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Bi- Univalent Functions Particular Subclass via Third Hankel Determinant

Authors Names	**ABSTRACT**
Majida H. Majeed ***Article History*** Publication data: 1/ 4 /2025 ***Keywords:*** Hankel determinant, boundaries, functions, limits, bi-univalent.	The current work tackles bi-univalent functions' subclass $\mathcal{S}(\lambda)$ inside the reign U (which is a rather open disk). In addition, this study involves complicated analysis and geometric function theory as an attempt to limit the rating the Hankel determinant 3[ed] for the class $\mathcal{S}(\lambda)$. And this is of highly significance in multiple mathematical fields. Consequently, be-univalent functions are used as $\mathcal{S}(\lambda)$ and the constrains or limits are put on the coefficients $\lvert e_n \rvert$. The goal of the current study is to restrict the confines of order's 3 Hankel determinant, ($H_3(1)$. The researcher infers the third Hankel determinant of the recently progressed subclass (n=2,3,4 and 5), and the result is many interested outcomes. These results enlarge greater meaning and usages about functions of be-univalent type in multiple fields of mathematics. Moreover, these results might open the possibility for further studies concerning be-univalent functions (their use and their features).

1.Introduction

Assume A is the functions' collection $\hbar$ which is analytic in the open unit disk $\boldsymbol{U} = \{\boldsymbol{z}: \boldsymbol{z} \in \mathbb{C}\ \lvert \boldsymbol{z} \rvert < \mathbf{1}\}$. It is clear that analytic functions $\hbar \in$A have forms which are extended by Taylor series.

$$\hbar(z) = z + \sum_{\mathbf{n}=\mathbf{2}}^{\infty} \mathbf{e_n} z^{\mathbf{n}}, \ (z \in \mathbf{U}). \qquad (1)$$

It is worthy to mention that all functions' class in A indicated by K that are univalent in U. Theorem [13] of the Koebe-One-Quarter assures that U picture which lies beneath every $\hbar \in$K has a radius disk $\frac{1}{4}$. It is clear that each $\boldsymbol{\hbar} \in \mathbf{K}$ involves an Inverse function $\boldsymbol{\hbar}^{-1}$ agreeing with $\boldsymbol{\hbar}^{-1}(\boldsymbol{\hbar}(z)) = z$ and

$$\boldsymbol{\hbar}\left(\boldsymbol{\hbar}^{-1}(w)\right) = w, \left(\lvert w \rvert < r_{\circ}(\boldsymbol{\hbar}), r_{\circ}(\boldsymbol{\hbar}) > \tfrac{1}{4})\right), \text{ where}$$

$$G(w) = \boldsymbol{\hbar}^{-1}(w) = w - e_2 w^2 + \left(2e_2^2 - e_3\right) w^3 - \left(5e_2^3 - 5e_2 e_3 + e_4\right) w^4 + \cdots. \ (w \in U), (2)$$

$\boldsymbol{\hbar} \in \sum$ function is considered as be bi-univalent in $\boldsymbol{U}$ in case that $\boldsymbol{\hbar}(z)$ and $\boldsymbol{\hbar}^{-1}(z)$ are univalent in $\boldsymbol{U}$.

Lewin [20] In 1967 got a coefficient tied or bound which is gain by $\lvert e_2 \rvert < 1.51$ for all $\hbar \in \sum$ function form (1). This researcher examines the class Σ of bi-univalent functions in U. In the same year, Clunie and Brannan [7] size up the following $\lvert e_2 \rvert \leq \sqrt{2}$ is to $\hbar \in \sum$. And wat is stated by Netenyahu

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[21] is $|e_2| = \frac{4}{3}$. Additionly, Kiedzerawski [18] in 1985 agrees with his colleagues' view (Brannan-Clunie's project) concerning bi-starlike function. Another study by Brannan and Taha records evaluation for the initial coefficients $|e_2|$ and $|e_3|$. Their evaluation is that functions in the classes of order $\wp$ bi-starlike functions signify $\mathcal{T}_{\Omega}^{*}(\wp)$ as well as order $\wp$ bi-convex functions symbolized as $\Psi_{\Omega}(\wp)$. It is noted that unclear speculation for 1st two Taylor-Maclaurin Coefficients are established of all function classes $\mathcal{T}_{\Omega}^{*}(\wp)$ and $\Psi_{\Omega}(\wp)$, (see[2-10]). Many writers (see [15-24]) presented primary Maclaurin coefficients determiners of bi-univalent functions subclass. Moreover, number of scholars ([15, 26,27]) examined devised the subclass of the bi-univalent function class Ω and noted that unclear boundaries were in the 1st two Taylor-Maclaurin coefficients. Besides, the Taylor-Maclaurin coefficient problem $|e_n|$, n = 3,4,... stills unsolved problem till now [20]. As a consequence, if $\mathcal{P}$ signifies analytic functions class $\wp$ which have equalized as the following:

$$\boldsymbol{\wp(z) = 1 + \wp_1 z + \wp_2 z^2 + \cdots. Re(\wp(z)) > 0. z \in U}.$$

In 1976, $\boldsymbol{q^{th}}$ Hankel determinant of $\boldsymbol{h}$ is studied and defined by Noonan and Thomas [22], where $\boldsymbol{n \geq 1}$ and $\boldsymbol{q \geq 1}$ as the following:

$$\boldsymbol{H_q(n) = \begin{vmatrix} e_n & e_{n+1} & \cdots & e_{n+q-1} \\ e_{n+1} & e_{n+2} & \cdots & e_{n+q} \\ \vdots & \vdots & & \vdots \\ e_{n+q-1} & e_{n+q} & \cdots & e_{n+2q-2} \end{vmatrix}, (e_1 = 1)}$$

Where $q = 2$ and $n = 1$, it is diagnosed that the function $H_2(1) = e_3 - e_2^2$. $|H_2(2)| = |e_2 e_4 - e_3^2|$ gives 2nd Hankel determinant $H_2(2)$ is given through $|H_2(2)| = |e_2 e_4 - e_3^2|$ for the bi-convex classes and bi-starlike classes ([6-9]). Another study about 2nd Hankel determinant for bi-univalent functions particular subclasses is made by Al-Ameadea et al. while, Atishan et al. [1], where m-fold symmetric bi-univalent functions the Hankel determinant is tackle and unprecedented factor is used. Besides, Fekeate and Sizegö [14] in their study focus upon the Hankel determinant of h and they update previous study about estimates of $|e_3 - \alpha e_2^2|$, where $e_1 = 1$ and $\alpha \in \mathbb{R}$ As examples for this case are the third Hankel determinant, and $|e_3 - \alpha e_2^2|$ see [17]. All functions that are mentioned are tackled by ([6-11-19]) and functional given as the following:

$$\boldsymbol{H_3(1) = \begin{vmatrix} e_1 & e_2 & e_3 \\ e_2 & e_3 & e_4 \\ e_3 & e_4 & e_5 \end{vmatrix}}. \quad \boldsymbol{(e_1 = 1)} \text{ and } \boldsymbol{(n = 1. q = 3)}.$$

Unequal triangle is adapted to $\boldsymbol{H_3(1)}$, and the result was

$$|\boldsymbol{H_3(1)}| \leq |\boldsymbol{e_3}||\boldsymbol{e_2e_4} - \boldsymbol{e_3^2}| - |\boldsymbol{e_4}||\boldsymbol{e_4} - \boldsymbol{e_2e_3}| + |\boldsymbol{e_5}||\boldsymbol{e_3} - \boldsymbol{e_2^2}|. \tag{3}$$

Bi-univalent functions' subclass $\mathcal{S}(\lambda)$, inside the open unit disk region U is provided by the current research.

The following Lemmas are significant for fixing the outcomes of the current paper:

Lemma 1 [13]. Assume $\mathcal{P}$ be is all analytic functions $p(z)$ class of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{4}$$

and $Re\big(p(z)\big) > 0$ for whole $z \in U$. After that $|p_n| \leq 2$, for every $n = 1.2.\cdots$.

Lemma 2 [16]. Where $p \in \mathcal{P}$ function was obtained by (4), so,

$$2p_2 = p_1^2 + (4 - p_1^2)x$$
$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some $x.z$ with $|x| \leq 1$ and $|z| \leq 1$.

2- Major Findings

Definition 1. A function $h \in \Sigma$, which is obtained by (1) found in the class $\mathcal{S}(\lambda)$, in case this function agrees with the coming condition:

$$Re\left(1 + \frac{zh''(z)}{h'(z)} + z^2 h'''(z)\right) > \lambda, \tag{5}$$

$$Re\left(1 + \frac{wg''(w)}{wg'(w)} + w^2 g'''(w)\right) > \lambda, \tag{6}$$

where $(0 < \lambda \leq 1).z.w \in U$ and $g = h^{-1}$.

Theorem 1. Assume the function $h(z)$ while it appears in equation (1), (class' $\mathcal{S}(\lambda)$ element), where $0 < \lambda \leq 1$. consequently, the researcher got the following:

$$|e_2 e_4 - e_3^2| \leq (1-\lambda)^2 \left[\frac{107}{72}(1-\lambda)^2 + \frac{3}{4}(1-\lambda) + \frac{5}{36}\right]. \qquad (7)$$

It is proof. from (5) and (6), we have

$$1 + \frac{z h''(z)}{h'(z)} + z^2 h'''(z) = \lambda + (1-\lambda) p(z) \qquad (8)$$

and

$$1 + \frac{w g''(w)}{w g'(w)} + w^2 g'''(w) = \lambda + (1-\lambda) q(w), \qquad (9)$$

where $(0 < \lambda \leq 1;\ p.q \in \mathcal{P}).z.w \in U$ and $g = h^{-1}$.

Then there exist $u, v: U \to U$ with $u(0) = v(0) = 0.\ |u(z)| < 1.|v(w)| < 1$ as well as make the function $p.q \in \mathcal{P}$, like

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \sum_{n=1}^{\infty} r_n z^n$$

and

$$q(z) = \frac{1 + v(w)}{1 - v(w)} = 1 + \sum_{n=1}^{\infty} k_n w^n.$$

It follows that

$$\lambda + (1-\lambda) p(z) = 1 + \sum_{n=1}^{\infty} (1-\lambda) r_n z^n \qquad (10)$$

and

$$\lambda + (1-\lambda) q(w) = 1 + \sum_{n=1}^{\infty} (1-\lambda) k_n w^n \ . \qquad (11)$$

Since $\hbar \in \sum$ has the Maclurian series defined by (1), a counting indicates its opposite$g = \hbar^{-1}$ get the expansion by (2), we have

$$1+\frac{z\hbar''(z)}{\hbar'(z)}+z^2\hbar'''(z) = 1+32z+(12e_3-4e_2^2)z^2+(36e_4-18e_2e_3+8e_2^3)z^3+$$
$$(80e_5-32e_2e_4+48e_3e_2^2-18e_3^2-16e_2^4)z^4+\cdots, \quad (12)$$

and

$$1+\frac{wg''(w)}{wg'(w)}+w^2g'''(w) = 1-2e_2w+(20e_2^2-12e_3)w^2-(152e_2^3-162e_2e_3+36e_4)w^3$$
$$+(140e_2e_4+(448e_2e_3+968e_2^4-1680e_2e_3+222e_3^2+184e_3e_2^2-60e_5)w^4+\cdots. \quad (13)$$

Now using (10) and (12) comparing the coefficients of $z.z^2.z^3$ and z^4, we get

$$2e_2 = (1-\lambda)r_1, \quad (14)$$
$$12e_3-4e_2^2 = (1-\lambda)r_2, \quad (15)$$
$$36e_4-18e_2e_3+8e_2^3 = (1-\lambda)r_3 \quad (16)$$

and

$$80e_5-32e_2e_4+48e_3e_2^2-18e_3^2-16e_2^4 = (1-\lambda)r_4. \quad (17)$$

Also by using (11) and (13) comparing the coefficients of $w.w^2.w^3$ and w^4, we get

$$-2e_2 = (1-\lambda), \quad (18)$$
$$20e_2^2-12e_3 = (1-\lambda)\mathrm{K}_2, \quad (19)$$
$$-(15e_2^3-162e_2e_3+36e_4) = (1-\lambda)\mathrm{K}_3 \quad (20)$$

and

$$448e_2e_4+968e_2^4-1680e_2e_3+222e_3^2+184e_3e_2^2-60e_5 = (1-\lambda)\mathrm{K}_4\ . \quad (21)$$

From (14) and (18), we have

$$\frac{(1-\lambda)r_1}{2} = e_2 = -\frac{(1-\lambda)K_1}{2} \ . \qquad (22)$$

It follows that its

$$r_1 = -K_1 \ . \qquad (23)$$

Subtracting (15) from (19) and (16) from (20), we get

$$e_3 = \frac{(1-\lambda)^2 r_1^2}{4} + \frac{(1-\lambda)(r_2-K_2)}{24} \qquad (24)$$

and

$$e_4 = \frac{179(1-\lambda)^3 r_1^3}{576} + \frac{5(1-\lambda)^2 r_1(r_2-K_2)}{24} + \frac{(1-\lambda)(r_3-K_3)}{72} \ . \qquad (25)$$

Thus, using (22), (24) and (25), we get

$$e_2e_4 - e_3^2 = \frac{107}{1152}(1-\lambda)^4 r_1^4 + \frac{1}{12}(1-\lambda)^3 r_1^2(r_2-K_2) + \frac{1}{144}(1-\lambda)^2 r_1(r_3-K_3)$$
$$-\frac{1}{576}(1-\lambda)^2(r_2-K_2)^2. \qquad (26)$$

According Lemma (2) and $r_1 = -K_1$, we get

$$r_2 - K_2 = \frac{4-r_1^2}{2}(x-y) \qquad (27)$$

and

$$r_3 - K_3 = \frac{r_1^3}{2} + \frac{(4-r_1^2)r_1}{2}(x+y) - \frac{(4-r_1^2)r_1}{4}(x^2+y^2)$$
$$+\frac{4-r_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w]. \qquad (28)$$

for some $x.y.z$ and w with $|x| \leq 1. |y| \leq 1$ and $|w| \leq 1$.

Allow the $p \in \mathcal{P}$, the researcher gets $|r_1| \leq 2$. Allowing $r_1 = r$, as assume, without generality's lost, that $r \in [0.2]$. Thus, s the expressions' replacement (27) and (28) in (26), letting $\zeta = |x| \leq 1$ and $\varsigma = |y| \leq 1$, the researcher gets:

$$|e_2 e_4 - e_3^2| \leq J_1 + J_2(\zeta + \varsigma) + J_3(\zeta^2 + \varsigma^2) + J_4(\zeta + \varsigma)^2 = J(\zeta.\varsigma),$$

where

$$J_1 = J_1(\lambda.r) = (1-\lambda)^2 r^4 \left[\frac{107(1-\lambda)^2}{1152} + \frac{1}{228}\right] \geq 0,$$

$$J_2 = J_2(\lambda.r) = \frac{(1-\lambda)^2(4-r^2)r^2}{2}\left((1-\lambda) + \frac{1}{144}\right) \geq 0,$$

$$J_3 = J_3(\lambda.r) = \frac{(1-\lambda)^2(4-r^2)r}{288}\left(\frac{r}{2} - 1\right) \leq 0$$

and

$$J_4 = J_4(\lambda.r) = \frac{(1-\lambda)^2(4-r^2)^2}{2304} \geq 0.$$

Now, we need to maximize of $J(\zeta.\varsigma)$ in $[0.1] \times [0.1]$ the closed square for $r \in [0.2]$, since $J_3 \leq 0$ and $J_3 + 2J_2 \geq 0$ the researcher reaches the following conclusion: $r \in (0.2)$, $G_{\zeta.\varsigma}G_{\varsigma.\varsigma} - \left(G_{\zeta.\varsigma}\right)^2 < 0$. We search the extreme of G above boarders of closed square because there is lack of municipal extreme inside closed square in function G. If $\zeta = 0$ and $0 \leq \varsigma \leq 1$, we have

$$G(0.\varsigma) = \theta(\varsigma) = J_1 + J_2\varsigma + (J_3 + J_4)\varsigma^2 .$$

Next, we are going to be dealing with following two cases.

Case 1: Let $J_3 + J_4 \geq 0$. Such case we get, $0 \leq \varsigma \leq 1$, and whatever stable r with $0 \leq r < 2$, this by itself indicates that

$$\theta'(\varsigma) = J_2 + 2(J_3 + J_4)\varsigma > 0 .$$

Consequently, $\theta(\varsigma)$ is a function that gradually increased. So, to stable $r \in [0.2]$, extreme of $\theta(\varsigma)$ happened in$\varsigma = 1$ and

$$\max \theta(\varsigma) = \theta(1) = J_1 + J_2 + J_3 + J_4$$

Case 2: Let $J_3 + J_4 < 0$. Since $2(J_3 + J_4) + J_2 \geq 0$ for $0 < \varsigma < 1$ with $0 < r < 2$, it is clear that $2(J_3 + J_4) + J_2 < 2(J_3 + J_4)\varsigma + J_2 < J_2$ and so $\theta(\varsigma) > 0$. Accordingly, the maximum of $\theta(\varsigma)$ happened at $\varsigma = 1$ and $0 \leq \varsigma \leq 1$, we reached to

$$J(1.\varsigma) = \varphi(\varsigma) = (J_3 + J_4)\sigma^2 + (J_2 + 2J_4)\varsigma + J_1 + J_2 + J_3 + J_4$$

Depending on the cases of $J_3 + J_4$, we reached to

$$\max \varphi(\varsigma) = \varphi(1) = J_1 + 2J_2 + 2J_3 + 4J_4.$$

From $\theta(1) \leq \varphi(1)$, the researcher obtains $\max\big(J(\zeta.\varsigma)\big) = J(1.1)$ on the square borders $[0.1] \times [0.1]$. And the real- actual function $\mathcal{J}$ on (0.1) is defined as

$$\mathcal{J}(r) = \max\big(J(\zeta.\varsigma)\big) = J(1.1) = J_1 + 2J_2 + 2J_3 + 4J_4.$$

Now, putting $J_1.J_2.J_3$ and J_4 in the function $\mathcal{J}$, we obtain

$$\mathcal{J}(r) = (1-\lambda)^2[B + E],$$

where

$$B = \left[\frac{107(1-\lambda)^2}{1152} + \frac{1}{228}\right] r^4$$

and

$$\mathrm{E} = \left[\frac{r^2}{96} + (1-\lambda)r^2 - \frac{r}{144} + \frac{(4-r^2)}{576}\right](4 - r^2),$$

by elementary calculations, we obtain $\mathcal{J}(r)$ is an increasing function of r. Therefore, the researcher gets the maximum of $\mathcal{J}(r)$ on $r = 2$ and

$$\max \mathcal{J}(r) = \mathcal{J}(2) = (1-\lambda)^2 \left[\frac{107}{72}(1-\lambda)^2 + \frac{4}{57}\right].$$

This completes the proof.

Theorem 2. If $\hbar(z) \in \mathcal{S}(\lambda). 0 < \lambda \leq 1$.So, we have

$$|e_2 e_3 - e_4| \leq \begin{cases} 8(1-\lambda)\left[\dfrac{43(1-\lambda)^2}{144} + \dfrac{1}{1444}\right] & . \ n \leq r \leq 2 \\ \dfrac{1}{18}(1-\lambda) & . \ 0 \leq r \leq n. \end{cases} \quad (29)$$

where

$$n = \frac{m_3 \pm \sqrt{m_3^2 - 12 m_2 (m_1 - m_2)}}{3(m_1 - m_2)},$$

$$m_1 = (1-\lambda)\left[\frac{43(1-\lambda)^2}{144} + \frac{1}{144}\right],$$

$$m_2 = (1-\lambda)\left[\frac{29(1-\lambda)}{168} + \frac{3}{144}\right]$$

and

$$m_3 = \frac{1}{72}(1-\lambda),$$

Proof. From (22), (24) and (25), we obtain

$$|e_2 e_3 - e_4| = \left|\frac{172(1-\lambda)^3 r_1^3}{576} - \frac{29(1-\lambda)^2 r_1 (r_2 - \mathrm{K}_2)}{168} - \frac{(1-\lambda)(r_3 - \mathrm{K}_3)}{72}\right|$$

Depending on Lemma 2, we assume with no limitations, $r \in [0.2]$, where $r_1 = r$, thus for $\tau = |x| \leq$ 1 and
$\epsilon = |y| \leq 1$, we have

$$|e_2 e_3 - e_4| \leq D_1 + D_2(\tau + \epsilon) + D_3(\tau^2 + \epsilon^2) = D(\tau . \epsilon),$$

where

$D_1(\lambda.r) = (1-\lambda)r^3\left[\frac{172(1-\lambda)^2}{576} + \frac{1}{144}\right] \geq 0,$

$D_2(\lambda.r) = (1-\lambda)(4-r^2)r\left[\frac{29(1-\lambda)}{336} + \frac{1}{144}\right] \geq 0$

and

$D_3(\lambda.r) = (1-\lambda)(4-r^2)\left[\frac{r}{288} + \frac{1}{144}\right] \geq 0 .$

Depending on the exact method of Theorem 2, therefore extreme happens in$\tau = 1$ and $\epsilon = 1$ in closed square [0.2],

$$\varphi(r) = \max\big(D(\tau.\epsilon)\big) = D_1 + 2(D_2 + D_3) .$$

By replacing the value of $D_1.D_2$ and D_3 in $\varphi(r)$, we obtain

$$\varphi(r) = m_1 r^3 + m_2 r(4-r^2) + m_3(4-r^2),$$

where

$m_1 = (1-\lambda)\left[\frac{43(1-\lambda)^2}{144} + \frac{1}{144}\right],$

$m_2 = (1-\lambda)\left[\frac{29(1-\lambda)}{168} + \frac{3}{144}\right]$

and

$m_3 = \frac{1}{72}(1-\lambda).$

We have

$$\varphi'(r) = 3(m_1 - m_2)r^2 - 2m_3 r + 4m_2,$$

$$\varphi''(r) = 6(m_1 - m_2)r - 2m_3,$$

if $m_1 - m_2 > 0$, that is $m_1 > m_2$. Therefore, we get $\varphi'(r) > 0$. Thus, $\varphi(r)$ is considered as a growing function on the closed interval [0,2] and so and so $\varphi(r)$ get the high prominence at $r = 2$, as it clarified below:

$$|e_2 e_3 - e_4| \leq \varphi(2) = 8(1-\lambda)\left[\frac{43(1-\lambda)^2}{144} + \frac{1}{144}\right],$$

if $m_1 - m_2 < 0$, let $\varphi'(r) = 0$. Then we get

$$r = n = \frac{m_3 \pm \sqrt{m_3^2 - 12m_2(m_1 - m_2)}}{3(m_1 - m_2)}.$$

If $n < r \leq 2$. So, it is got $\varphi'(r) > 0$, this by itself implies one thing [0.2] is the closed interval function. Then, the maximum value is for $\varphi(r)$, at $r = 2$. This signifies one thing, that this kind of function is a diminishing one on the closed interval [0.2], consequently, $\varphi(r)$ has the highest value at $r = 0$. We have

$$|e_2 e_3 - e_4| \leq \varphi(0) = \frac{1}{18}(1 - \lambda).$$

The proof is confirming.

Theorem 3. If $h(z) \in \mathcal{S}(\lambda). 0 < \lambda \leq 1$. there are

$$|e_3 - e_2^2| \leq \frac{1}{6}(1 - \lambda), \tag{30}$$

$$|e_3| \leq (1 - \lambda)^2 + \frac{1}{6}(1 - \lambda). \tag{31}$$

It is confirmed by adapting (24) and using Lemma 1, we get (31).

Below is Fekete-Szegö functional, for α ∈ ℂ and $h \in \mathcal{S}(\lambda)$,

$$e_3 - \alpha e_2^2 = \frac{(1-\lambda)^2 r_1^2}{4}(1 - \alpha) + \frac{(1-\lambda)(r_2 - \mathrm{K}_2)}{24}.$$

By Lemma 1), we obtain

$$|e_3 - \alpha e_2^2| \leq (1 - \lambda)^2(1 - \alpha) + \frac{1}{6}(1 - \lambda),$$

for $\alpha = 1$. we obtain (30).

Theorem 4. If $\hbar(z) \in \mathcal{S}(\lambda). 0 < \lambda \leq 1$. So, we have

$$|e_4| \leq (1-\lambda)\left[\frac{179}{72}(1-\lambda)^2 + \frac{5}{3}(1-\lambda) + \frac{1}{6}\right], \qquad (32)$$

$$|e_5| \leq (1-\lambda)^2\left[\frac{383}{21}(1-\lambda)^2 + \frac{677}{105}(1-\lambda) + \frac{13}{21}\right] + \frac{2}{35}(1-\lambda)\ . \quad (33)$$

It is Proofed from (25) and by Lemma 1, we obtain (32).

By remove (17) from (21), we get

$$68e_5 = 480e_2e_4 + 136e_3e_2^2 + 240e_{3+}^2 984e_2^4 - 1680e_2e_3 + (1-\lambda)(r_4 - \mathrm{K}_4),$$

by removed (22), (24) and (25), we get

$$140e_5 = \frac{5745}{36}(1-\lambda)^4 r_1^4 + \frac{677}{12}(1-\lambda)^3 r_1^2(r_2 - \mathrm{K}_2) + \frac{10}{3}(1-\lambda)^2 r_1(r_3 - \mathrm{K}_3)$$
$$+\frac{5}{12}(1-\lambda)^2(r_2 - \mathrm{K}_2)^2 + (1-\lambda)(r_4 - \mathrm{K}_4),$$

by applying Lemma 1, we get (33).

Theorem 5. If $\hbar(z) \in \mathcal{S}(\lambda). 0 < \lambda \leq 1$. So, there are

$$|H_3(1)| \leq \begin{cases} \mathcal{N}\mathcal{N}_1 - \mathcal{N}_2\left(8(1-\lambda)\left[\frac{43(1-\lambda)^2}{144} + \frac{1}{1444}\right]\right) + \mathcal{N}_3\mathcal{N}_4. & n \leq r \leq 2 \\ \mathcal{N}\mathcal{N}_1 - \frac{1}{18}(1-\lambda)\mathcal{N}_2 + \mathcal{N}_3\mathcal{N}_4. & 0 \leq r \leq n, \end{cases} \qquad (34)$$

where $\mathcal{N}.\mathcal{N}_1.\mathcal{N}_2.\mathcal{N}_3$ and $\mathcal{N}_4$ are given by (31), (7), (32), (33), and (30) respectively.

Proof. Since

$$|H_3(1)| = e_3(e_2e_4 - e_3^2) - e_4(e_4 - e_2e_3) + e_5(e_3 - e_2^2),$$

by using the triangle inequality, we obtain (3).

Replacement $|e_3| \le (1-\lambda)^2 + \frac{1}{6}(1-\lambda)$,

$$|e_2e_4 - e_3^2| \le (1-\lambda)^2\left[\frac{107}{72}(1-\lambda)^2 + \frac{3}{4}(1-\lambda) + \frac{5}{36}\right],$$

$$|e_4| \le (1-\lambda)\left[\frac{179}{72}(1-\lambda)^2 + \frac{5}{3}(1-\lambda) + \frac{1}{6}\right],$$

and

$$|e_3 - e_2^2| \le \frac{1}{6}(1-\lambda)$$

in

$$|H_3(1)| \le |e_3||e_2e_4 - e_3^2| - |e_4||e_4 - e_2e_3| + |e_5||e_3 - e_2^2|.$$

We obtain (34).

The proof is complete.

3- Conclusion

3rd Hankel determinant ($H_3(1)$ is the center of this study. It is for a bi-univalent functions particular subclass, $\mathcal{S}(\lambda)$. It has a great importance in different branches of math, including engineering theory and complex analysis. The researcher determines the bi-univalent functions $\mathcal{S}(\lambda)$ as well as put constrains (bonds) on the grade $|e_n|$. The results that have been reached the top boundaries of the bi-univalent functions of the updated subclass, particularly for n=2,3,4, and 5. In addition, this new vision develops understanding these functions by derecognizing 3rd Hankel for that specific category. All this reveals many interested studies. This accomplishment leads to the prove of 3rd Hankel determinant's bound which is for bi-univalent functions class $\mathcal{S}(\lambda)$. The current research highlights further comprehension of the bi-univalent functions for this subclass and their possible use in different mathematical field. Moreover, the outcomes of the current research prepare the job for further researches and development about bi-univalent functions and their subclass in the future. The coming efforts in the future might open insight towards more boundaries' modifications and progress. In addition, the future studies might tackle other subclass of bi-univalent functions, specially those subclass which are never tackled before, their features and possible usages.

In the end, the current study prepares for many other studies and investigations concerning the wonderful world of bi-univalent advantages in multiple mathematical fields.

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