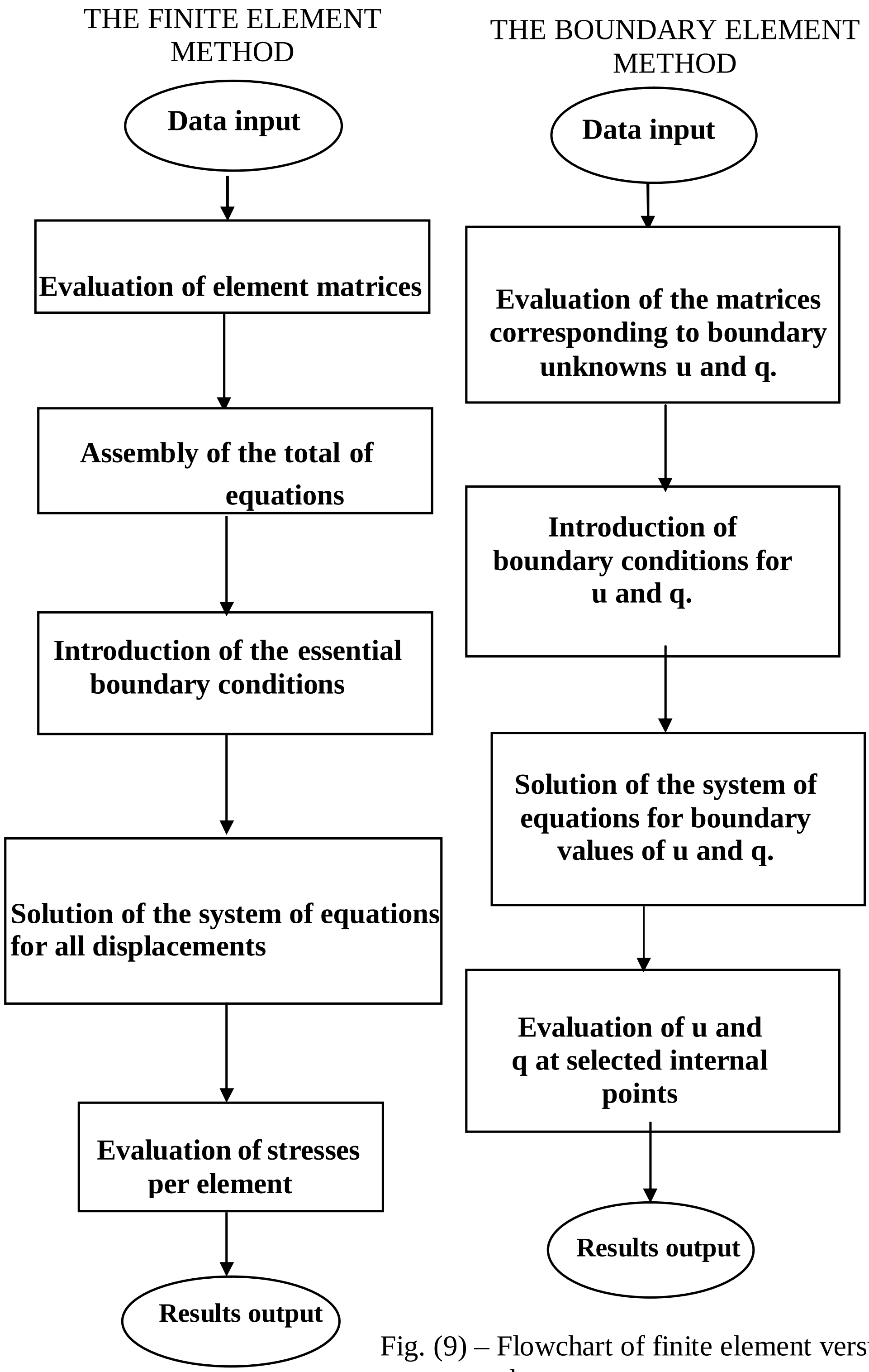

Fig. (9) – Flowchart of finite element versus boundary element computer programs.