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The Best Multiplier approximation of k-Monotone of $f \in L_{p,\lambda_n}(\mathrm{X})$

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Abstract The aim of this paper is to obtain of the degree of the best multiplier approximation unbounded monotone functions $f \in L_{p,\lambda_n}$ $[-1,1]$ in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{p,\lambda_n}$.

Keywords: Multiplier Integral, Multiplier Averaged Modulus of Smoothness, Multiplier Norm.

1. Introduction

Many researchers and specialists have worked in the field of approximation theory for example; In 1995 [1] , Kopotun, Kirill A. , introduced a paper on k −monotone polynomial and spline approximation L_p , $0 < \mathrm{p} < \infty$ quasi norm. Also, 2001 [2], Kopotun, K.A., had studied and got several results about approximations of bounded functions in $L_p(X)$ −space, where $X = [a, b]$,and $p \geq 1$ by utilizing Whitny's theorem. In 2004 [3] N.M.Kassim had studied the monotone and comonotone approximation. . In 2013 [4] , Eman Samir Bhaya and Munther Salman Al-Lami have obtained the degree of comonotone polynomials approximation of continuous functions f in $L_p[-1,1]$ −space. In 2014 [5] , Hadi, J.M., obtained some results of Bivariate monotone and comonotone approximation of function. In 2015 [6] Saheb Al-Saidy and Noor Saad have studied k −monotone approximation of unbounded functions in $L_{p,w_\alpha} -$ space s. In our research we will find degree of the best multiplier approximation unbounded monotone functions, $f \in L_{p,\lambda_n} - space$.

2. Definitions and Concepts

Definition (2.1) [8]

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent series if there is a convergent sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} a_n \lambda_n \leq \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is called multiplier for the convergence.

Definition (2.2)

For any real valued function $f \in L_{p,\lambda_n}(X)$, where $X = [-1,1]$,if there is a sequence $\{\lambda_n\}_{n=0}^{\infty}$,such that:

$$\int_{-1}^{1} f(x)\lambda_n dx < \infty, \qquad (2.1)$$

then f is called a Multiplier integrable function, λ_n,is called a Multiplier integrable sequence.

Definition (2.3)

A. [7] Let $f \in L_p[a,b]$, where $1 \leq p \leq \infty$, be the space of all bounded funaction with the norm

$$\| f \|_p = [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} < \infty \qquad (2.2)$$

B. Let $f \in L_{p,\lambda_n}(X)$,where $X = [-1,1]$ then: $\| f \|_{p,\lambda_n}$,is given by the below definite Multiplier integral norm :

$$\| f \|_{p,\lambda_n} = [\int_{-1}^{1} |(\lambda_n f)(x)|^p dx]^{\frac{1}{p}}. \qquad (2.3)$$

Definition (2.4)

A.[7] Let $f \in L_p[a,b]$, where $1 \le p \le \infty$, then the integral modulus(L_p -modulus or p -modulus) of order k of the function f is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f;\delta)_{L_p} = \sup_{0\le h\le\delta}\{\int_a^{b-kh} \left|\Delta_h^k f(x)\right|^p dx\}^{1/p} \qquad (2.4)$$

B. The Multiplier integral modulus of order k of the function $f \in L_{p,\lambda_n}(X)$, where $X = [-1,1], 1 \le p < \infty$ is defined by:

$$\omega_k(f,\delta)_{p,\lambda_n} = \sup_{h\in[0,\delta]} (\int_a^{b-kh} | \Delta_h^k (\lambda_n f)(x)|^p dx)^{\frac{1}{p}},\ 0 \le \delta \le b - ak, \qquad (2.5)$$

where

$$\Delta_h^k (\lambda_n f)(x) = \sum_{m=i}^{k} (-1)^{m+k} \binom{k}{m} (\lambda_n f)(x + mh);\ \binom{k}{m} = \frac{k!}{m!(k-m)!}. \qquad (2.6)$$

Definition (2.5)

A.[7] Let $f \in L_p(X)$; where $X = [a,b]$ and $1 \le p \le \infty$. The local modulus of smoothness of the function f of order k at a point $x \in [a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f,x;\delta) = \sup\{\left|\Delta_h^k f(t)\right| : t, t+kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}] \bigcap [a,b]\} \qquad (2.7)$$

B. The multiplier local modulus of smoothness of a function f of order k at a point $x \in [a,b]$,

$$0 \le \delta \le \frac{b-a}{k},$$

is defined by:

$$\omega_k(f,x,\delta)_{p,\lambda_n} = \sup_{h\in[0,\delta]} \{\Delta_h^k (\lambda_n f)(t) : t, t + kh \in [x - \tfrac{k\delta}{2}, x + \tfrac{k\delta}{2}] \cap [a,b]\}. \qquad (2.8)$$

Definition (2.6)

A. [7] The averaged modulus of smoothness of order k (or τ -modulus) of the function $f \in M[a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\tau_k(f;\delta)_p = \left\|\omega_k(f,.,;\delta)\right\|_{L_p} = [\int_a^b (\omega_k(f,x;\delta))^p dx]^{1/p} \qquad (2.9)$$

B. The multiplier averaged modulus of smoothness of order k of $f \in L_{p,\lambda_n}(X)$, where $X = [-1,1]$, is defined by:

$$\tau_k(f,\delta)_{p,\lambda_n} = \| \omega_k(f,.,\delta) \|_{p,\lambda_n} = (\int_a^b [\omega_k(\lambda_n f,x,\delta)]^p dx)^{\frac{1}{p}}. \qquad (2.10)$$

Definition (2.7)[9]

Let $f \in L_{p,\lambda_n}(X)$,where $X = [-1,1]$, then the Ditzian-Totic moduli of smoothness of the function f is defined by:

$$\omega_{k,\phi(x)}(f,\delta)_{p,\lambda_n} = \sup_{h\in[0,\delta],\delta>0} \| \Delta^k_{h\phi(.)}(f,.,X)\|_{p,\lambda_n}, \quad (2.11)$$

where

$$\phi(x) = \sqrt{1-x^2}, x \in [-1,1]. \quad (2.12)$$

Then $\Delta^k_{h\phi(x)} f(x)$ is given by the below finite summation:

$$\Delta^k_{h\phi(x)} f(x) = \sum_{i=0}^k (-1)^{k-i}\binom{k}{i}[\lambda_n f(x+(i-\frac{k}{2})h\phi(x)], \quad (2.13)$$

provided that:

$$x - \frac{k}{2}h\phi(x), x + \frac{k}{2}h\phi(x) \in [-1,1]. \quad (2.14)$$

Definition (2.8)

A.[7] Let $f \in L_{p,\lambda_n}(X), X = [a,b]$, then:

$$E_n(f)_p = \inf\{\|f - P_n\|_p : P_n \in P\} \quad (2.15)$$

Such that $E_n(f)_p$ is called the degree of the best monotone multiplier approximationof f by polynomial P_n.

B. Let $f \in L_{p,\lambda_n}(X), X = [-\mathbf{1},\mathbf{1}]$, then:

$$E_n(f)_{p,\lambda_n} = \inf\{\|f - S_n\|_{p,\lambda_n} : S_n \in P\} \quad (2.16)$$

Such that $E_n(f)_{p,\lambda_n}$ is called the degree of the bestmonotone multiplier approximationof f by polynomial S_n.

Definition (2.9)

A.[7] Let $f \in L_p(X)$, the best one-sided approximation of f by means of trigonometric polynomails of order n in $L_p(X)$ is given by:

$$\tilde{E}_n(f)_p = \inf\{\|P-Q\|_{L_p} : P,Q \in T, Q(x) \le f(x) \le P(x); \forall x\} \quad (2.17)$$

B. Let $f \in L_{p,\lambda_n}(X)$, $X = [-\mathbf{1},\mathbf{1}]$, then:

$$\tilde{E}_n(f)_{p,\lambda_n} = \inf\{\|S_n - G_n\|_{P,\lambda_n} : S_n, G_n \in T, G_n(x) \le f(x) \le S_n(x); \forall x\} \quad (2.18)$$

Such that $\tilde{E}_n(f)_{p,\lambda_n}$ is called the degree of the best one-sided monotone multiplier approximation of f by polynomials S_n and G_n.

Definition (2.10)[11]

A funtion $f: [a,b] \longrightarrow \mathbb{R}$ is said to be $k-$monotone, $k \ge 1$, on $[a,b]$ if and only if for all choices of $(k+1)$ distinct, $x_0, x_1, \dots, x_k$, in $[a,b]$ the inequality

$$[x_0, x_1, \dots, x_k]f \ge 0, \quad (2.19)$$

holds. Where

$$[x_0, x_1, \dots, x_k]f = \sum_{j=0}^k \left(\frac{f(x_j)}{m'(x_j)}\right), \quad (2.20)$$

denotes the k-th divided difference of the funtion f at $x_0, x_1, \dots, x_k$, and

$$m(x) = \prod_{j=0}^k (x - x_j). \quad (2.21)$$

Moreover, let $f \in L_{p,\lambda_n}, 1 \le p \le \infty$, f is any real valued function and let

$$\Omega = X = [-1,1] \subset \mathbb{R}.$$

Define $N^\star(\delta,x)$ to be:

$$N^\star(\delta,x) = \delta N(x) + \delta^2, \quad (2.22)$$

where $N(x)$ is given by the piecewise function:

$$N(x) = \begin{cases} (1-x^2)^{\frac{1}{2}} & if\ \Omega=[-1,1] \\ x(1-x^2) & if\ \Omega=[0,1] \end{cases}. \quad (2.23)$$

Assume that

$$Z = \{0,1,2,\dots,N-1\}. \quad (2.24)$$

$$z_\nu = \cos\frac{(\pi - \pi\nu)}{N};\ \nu = 1,2,\dots,N. \quad (2.25)$$

Suppose that

$$z_{-1} = z_0 = -1; z_{N+1} = z_N = 1. \quad (2.26)$$

For $j \in Z$, put:

$$\Omega_j = [z_j, z_{j+1}]. \quad (2.27)$$

For $\nu = 0,1,2,\dots,N-1$. Put:

$$u_\nu = \pi - \frac{(2\nu+1)\pi}{2N}, \quad (2.28)$$

and

$$\phi_\nu = \phi_\nu(u) = \sin^4\left(\frac{\pi}{4N}\right)\left\{\frac{[\sin^4 N(u-u_\nu)]}{\left[\frac{\sin^4(u-u_\nu)}{2}\right]} + \frac{[\sin^4 N(u+u_\nu)]}{\left[\frac{\sin^4(u+u_\nu)}{2}\right]}\right\}. \quad (2.29)$$

Thus ϕ_ν are all even trigonometric polynomials of degree $4N-2$ and $\phi_\nu(u) \geq 1$ for $u \in \left[\pi - \left(u + \frac{\nu\pi}{N}, \pi - \frac{\nu\pi}{N}\right)\right]$. Now let

$$u = \arccos\nu;\ \nu \in [-1,1], \quad (2.30)$$

then

$$F_\nu(u) = F_{\nu,N}(u) = \phi_\nu(\arccos\nu). \quad (2.31)$$

By Jensen inequality, and for $\alpha_j \geq 0, j \in Z$, it is obtained that:

$$\| \sum_{j\in Z} \alpha_j \phi_{j,N} \|_{p(\Omega)} \leq c\left[\sum_{j\in Z} \alpha_j^p meas\ \Omega_j\right]^{\frac{1}{p}}. \quad (2.32)$$

Definition (2.11)[10]

Let R_n be an algebraic monotone polynomial which is the best approximation polynomial of $f \in L_{p,\lambda_n}$ such that for $N = \frac{n}{4}$

$$Q_n^{\pm}(f,x) = R_n(x) \pm \sum_{j\in Z} \phi_{j,m}(x) \| f(x) - R_n(x) \|_{\infty(\Omega_j)}. \quad (2.31)$$

It is clear that $Q_n^{\pm}(f,x)$ are algebraic monotone polynomial of degree less than or equal to n.

In the next section, significant lemmas will be proved.

3. Necessary Lemmas

Lemma (3.1)[10]

Let $f \in L_{p,\lambda_n}(X), X = [-1,1]$, then $Q_n^-(f) \leq f \leq Q_n^+(f)$. (3.1)

Proof

Start the proof with the below algebraic equation given by:

$$Q_n^+(f,x) = R_n(x) + \sum_{j\in Z} \phi_{j,m}(x) \| f(x) - R_n(x) \|_{\infty(\Omega_j)},$$

$$Q_n^+(f,x) \geq R_n(x) + \| f(x) - R_n(x) \|_{\infty(\Omega_j)},$$

$$Q_n^+(f,x) \geq R_n(x) + |f(x) - R_n(x)|,$$

$$Q_n^+(f,x) \geq R_n(x) + f(x) - R_n(x),$$

and then

$$Q_n^+(f,x) \geq f(x). \quad (3.2)$$

Then $Q_n^+(f,x) = f(x)$. Similarly,

$$Q_n^-(f,x) = R_n(x) - \sum_{j\in Z} \phi_{j,m}(x) \| f(x) - R_n(x) \|_{\infty(\Omega_j)},$$

$$Q_n^-(f,x) \leq R_n(x) - \| f(x) - R_n(x) \|_{\infty(\Omega_j)},$$

$$Q_n^-(f,x) \leq R_n(x) - |f(x) - R_n(x)|,$$

$$Q_n^-(f,x) = R_n(x) - f(x) - R_n(x),$$

and then

$$Q_n^-(f,x) = -f(x) \leq f(x) \Rightarrow Q_n^-(f,x) \leq f(x). \quad (3.3)$$

From (3.2) and (3.3) we get:

$$Q_n^-(f) \leq f \leq Q_n^+(f).$$

Lemma (3.2)

Let $f \in L_{p,\lambda_n}(X), X = [-1,1]$,and $f \in \Delta^2$ (x) then: $\omega_{2,\phi}(f,\delta)_{1,\lambda_n} \leq C\delta^2 \parallel f \parallel_{1,\lambda_n}$; C is constant. (3.4)

Proof.

By using definition (2.7)

$$\omega_{k,\phi}(f,\delta)_{p,\lambda_n} = \sup_{h\in(0,\delta]} \parallel \Delta_{h\phi}^k (f,X) \parallel_{p,\lambda_n} \quad (3.5)$$

Where $k = 2, p = 1$.

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \sup_{h\in(0,\delta]} \parallel \Delta_{h\phi(.)}^2 (f,.,X) \parallel_{1,\lambda_n},$$

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \sup_{h\in(0,\delta]} \int_{-1}^{1} | \Delta_{h\phi}^2 f(x)\lambda_n(x)|dx,$$

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \int_{-1}^{1} \left(\sum_{i\in[0,2]} \binom{2}{i}\right)\left(-1)^{2-i}\lambda_n f\big(x - h\phi(x) + ih\phi(x)\big)\big(x - h\phi(x) + ih\phi(x)\big)\right) dx.$$

And

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \int_{-1}^{1} \{((\tbinom{2}{0})(-1)^2\lambda_n f(x - h\phi(x))(x - h\phi(x)))$$

$$+((\tbinom{2}{1})\left(-1)^{2-1}\lambda_n f\big(x - h\phi(x) + h\phi(x)\big)\big(x - h\phi(x) + h\phi(x)\big)\right)$$

$$+((\tbinom{2}{2})(-1)^{2-2}\lambda_n f(x - h\phi(x) + 2h\phi(x))(x - h\phi(x) + 2h\phi(x)))\}dx,$$

then

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \int_{-1}^{1} \{(\lambda_n f(x - h\phi(x))(x - h\phi(x))$$

$$-2(\lambda_n f)(x)(x + \lambda_n f(x + h\phi(x))(x + h\phi(x)))\}dx. \quad (3.6)$$

By Whitney's theorem [7](for any continuous function f on [a,b] and for each integer n $\geq$. There is a number w_n and a polynomial P of degreen $n - 1$ such that :
$|f(x) - P(x)| \leq w_n\omega_n(f,[a,b])$, Where w_n is Whitney's constant) , we get:

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} = \int_{-1}^{1-\frac{3h^2}{1}+h^2} (f\lambda_n)(y)\frac{1}{1+h^2}(1 - yh(h^2 + 1 - y^2)^{-\frac{1}{2}})dy$$

$$-2\int_{-1+\frac{h^2}{1}+h^2}^{1-\frac{h^2}{1}+h^2} (\lambda_n f)(y)dy$$

$$+\int_{-1+\frac{3h^2}{1}+h^2}^{1} (\lambda_n f)\, y)\frac{1}{1+h^2}(1 + yh(h^2 + 1 - y^2)^{-\frac{1}{2}})dy. \quad (3.7)$$

Then

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} \leq Ch^2 \parallel f(.) \parallel_{1,\lambda_n} + \int_{-1+\frac{3h^2}{1}+h^2}^{1-\frac{3h^2}{1}+h^2} \frac{2h^2}{1+h^2}|(\lambda_n f)(y)|dy. \quad (3.8)$$

Finally,

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} \leq Ch^2 \parallel f(.) \parallel_{1,\lambda_n}. \quad (3.9)$$

Since $h^2 \leq \delta^2$, then

$$\omega_{2,\phi}(f,\delta)_{1,\lambda_n} \leq C\delta^2 \parallel f \parallel_{1,\lambda_n}.$$

The proof is completed.

Lemma (3.3)

Let $f, f^{(k)} \in L_{p,\lambda_n}(X), X = [-1,1]$, $p \geq 1$, then: $\tau_k(f,\delta)_{p,\lambda_n} \leq c_k \delta^k \parallel f^{(k)} \parallel_{p,\lambda_n}$.

Proof
To show that holds, start with the below inequality:

$$\tau_k(f;\delta)_{p,\lambda_n} \leq \delta \tau_{k-1}((\lambda_n f)', k(k-1)\delta)_p,$$

$$\leq \delta^2 \tau_{k-2}((\lambda_n f)'', (k-1)(k-2)\delta)_p,$$

$$\vdots$$

$$\vdots$$

$$\leq \delta^{k-1} \tau_1((\lambda_n f)^{(k-1)}, \delta)_p.$$

From

$$\tau_1(f;\delta)_{p,\lambda_n} \leq \delta \parallel f' \parallel_{p,\lambda_n},$$

then:

$$\tau_1(f;\delta)_{p,\lambda_n} \leq \delta^{k-1} \delta \parallel f^{(k)} \parallel_{p,\lambda_n},$$

$$= C_k \delta^k \parallel f^{(k)} \parallel_{p,\lambda_n},$$

and then:

$$\tau_k(f;\delta)_{p,\lambda_n} \leq C_k \delta^k \parallel f^{(k)} \parallel_{p,\lambda_n}. \quad (3.16)$$

The proof is completed.

Lemma (3.4)

If $f \in L_{p,\lambda_n}(X), X = [-1,1]$,then:

$$E_n(f)_{p,\lambda_n} \leq \tilde{E}_n(f)_{p,\lambda_n}. \quad \textit{(3.17)}$$

Proof.
Let θ is the best multiplier approximation of the function f and suppose that θ_1 and θ_2 are best one-sided multiplier approximations of f such that:

$$\theta_1 \leq f \leq \theta_2. \quad (3.18)$$

$$E_n(f)_{p,\lambda_n} = \parallel f - \theta \parallel_{p,\lambda_n} = \left(\int_X |(f-\theta)\lambda_n|^p dx\right)^{\frac{1}{p}};\ p \geq 1,$$

$$\leq \left(\int_X |(\theta_2 - \theta_1)\lambda_n|^p dx\right)^{\frac{1}{p}} = \parallel \theta_2 - \theta_1 \parallel_{p,\lambda_n} = \tilde{E}_n(f), \quad (3.19)$$

Finally,

$$E_n(f)_{p,\lambda_n} \leq \tilde{E}_n(f)_{p,\lambda_n}.$$

The proof is done.

4. Main Results

We introduce our main theorems in what follows:

Theorem (4.1)

For $f \in L_{p,\lambda_n}$, and for f is monotone, $1 \le p \le \infty$, then:

$$\widetilde{E}_n(f)_{p,\lambda_n} \le \tau(f,\delta)_{p,\lambda_n}. \quad (4.1)$$

Proof

Starting with the below inequality, $Q_n^{\pm}(f,x)$ are algebraic monotone polynomial of degree $\le n$

$$\tilde{E}_n(f)_{p,\lambda_n} \le \| Q_n^+ - Q_n^- \|_{p,\lambda_n} = 2 \| \sum_{j\in Z} \phi_{j,N}(x) \| f - R_n \|_{\infty(\Omega_j)} \|_p,$$

$$\le c[\sum_{j\in Z} meas\ \Omega_j \| f - R_n \|^p_{\infty(\Omega_j)}]^{\frac{1}{p}},$$

$$\le c[\sum_{j\in Z} \int_{\Omega_j} \| f - R_n \|^p_{\infty(\Omega_j)}\ dx]^{\frac{1}{p}},$$

$$\le c[\int_{\Omega_j} \| f - R_n \|^p_{\infty(N(x,\delta))}\ dx]^{\frac{1}{p}},$$

$$\le c[\int_{\Omega_j} \sup |f(x_j) - f(x_{j+1})|^p dx]^{\frac{1}{p}} = C_p \tau(f,\delta)_{p,\lambda_n}. \quad (4.2)$$

The proof is completed .

Notice that from lemma 3.4 and the theorem above **,** we get: $E_n(f)_{p,\lambda_n} \le \tau(f,\delta)_{p,\lambda_n}$.

Theorem (4.2)

For $f \in L_{p,\lambda_n}$,then $\tau(\delta,\frac{1}{n})_{p,\lambda_n} \le \frac{c}{n}\sum_{k=1}^{n} E_k(f)_{p,\lambda_n}$. (4.8)

Proof

Starting with the below inequality:

$$\tau_2(f,n^{-\frac{1}{2}})_{p,\lambda_n} \le \tau_2(f - R_n, n^{-\frac{1}{2}})_{p,\lambda_n} + \tau_2(R_n, n^{-\frac{1}{2}})_{p,\lambda_n}. \quad (4.9)$$

By lemma (3.3) we get:

$$\tau_2(f,n^{-\frac{1}{2}})_{p,\lambda_n} \le c \| f - R_n \|_{p,\lambda_n} + c \| \lambda_n R'' \|_{p,\lambda_n},$$

$$\le \frac{c}{n}\sum_{\frac{n}{2}\le k\le n} \| R_k - f \|_{p,\lambda_n} + \frac{c}{n}\sum_{k=1}^{n} \| R_k - f \|_{p,\lambda_n,}$$

$$\le \frac{c}{n}\sum_{k=1}^{n} \| R_k - f \|_{p,\lambda_n} = \frac{c}{n}\sum_{k=1}^{n} E_k(f)_{p,\lambda_n}. \quad (4.10)$$

The proof is completed.

5. Conclusion

The aim of this paper is to obtain of the degree of the best multiplier approximation unbounded monotone functions, $f \in L_{p,\lambda_n}(X), where\ X = [-\mathbf{1}, \mathbf{1}]$ in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{p,\lambda_n}$.

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