A New Kind of Topological Vector Space Via β-Approach Structure

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Abstract. The goal of this paper is to define β-distance on a non-empty set if it meets the conditions. A pair (X, β) is called β-approach space and we will also discuss solve various problems. The relationship between metric space and β-app-space is clarified. We define the β-contraction function and discuss some of its properties. The convergent sequence in β-approach space and sequentially convergent are discussed. We introduce the definition of β –semigroup, β –group in β –approach space, β –vector space and β –topological approach vector spaces. In addition, we identify corresponding between convergent and sequentially convergent with new results.

Keywords: distance, β-approach group, β-approach vector space, β-approach topological space.

1. Introduction

The concept of a topological vector space is central to modern functional analysis, and in recent years, applications in various other fields of mathematics have been studied in order to find and compare their properties. Approach space theory is important in quantitative domain theory; there are many examples of approach structure in functional analysis, measure theory, probability space, and approximation theory. As in the metric case. If an approach space is generated by a topological space, it is said to be "topological," and if it is generated by a metric space, it is said to be "metric." " The AP-product carries only that portion of the numerical data that is present," which can be retained if compatibility with the topological product of the family of underlying metric topologies is desired." The fundamental difference in existence There is a difference between approach and metric spaces. " in the fact that in an approach space, all the distances between the points are defined," where such a point-set distance does not have to bring the two together infimum over the considered set of all the point distances "As in the metric case, an approach space is defined. Lowen [13] found definition approach spaces were introduced in 1987. Lowen's monographs [14] can be used to set up an overall realization of approach spaces. The theory of approach spaces, a generalization of metric and topological spaces, is based on point-to-set distances rather than point-to-point distances. The most important motivation was to solve the problem of an infinite product of metric spaces. Another reason for introducing approach spaces is to unify metric, uniformity, topological, and convergence theories. Barn and Qasim [5, 6] characterized local distance-approach spaces, Approach spaces , and gauge-approach spaces and compared them with usual , approach spaces. Colebuders, Sion,… etc [1] show that some considerable consequences on real valued contractions. Martinez-Moreno1, Rpld'an2, …etc[3] found definition the concept of fuzzy approach spaces as spaces generalization of fuzzy metric spaces and demonstrate some Properties of fuzzy approach. Gutierres, Hofmann [2] calculated the concept of completeness for approach spaces and calculated some properties in completeness approach spaces. Van Opdenbosch [4] set up new isomorphic characterizations of approach spaces, pre-approach spaces , convergence approach spaces , uniform gauge spaces, topological spaces , and convergence spaces, pretopological spaces , metric spaces, and spaces that are consistent. Baekeland and Lowen [7] institute the measures of Lindelof and separability in approach spaces. Lowen and Verwulgen [14] institute define Approach vector spaces. Lowen and Windels [10] defined an approach groups spaces, semi-group spaces, and uniformly convergent. Lowen [16] detail of this book approach theory completely disband this by presentation properly those two new kinds of numerically form spaces which are wanted: approach spaces on the local level and uniform gauge spaces on the uniform level.

Lowen and Sion [12, 13] introduced the definitions of some separation axioms in the approach spaces and set up the correlation the axiom, the axiom, regular and completely regular and also calculated of normed linear spaces and from a normed real vector space $(X,\|\ \|)$,we a uniform approach structure on X Lowen, Van Olmen, …etc [17] introduced Functional Ideas and Topological Theories. Lowen and C. Van Olmen [11] explained some concepts and correlation in approach Theory. Lowen [15] studied the development of essential theory of approximation. Abbas and Hussein [9, 8] introduced topological approach space and found completion if the completeness is not satisfies. W. Li, Dexue Zhang [18] introduced the Smyth complete.

The goal of this paper is two - fold: first, we want to put approach group checking space in the proper perspective when approach vector spaces, and second, we want to use this topological approach structure, as we will call it, to create a canonical counterpart of the classical topological vector space. Both metric spaces and preorders are generalized in extended pseudo-quasi metric spaces.

This paper is divided into six sections: In Section 1, weintroduces the research and Preliminaries with basic definitions. In Section 2, we introduce new definition which is called β –distance and explains the relationship between metric space and -approach space; we proved that every metric space is -approach space but not the converse; and we proved that every symmetric - β -approach space is metric space. In Section 3, we demonstrated some properties of β -contractions. Section 4. We discuss convergent sequence in β –approach sepace with new results. Section 5 introduced the definitions of -approach group, -approach semi-group, -approach sub-group, and solved some examples in β -approach group, as well as introduced the definition of β -approach vector space and proved some examples in -approach vector space. Section 5 presented the definitions of topological vector space, -approach sub-space, and show that a new definition of convergent in -approach space, and sequentially - contraction

Definition 1.1[13]: Let X be a non-empty set. A function $\delta : X \times 2^X \to [0,\infty]$ is said to be distance on X if it satisfies the conditions:
(D1) $\forall m \in X: \delta(m,\{m\}) = 0$,
$(D2)$ $\forall m \in X: \delta(m,\emptyset) = \infty$,
$(D3)$ $\forall m \in X: \forall A,B \in 2^X : \delta(m, A \cup B) = \min\{\delta(m,A), \delta(m,B)\}$,
$(D4)$ $\forall m \in X: \forall A \in 2^X, \forall \varepsilon \in [0,\infty]: \delta(m,A) \leq \delta\left(m, A^{(\varepsilon)}\right) + \varepsilon$.
A pair (X, δ) is called an approach space and denoted by app. spaces.

2. Structre of β-Approach space

We benefit from the definition of distance in Lowen's paper in 1987 for a new definition:

Definition 2.1: Let X be a non-empty set. A function β: $2^X \times 2^X \to [0,\infty]$ is called β- distance on X if it is satisfies the conditions:

1) For all $M,N \in 2^X$, $if\ M \cap N \neq \emptyset$, then $\beta(M,N) = 0$
2) For all $M,N,P \in 2^X$, $\beta(M, N \cup P) = min\{\beta(M,N), \beta(M,P)\}$
3) $For\ all\ M,N \in 2^X, if\ M = \emptyset\ or\ N = \emptyset, then\ \beta(M,N) = \infty$
4) $For\ all\ M,N \in 2^X, \varepsilon \in [0\ \infty]\ and\ \curlyvee \in [0,\infty]$
$\beta(M,N) \leq \beta(M^{\varepsilon}, N\varepsilon) + \varepsilon + \curlyvee,\ where\ M^{\varepsilon} = \{x \in X: \beta(\{x\}, M) \leq \varepsilon\}$.

A pair (X, β) where β is a distance is called β-approach space and denoted by β-app. space.

Example 2.2: The discrete distance approach structure β on X is given as for all $x \in X$ and

$M \subseteq X$ and $N \subseteq X$: by $\beta(M, N) = \begin{cases} 0 & x \in M \cap N \\ \infty & x \notin M\ or\ x \notin N \end{cases}$

Proof:

1) If $x \in M$ and $x \in N$ that is $M \cap N \neq \emptyset$, then $x \in M \cap N$.

So $\beta(M,N) = 0, \beta(\{x\},N) = 0$ and $\beta(\{x\},M) = 0$

2) let $M, N \in 2^{x}$ .Such that If $M = \emptyset$ or $N = \emptyset \Rightarrow x \notin M$ or $x \notin N \Rightarrow \beta(M,N) = \infty$

If $x \notin M \Rightarrow \beta(\emptyset, N) = \infty$ or $x \notin N \Rightarrow \beta(M,\emptyset) = \infty$

3) For all $M,N,P \in 2^{x}$, $M \subseteq N$, $x \in (M \cap N) \cup (M \cap P) \Rightarrow x \in M \cap (N \cup P)$
$\beta(M, N \cup P) = 0 = min\{0,0\} = min\{\beta(M,N), \beta(M,P)\}$
If $x \notin M\ or\ x \notin N \cup P \Rightarrow \beta(M, N \cup P) = \infty$
$\beta(M, N \cup P) = min\{\infty,\infty\}$
$= min\{\beta(M,N), \beta(M,P)\}$

4) For all $M, N \in 2^{x}$, for all $\varepsilon \in [0,\infty]$, $\varepsilon \in [0,\infty]$

Where $M^{\varepsilon} = \{x \in X: \beta(\{X\}, M) \leq \varepsilon\}$

If $x \in M,\ x \in N\ then\ x \in M \cap N \Rightarrow M \cap N \neq \emptyset\ then\ \beta(M,N) = 0$

Then $\beta(M,N) \leq \beta(M^{\varepsilon}, N^{\varepsilon}) + \varepsilon + \curlyvee$

If $x \notin M\ or\ x \notin N\ then\ x \notin M^{\varepsilon}\ or\ x \notin N^{\curlyvee} \Rightarrow \beta(M,N) = \infty\ then\ \beta(M^{\varepsilon}, N^{\curlyvee}) = \infty$

$\beta(M,N) \leq \beta(M^{\varepsilon}, N^{\curlyvee}) + \varepsilon + \curlyvee = \infty$

Proposition 2.3: Let X be non-empty set and $\beta : 2^X \times 2^X \to [0,\infty]$ is distance on X. Then the following hold:

1) for all $M, N \in 2^X$, $x \in M$ then $\beta(M,N) = 0$
2) for all $M, N \in 2^X$, for all $x \in M$, $M \subset N$, then $\beta(\mathbb{A}, N) \leq \beta(\mathbb{A}, M)$
3) For all $M_i \in 2^X$, ψ is set, $\beta[\mathbb{A}, U\, Mi] = \sup\{ \min_{M_i \in \psi,\ M \in 2^X} \{\beta(\mathbb{A}, M_i)\}$

$M_i \in$,for all $M, N \in 2^X$, $\mathbb{A} \subset M$

4) $\beta(M,\mathbb{A}) \leq \beta(\mathbb{A}^{\varepsilon}, N^{\varepsilon}) + \{ sup_{b\in N\ \ N\in 2^X}\ \ sup_{c\in M\ \ M\in 2^X}\ \beta(\{b\},\{c\})\}$

Proof:

1)For all $M, N \in 2^X$, If $x \in M$ and $x \in N$ then $x \in M \cap N \Rightarrow \beta(M,N) = 0$

If $x \in M$ and $x \notin N \Rightarrow x \in M \cap N^c \Rightarrow \beta(M, N^c) = 0 \Rightarrow \beta(M,\{x\}) = 0$,$x \in N^c$

Similarly, $x \notin M$ and $x \in N$.

2) Let $M, N \in 2^X$, $M \subseteq N$, $\beta(\mathbb{A}, N) = \beta(\mathbb{A}, M \cup N) = \min\{\beta(\mathbb{A}, M), \beta(\mathbb{A}, N)\} \leq \beta(\mathbb{A}, M)$

3) let $\psi \in 2^X$, $\psi = \{M_1, \ldots, M_n\}$, $N \in 2^X$, ψ is finite

$\beta(N, \cup M_i) = min$
$\beta(N, M1)$,
$\beta(N, M1 \cup M2 \cup \ldots) = min(\beta(N,M1), \beta(N, M2 \cup M3 \cup \ldots) = min(\beta(N,M1), min\, \beta(N,M2), \beta(N, M3 \cup \ldots)$

$= \ldots = min\{(\beta(N,M1), \ldots, \beta(N,Mn),.\} = \min_{M\in\psi} \beta(N,M)$

4) $let\ M, \mathbb{A} \in 2^X$, $\varepsilon = inf\{\varepsilon \in [0\ \infty], N \subset \mathbb{A}\,\varepsilon\}$,

$$\beta(M,\mathbb{A}) \leq \beta(M,N) + \sup_{S\in 2^X} \sup_{\mathbb{A}\in 2^X} \beta(S,\mathbb{A})$$

$$\beta(M,\mathbb{A}) \leq \beta(M, \mathbb{A}\,\varepsilon) + \beta(\mathbb{A}\,\varepsilon, N) + \varepsilon + \gamma \leq \beta(M\,\varepsilon, N\,\varepsilon) + \varepsilon + \gamma$$

$$\leq \beta(M,N) + \sup_{S\subseteq N} sup\ \beta(S,\mathbb{A})$$

Proposition 2.4: Every metric space is β – app. space.

Proof: Let (X, d) be a metric space, we define: $\beta_d : 2^X \times 2^X \to [0,\infty]$ by

$$\beta_d(M,N) = \begin{cases} \infty & , M = \emptyset \text{ or } N = \emptyset \\ \inf_{x\in M} \inf_{y\in N} d(x,y), & M \neq \emptyset \text{ and } N \neq \emptyset \end{cases}$$

To prove β_d is distance on X

1) If $M = \emptyset$ or $N = \emptyset$ then $\beta_d(M,N) = \infty$

2) For all $M, N \in 2^X$, $M \cap N \neq \emptyset \Rightarrow x \in M$ and $x \in N$

$$\beta_d(M,N) = \inf_{x\in M} \inf_{x\in N} d(x,x) = \inf_{x\in M}\{0\} = 0$$

3) For all $M, N, P \in 2^X$

$$\beta_d(M, N \cup P) = \inf_{x\in M} \inf_{y\in N\cup P} d(x,y) = \inf_{x\in M}\{\min\{\inf_{y\in N} d(x,y), \inf_{y\in P} d(x,y)\}\}$$

$$= \min_{x\in M}\{\inf \inf_{y\in N} d(x,y), \inf_{x\in M} \inf_{y\in P} d(x,y)\} = \min\{\beta_d(M,N), \beta_d(M,P)\}$$

If $M = \emptyset$ or $N = \emptyset \Rightarrow \beta_d(M, N \cup P) = \infty$

If $M = \emptyset \Rightarrow \beta_d\ (M, N\ U\ P) = \infty = min\ \{\infty, \infty\} = min\{\beta_d\ (M, N), \beta_d\ (N, P)\}$

If $N = \emptyset \Rightarrow \beta_d\ (M, N\ U\ P) = \infty = min\ \{\infty, \infty\} = min\{\beta_d\ (M, N), \beta_d\ (N, P)\}$

4) For all $M, N \in 2^X$, for all $\varepsilon \in [o, \infty]$, where $M^\varepsilon = \{x \in X \mid \beta_d\ (\{x\}, M) \leq \varepsilon\}$

$$\beta_d\ (M, N) = \inf_{x\in N} \inf_{a\in M} d\ (x, a) \leq \inf_{x\in N} \inf_{a\in M} d\ (x, y) + \varepsilon + \gamma \leq \inf_{x\in N^\varepsilon} \inf_{a\in M^\varepsilon} d\ (x, y) + \varepsilon + \gamma$$

$$\leq \beta_d\ (M^\varepsilon, N^\varepsilon) + \varepsilon + \gamma$$

Then (X, β_d) is β −app. space. A pair (X, β_d) is said to be metric approach space.

Example 2.5: For all $M, N \in 2^{[0,\infty]}$

$$\beta\ (M, N) = \begin{cases} max\{\sup_{N\in 2^{[0,\infty]}} N - sup_{M\in 2^{[0,\infty]}}\ M, 0\} & if\ M \neq \emptyset\ and\ N \neq \emptyset \\ \infty & if\ M = \emptyset\ or\ N = \emptyset \end{cases}$$

Proof:

1) For all $M, N \in 2^{[0,\infty]}$ if $M \neq \emptyset\ and\ N \neq \emptyset, M \cap N \neq \emptyset$
$\Rightarrow x \in M\ and\ x \in N$

$\beta\ (M, N) = max\{\sup_{\substack{N\in 2^{[0,\infty]}\\ x\in N}} \{x\} - sup_{\substack{M\in 2^{[0,\infty]}\\ x\in M}}\{x\}, 0\} = 0$ or $sup\ M > supN \Rightarrow \beta\ (M, N) = 0$

2)If $M = \emptyset\ or\ N = \emptyset$

if $M=\emptyset$, $\beta\ (M, N) = max\{\sup_{N\in 2^{[0,\infty]}} N - sup_{\emptyset\in 2^{[0,\infty]}}\ \emptyset, 0\} = \infty = max\{\infty, 0\} = \infty$.

If $N = \emptyset$, $\beta\ (M, N) = max\{\sup_{\emptyset\in 2^{[0,\infty]}} \emptyset - sup_{M\in 2^{[0,\infty]}}\ M, 0\} = \infty = max\{0, \infty\} = \infty$.

Then, $\beta\ (M, N) = \infty$.

2) For all $M, N, P \in 2^{[0,\infty]}$,if $M \cap (NU\ P) \neq \emptyset$
$\beta\ (M, N\ U\ P) = max\{\sup_{x\in N\ U\ P} NU\ P - sup_{x\in M}\ M, 0\} = 0$

$= min\{max\{\sup_{x\in N} N - \sup_{x\in M} M, 0\}, max\{\sup_{x\in P} P - sup_{x\in M}\ M, 0\}\} = min\ \{\beta\ (M, N), \beta\ (M, P)\}$
If $M = \emptyset\ or\ N = \emptyset$
If $M = \emptyset$, $\beta\ (M, N\ U\ P) = max\{\sup_{x\in N\ U\ P} NU\ P - sup_{x\in\emptyset}\ M, 0\} = \infty$

$= min\{max\{\sup_{x\in N} N - \sup_{x\in\emptyset} M, 0\}, max\{\sup_{x\in P} P - sup_{x\in\emptyset}\ M, 0\} = min\ \{\beta\ (M, N), \beta\ (M, P)\}$

If $N = \emptyset$, $\beta\ (M, N\ U\ P) = max\{\sup_{x\in\emptyset\ U\ P} NU\ P - sup_{x\in M}\ M, 0\} = \infty$

$= min\{max\{\sup_{x\in\emptyset} N - sup_{x\in M}\ M, 0\}, max\{\sup_{x\in P} P - sup_{x\in M}\ M, 0\}\} = min\ \{\infty, \infty\}$

$= min\ \beta\ (M, N), \beta\ (M, P)$

3) For all $M, N \in 2^{[0,\infty]}$ and for all $\varepsilon \in [0, \infty]$, where $M^\varepsilon = \{x \in X \mid \beta\ (\{x\}, M) \leq \varepsilon\}$

If $M \neq \emptyset\ and\ N \neq \emptyset \Rightarrow M \cap N \neq \emptyset$, $x \in M$ and $x \in N$

$\beta\ (M, N) = max\{\sup_{x\in N}\{x\} - Sup_{a\in M}\{a\}, 0\} \leq max\{\sup_{x\in N}\{x\} - sup_{a\in M}\{a\}, 0\} + \varepsilon + \gamma$

$\leq \beta\ (M^\varepsilon, N^\varepsilon) + \varepsilon + \gamma$. Thus $\beta\ (M, N)$ is β –app. space.

$d_\beta(x,y) = \beta(\{x\},\{y\}) \neq \beta(\{y\},\{x\}) = d_\beta(y, x)$. Therefore X is not metric space

Example 2.6 : Define $\beta_{\breve{Ȩ}}(M,N) = \begin{cases} 0 & M\cap N \neq \emptyset, M,N \text{ unbounded} \\ \infty & M\cap N = \emptyset, M,N \text{ bounded} \\ inf \; inf_{y\in N}|x-a| & M<\infty, \quad N<\infty \end{cases}$

$\breve{Ȩ}$ =[0, ∞]

Proof: it is clear (X,β) is an β-app-space.

Definition 2.7: Let (X,β) be A β – app- space. We say that (X,β) is symmetric if β(M,N) = β(N,M) for all $M,N \in 2^X$.

Proposition 2.8 : Every symmetric β – app. space is metric space.

Proof: Let (X, β) is symmetric β – app. space.

$d_\beta(x,y)$ =β($\{x\},\{y\}$)

It is clear (X, β) is metric space generated by β – app- space.

Proposition 2.9: Let (X, β_1), (X, β_2) be β −app. space, then ($X\times X$, β) is β −app- space. Where:

β (($M\times N$), ($S\times T$)) = $min\,\{\beta_1(\mathrm{M},S),\beta_2\,(N,T)\}$, for all $S,T \in 2^X$, for all $M,N \in 2^X$

Proof:

1) for all $x,y\in X$ and for all $S,T \in 2^{\times}$, for all $M,N \in 2^{\times}$
 $(M\times N)\cap(S\times T) \Rightarrow (x,y) \in (M\times N)$ and $(x,y)\in(S\times T)$.
 Then, $x \in M\cap S$ and $y\in N\cap T$

 β ($M\times N,S\times T$) = $min\,\{\beta_1(\mathrm{M},S),\beta_2\,(N,T)\}$, $= min$ {(0, 0)} = 0
2) If $S=\emptyset$ or $T=\emptyset$

β (($M\times N$), ($S\times T$) = β (($M\times N$) , $\emptyset\times\emptyset$) = $min\,\{\beta_1\,(M,\emptyset\,),\beta_2\,(N,\emptyset)\}$

$= min\,\{\,\infty,\infty\} = \infty$

3) $\beta(\,(M\times N)\,,(S_1\times S_2\cup T_1\times T_2))\;for\;all\;S_1\,,S_2\in 2^X and T_1,T_2\in 2^X$

$\beta(M\times N\,,S_1\cup T_1\times S_2\cup T_2) = min\,\{\beta 1(M,S_1\cup T_1),\beta 1\,(N\,,S_2\cup T_2)\}$

$= min\,\{min\{\beta_1(M,S_1),\beta_1\,(M,T_1)\},min\,\{\beta_2\,(N,S_2),\beta_2(N,T_2)\}\}$

$= min\,\{\beta_1(M,S_1),\beta_2\,(N,S_2)\},min\,\{\beta_1\,(M,T_1),\beta_2(N,T_2)\}$

$= min\,\{(\beta\,(M\times N\,,(S_1\times S_2)\,,\beta(M\times N\,,T_1\times T_2\,)\}$

4) $For\;all\;x,y\in X\,,for\;all\;M\,,N\,,S\,,T\in 2^X\;for\;all\;\varepsilon\in[0\,,\infty]$

$\beta\,(M\times N\,,S\times T) = min\,\{\,\beta_1(M,S),\beta_2\,(N,T)\}\leq min\,\{\,\beta_1\,(M,S^{\varepsilon})+\varepsilon+\gamma\;\;,\beta 2\,(N,T^{\text{Ɣ}})+\varepsilon+\gamma\,\}$

$= min\,\{\,\beta_1(M,S^{\varepsilon}),\beta_2\,(N,T^{\varepsilon})\}+\varepsilon+\gamma = \beta\left((M\times N),(S^{\varepsilon}\times T^{\text{Ɣ}})\right)+\varepsilon+\gamma.$

Thus ($X\times X$, β) is β- app. space.

Proposition 2.10: Let (X_i, β_i), i ∈ I be family of β −app- space. Define β : $2^{\prod X_i}\times 2^{\prod X_i}$ →[0,∞] as follows:
β($\prod_{i\in I}M_i\,,\prod_{i\in I}N_i$) = $\underset{,N_i\in 2^{\prod x_i}}{inf}\;\underset{M_i\in 2^{\prod x_i}}{inf}\;\beta(M_i\,,N_i)$ for all $M_i\,,N_i \in 2^{\prod x_i}$, then ($\prod_{i\in I}M_i$, β) is β-app. space.

Proof:

1) for all $M_i\,,N_i \in 2^{\prod x_i}$

If $M_i\cap N_i\neq\emptyset \Rightarrow (m_1,m_2,\ldots,m_n)\in M_i\;and\;(m_1,m_2,\ldots,m_n)\in N_i,for\;all\;i\in I$

 $\Rightarrow(m_1,m_2,\ldots,m_n)\in M_i\cap N_i \Rightarrow (m_1,m_2,\ldots,m_n)\in\prod_{i\in I}M_i$, and
 $(m_1,m_2,\ldots,m_n)\in\prod_{i\in I}N_i$,
 then β($\prod_{i\in I}M_i\,,\prod_{i\in I}N_i$) = $\underset{,N_i\in 2^{\prod x_i}}{inf}\;\underset{M_i\in 2^{\prod x_i}}{inf}\;\beta(M_i\,,N_i) = \underset{,N_i\in 2^{\prod x_i}}{inf}\;\underset{M_i\in 2^{\prod x_i}}{inf}\;\{0\}$ =0
2) β($\prod_{i\in I}M_i\,,\prod_{i\in I}N_i\cup\prod_{i\in I}R_i$)= $inf_{(m_i)\in M_i}$ inf $_{(m_i)\in N_i\cup R_i}\beta\,(M_i\,,N_i\cup R_i)$
 = min { $inf_{(m_i)\in M_i}$ inf $_{(m_i)\in N_i}\beta\,(M_i\,,N_i)$, $inf_{(m_i)\in M_i}$ inf $_{(m_i)\in R_i}\beta\,(M_i,R_i)$ }

= min { $inf_{(m_i)\in M_i}$ inf $_{(m_i)\in N_i,(m_i)\in R_i}$ (β (M_i , N_i) , β (M_i , R_i)) }
= min {β ($\prod_{i\in I} M_i$, $\prod_{i\in I} N_i$) , β ($\prod_{i\in I} M_i$, $\prod_{i\in I} R_i$)}

3) If $M_i = \emptyset$ or $N_i = \emptyset \Rightarrow (m_i) \notin M_i$ or $(m_i) \notin N_i$
If $M_i = \emptyset \Rightarrow$ β($\prod_{i\in I} M_i$, $\prod_{i\in I} N_i$) $= inf_{(m_i)\in \emptyset}$ $inf_{(m_i)\in N_i}$ $\beta(M_i, N_i)$ =∞
If $N_i = \emptyset \Rightarrow$ β($\prod_{i\in I} M_i$, $\prod_{i\in I} N_i$) $= inf_{(m_i)\in M_i}$ $inf_{(m_i)\in \emptyset}$ $\beta(M_i, N_i)$ =∞

4) For all M_i , N_i $\in 2^{\prod x_i}$ for all ε ∈ [o , ∞] and Ɣ∈ [o , ∞]
where M^{ε}={$x\in X$: $\beta(\{x\}, M) \leq$ ε}
β($\prod_{i\in I} M_i$, $\prod_{i\in I} N_i$) $= inf_{(m_i)\in M_i}$ $inf_{(m_i)\in N_i}$ $\beta(M_i, N_i)$
$\leq inf_{(m_i)\in M_i}$ $inf_{(m_i)\in N_i}$ $\beta(M_i^{\varepsilon}, N_i^{\varepsilon})$ + ε +Ɣ
≤ β ($\prod_{i\in I} M_i^{\varepsilon}$, $\prod_{i\in I} N_i^{\varepsilon}$) + ε + Ɣ

Thus ($\prod_{i\in I} M_i$, β) is β- app. space.

2.New Results of β-Contractions on β-Approach spaces

Definition 3.1: Let (X, β) and (Y, β) be β–app. space. The function £ : $X \to Y$ is said to be β –contraction if β′(£ (M) , £(N)) ≤ β (M, N) , for all $x \in X$ and for all $M, N \in 2^X$.

Proposition 3.2: Let (X,β) be β –app. space and £: $(X,\beta) \to (X,\beta)$ then for all M , $N \in 2^X$

1- I: $(X,\beta) \to (X,\beta)$ is β – contraction.

2- The constant map is β- contraction.

Proof: It is clear

Proposition 3.3: Let (X,β), (X\`, β\`) and (X\`\`, β\`\`) be β –app.spaces. The function £: (X,β) → (X\`, β\`)

g: (X\`, β\`) → (X\`\`, β\`\`) are β - contraction. Then g of: (X,β) → (X\`\`, β\`\`) is β – contraction.

Proof: Let $M, N \in 2^X$ then β '' (g o £ (N), g o£(M) ≤ β' (£(N), £(M))

since £ is β- contraction ,so β' (£(N), £(M)) ≤ β (N, M).

Thus β'' (g o £(N), g o £(M)) = β'' (g (£(N), g (£(M) ≤ β' (£(N), £(M)) ≤ β (N, M)

g o £ is β- contraction.

If the β -distance is defined as a function in two sets, we can prove distance functional as follows:

Proposition 3.4: Let (X,β) be a β −app. space. For M ⊂ X the distance functional defined as:

β_M: $X \to$ Ȩ = [0 , ∞] , by: $\beta_M(x) = \beta$ ({x}, M) is β-contraction.

Proof: We will prove β_M is well define :

$x_1 = x_2$ then {x_1 }={x_1} and so ({x_1 },M)= ({x_2},M). Therefore β({x_1, M) = β({x_2, M)

Thus β_M (x_1) = β_M (x_2)

2) It is clear β_M is β-contraction.

Proposition 3.5 : Let (X,β) and (X\`, β \`) be β –app. space and £:(X,β) → (X\`, β\`) is β- contraction. Then the restriction £ $|_{Ķ}$ is the β- contraction for Ķ ⊂ X.

Proof: Suppose £: (X,β) → (X\`, β\`) is β- contraction and Ķ ⊂ X.

Define g: Ķ → X\` by $g(\{m\})$ = £({m}) for all m ∈ Ķ

β\` ($g$ {m}), $g$ (Ѧ)) = β\` (£ ({m}), £ (Ѧ)) ≤ β ({m}, Ѧ).

Proposition 3.6: Let (X_i, β_i) be a family of β –app. spaces that any Ķ ∈ I . Then, the projection

pr: Л $x_i \to x_i$ is β – contraction.

Proof: Let $x_i \in X_i$, $M \in 2^X$, Pr : Л $x_i \to x_i$ projection function .

$\beta'_i (Pr(x_i), Pr(M)) = \beta'_i (Pr(x_1, \ldots, x_i), Pr(M_i))$ for $k \in I$

$\beta'_i ((x_i), (M) \leq (\beta_1 ((x_1), M_1) \times \beta_2 ((x_2), M_2) \times \ldots \times \beta_i((x_i), (M_i) = \prod_{i \in I} \beta i (\prod_{i \in I} x_i, \prod_{i \in I} M_i))$

$= \beta_i (\prod_{i \in I} x_i, \prod_{i \in I} M_i)$. Hence $\beta_i`(Pr(x_i), Pr(M)) \leq \beta (\prod_{i \in I} x_i, M)$. Then $Pr(x)$ is β - contraction.

Proposition 3.7: Let £ : Ш → Ш` be $\beta$– contraction. Then, the map contraction £ x $I_V$: Ш x V → Ш ` x V is β– contraction.

Proof: For all w ∈ Ш, v ∈ V and M ∈ 2^x
β' (£ (Ш, V), £ (M, V))) = β' (£{w}) x I_V, £ ({m}) x I_V = β' ((£{w} xV, £ (M) x V)
= min β' ((£{w}), £ (M))), β'' (V, V)} ≤ min {β ({w}, M), β'' (V, V)}
= β (({w}, V), (M, V)). So ′ ((£ ({w}, V), £ (M, V)) ≤ β ({w}, V), (M, V)). Thus £ x I_V is β- contraction.

4. Convergent Results in β-Approach spaces.

In this section, we define the convergent of sequence in β –Approach space by start the following definition:
Definition 4.1: Let (X, d) be a metric space, then a sequence$\{ x_n\}_{n=1}^{\infty}$ in X is said to be a right Cauchy sequence if for all ε > 0 there exists $k \in Z^+$ such that

$d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$,$m \leq n$. Left Cauchy sequence if for all ε > 0 there exists $k \in Z^+$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$,$n \leq m$. If a sequence is left and right Cauchy is called Cauchy sequence.

Definition 4.2: A set $M \in 2^X$ is said to be a cluster point in an approach space (X, β) if there exists disjoint sequence $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ in X such that $\inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n \}, M) = 0$, which is written by

$\{ \mathbb{A}_n\}_{n=1}^{\infty} \to M$. We denoted the set of all cluster point in approach space $\Gamma(X)$.

Definition 4.3: A sequence $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ in X is said to be Cauchy sequence in approach space $\beta -$ Cauchy if for every cluster point M , $\lim_{n\to\infty} \inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n \}, M) = 0$ sequence$\{ \mathbb{A}_n\}_{n=1}^{\infty}$ in X is said to be β- convergent sequence in approach space if there exist $x \in X$ for all

$M \in \Gamma(X)$, $\beta(\{\mathbb{A}_n \}, M) = 0$.

Proposition 4.4: Let (X, β) be β-app. space, then the following are equivalent:

1) β- Convergent sequence in β-app. space.
2) $\lim_{n\to\infty} inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\mathbb{A}_n, M) = 0$ and $\lim_{n\to\infty} sup_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\mathbb{A}_n , M) = 0$

Proof: Let $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ be disjoint β- convergent sequence in approach space.
There exist $x \in X$ for all $M \in \Gamma(X)$: $\beta(\{ \mathbb{A}_n\}, M) = 0$
For all $M \in \Gamma(X)$: $inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$ and $\sup_{x \in M} \beta(\{ \mathbb{A}_n\}, M) = 0$
For all $M \in \Gamma(X)$: $lim_{n\to\infty} inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$
And $\lim_{n\to\infty} sup_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$
Conversely, suppose the condition (2) is true. $lim_{n\to\infty} inf_{\substack{x \in M \\ x_{i \in A_i}}} \beta(\{ x_n\}, M) = 0$ And $\lim_{n\to\infty} sup_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$
Then M is cluster point, that is $inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$
Then there exist $x \in X$ for all $M \in \Gamma(X)$: $\beta(\{ \mathbb{A}_n\}, M) = 0$
Thus $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ be β- convergent sequence in β- app. space.
Remark 4.5: Every β- convergent sequence is $\beta -$ Cauchy (Cauchy β-app. space).

Proposition 4.6: If (X, β) is β –app. Space then following are equivalent:

1) $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ is β- convergent sequence in β –app. space.
2) $sup_{M \in \Gamma(X)} \ inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} d_{\beta}(\{ \mathbb{A}_n\}, \{x\}) = 0$

Proof: Suppose that $\{ \mathbb{A}_n\}_{n=1}^{\infty}$ is disjoint β- convergent sequence in β -app. space. There exist $x \in X$ for all $M \in \Gamma(X)$: $\beta(\{ \mathbb{A}_n\}, M) = 0$
And $inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$, then $lim_{n\to\infty} inf_{\substack{x \in M \\ x_{i \in \mathbb{A}_i}}} \beta(\{ \mathbb{A}_n\}, M) = 0$

And $sup_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = 0$ that is $\lim_{n\to\infty} sup_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = 0$
Then $sup_{M\in \Gamma(X)} \; inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} d_\beta(\{\mathbb{A}_n\}, \{x\}) = 0$
Conversely, it is clear.

Proposition 4.7: If (X, β_d) is approach metric space and $\{A_n\}_{n=1}^{\infty}$ be disjoint sequence in X, then it is Cauchy sequence in (X, d) if and only if is β- Cauchy sequence in (X, β_d) .
Proof:
Let $\{\mathbb{A}_n\}_{n=1}^{\infty}$ be Cauchy sequence in (X, β_d) , then we have that $inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = 0$

$inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, \{\mathbb{A}_m\}) = inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} inf_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_n\}, \{\mathbb{A}_m\}) = 0$
That is $d(\{\mathbb{A}_n\}, \{\mathbb{A}_m\}) = 0$
Then $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is left Cauchy sequence.
Also $inf_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\}, \{\mathbb{A}_n\}) = inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} inf_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\}, \{\mathbb{A}_n\}) = 0$
That is $d(\{\mathbb{A}_m\}, \{\mathbb{A}_n\}) = 0$. Then $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is right Cauchy sequence.
Thus $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is Cauchy sequence in (X, d).
Conversely, if $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is Cauchy sequence in (X, d).
Then it is left and right Cauchy sequence, for all ε>0, there exists k∈ Z^+ such that $d(\{\mathbb{A}_m\}, \{\mathbb{A}_n\}) < \varepsilon$, for all $m, n \leq N$, $m \geq n$ and for all $\varepsilon < 0$ there exists $k \in Z+$ such that $d(\{\mathbb{A}_n\}, \{\mathbb{A}_m\}) < \varepsilon$, for all $m, n \leq N$, $n \geq m$
$inf_{\substack{A_n\subset M \\ x_{i\in \mathbb{A}_i}}} \beta(\{\mathbb{A}_n\}, M) = inf_{\substack{x\in M \\ x_{i\in \mathbb{A}_i}}} inf_{\substack{\mathbb{A}_m\subset M \\ \mathbb{A}_n\subset M}} \beta(\{\mathbb{A}_m\}, \{\mathbb{A}_n\}) = 0$
Hence $\{\mathbb{A}_n\}_{n=1}^{\infty}$ is β- Cauchy sequence in approach space.

Theorem 4.8: Let (X, β) be an approach space, $\{ V_n\}$ and $\{ U_n\}$ is an β – converge Sequence in (X, β) to $\{V\}$, $\{U\}$ Respectively, then:
1) $\{ V_n + U_n\}$ is an β – convergence to $\{V + U\}$
2) $\{\lambda V_n\}$ is an β – convergence to $\{\lambda V\}$
3) $\{ V_n . U_n\}$ } is an β – convergence to $\{V . U\}$

Proof (1)
Since $\{ V_n\}$, $\{U_n\}$ β – convergence to $\{V\}$, $\{U\}$.
Thus $\lim_{n\to\infty} \inf_{V\subset M} \beta(\{V_n\}, M) = 0$ and $\lim_{n\to\infty} \sup_{V\subset M} \beta(\{V_n\}, M) = 0$
So $\lim_{n\to\infty} \inf_{U\subset M} \beta(\{U_n\}, M) = 0$ and $\lim_{n\to\infty} \sup_{V\subset M} \beta(\{V_n\}, M) = 0$
Then so $\lim_{n\to\infty} \beta(\{U_n\}, \{U\}) = 0$ that is $\lim_{n\to\infty} inf\; d(\{U_n\}, \{U\}) = 0$
$\lim_{n\to\infty} inf\; d(\{V_n\}, \{V\}) = 0$
There for , $\lim_{n\to\infty} \inf_{\{V\},\{U\}\subset M} \beta(\{V_n + U_n\}, M)$
$= lim \inf_{V,U\subset M} \beta(\{V_n + U_n\}, \{V\} \cup \{V\})$
$= \lim_{n\to\infty} \inf_{V,U\subset M} \min\{\beta(\{V_n\}, \{V\}), \beta(\{U_n\}, \{U\}) = 0$
$= \lim_{n\to\infty} \inf_{V,U\subset M} \beta(\{V_n + U_n\}, M) = 0$
$= \lim \inf_{V\subset M} \beta(\{V_n\}, M) + \lim_{n\to\infty} \inf_{U\subset M} \beta(\{U_n\}, M))$
And $\lim \sup \beta(\{V_n + U_n\}, M)$
$= \lim \sup_{V\subset M} \beta(\{V_n\}, M) + \lim \sup_{U\subset M} \beta(\{U_n\}, M) = 0$
$\lim \inf \beta(\{V_n + U_n\}, M) = 0$
and $\lim_{n\to\infty} \sup_{V,U\subset M} \beta(\{V_n + U_n\}, M) = 0$
$= lim\; inf\; \beta(\{V_n + U_n\}, M) = 0$ and $\lim_{n\to\infty} \sup_{V,U\subset M} \beta(\{V_n + U_n\}, M) = 0$
Then $\{V_n + U_n\}$ is β-approach convergence sequence to {V+ U}
Proof (2):
Since $\{V_n\}$ approach convergence sequence
To {Vx} , $\lambda \in F$
So $\lim_{n\to\infty} \inf_{V\subset M} \beta(\{V_n\}, M) = 0$ and $\lim_{n\to\infty} \sup_{V\subset M} \beta(\{V_n\}, M) = 0$
if $\lambda \in F$, $\lambda . \lim_{n\to\infty} \inf_{V\subset M} \beta(\{V_n\}, M) = 0$
and $\lambda . \lim_{n\to\infty} \sup_{V\subset M} \beta(\{V_n\}, M) = 0$ then $\lim_{n\to\infty} \inf_{V\subset \lambda M} \beta(\{\lambda V_n\}, \lambda M)$
$= \lim_{n\to\infty} \inf_{V\subset M} d(\{\lambda V_n\}, \{\lambda V\})$
Therefore, there exist $V \subset M$ for all , $\beta(\{\lambda V_n\}, \lambda M) = 0$
Thus there exist $V \subset M$ for all M such that $\inf_{V\subset M} d(\{\lambda V_n\}, \{\lambda V\}) = 0$
And $\lim_{n\to\infty} \sup_{V\subset \lambda M} \beta(\{\lambda V_n\}, \lambda M) = \lim_{n\to\infty} \sup_{V\subset M} inf\; d(\{\lambda V_n\}, \{\lambda V\}) = 0$
Then $\{\lambda V_n\}$ β-app- convergence sequence to $\{\lambda V\}$
Proof (3)

Let $\{V_n\}$, $\{U_n\}$ β –app. convergent sequence to $\{V\}$, $\{U\}$.
for all $\in M$. There exists $M \in \Gamma(M)$ such that $\beta(\{V_n\}, M)$ =0. For all $x \in N$ there exists $N \in \Gamma(N)$ such that $\beta(\{U_n\}, N)$ =0. Let $C = MN$ and $V, U \subseteq C$ such that $\beta(\{V_n U_n\}, C) = \inf_{V \subset M} d(\{V_n U_n\}, \{V_n\})$ =0. Thus $\beta(\{V_n U_n\}, C)$ =0
Then $\{V_n U_n\}$ β –app. convergence sequence to $\{V, U\}$.

5. New Structure of β-Approach vector space

Definition 5.1: We say $(X, \beta, *)$ is an β –app. semi group if and only if:

1) (X, β) is β –app. space.

2) $(X, *)$ is a semi – group.

$* : X \times X \to X$, $(x, y) = x * y$ is β- contraction .

Definition 5.2: We say $(X, \beta, *)$ is an β –app. group if it satisfies:

1) (X, β) is approach space.

2) $(X, *)$ is group.

3) $* : X \otimes X \to X$, $x + y$ is β – contraction .

4) $- : X \to X, x \to -x$ is β – contraction.

Example 5.3: Let ℝ be set of real number and (R^n, β) is approach space with usual distance

Proof: $(R^n, \beta, +)$ is β –approach group with usual distance and addition for $i = 1, \dots, n$

For all $X \in R^n$, for all $M \in 2^{R^n}$

$\beta: 2^{R^n} \times 2^{R^n} \to [0, \infty]$ define as:

$$\beta: (M, N) = \begin{cases} \inf_{x_i \in M} \inf_{y \in N} d(x_i, y_i), & M \neq \emptyset \text{ and } N \neq \emptyset \\ \infty & M = \emptyset \text{ or } N = \emptyset \end{cases}$$

Example 5.4: Let M_2 is denoted the set of all orthogonal 2× 2 matrixes . M_2 subset of R^4 with the Euclidean metric, then $(M_2, \beta(d), \Theta)$ is approach group.

$$d(M, N) = [\textstyle\sum_{i,j=1}^{4}(aij - bij)^2]^{1/2} = d\left(\begin{bmatrix} cosx & sinx \\ sinx & -cosx \end{bmatrix}, \begin{bmatrix} cosy & siny \\ siny & -cosy \end{bmatrix}\right)$$

$$= 2\sqrt{1 - \cos(x - y)}$$

And $d(M^{-1}, N^{-1}) = d\left(\begin{bmatrix} -cosx & -sinx \\ -sinx & cosx \end{bmatrix}, \begin{bmatrix} -cosy & -siny \\ -siny & cosy \end{bmatrix}\right)$

$$= [4 - 4(cos\,x\ cos\,y - sinx\ sin\,y)]^{1/2} = 2\sqrt{1 - cos(x - y)}$$

Thus $d(M, N) = d(M^{-1}, N^{-1})$

a− (M_2, β) is β –app. space where β –distance defined as followes:

$$1)\ \beta(M, N) = \begin{cases} \infty & \text{if } M = \emptyset \text{ or } N_2 = \emptyset \\ \inf_{T \in N} |M - T| & \text{if } M \neq \emptyset \text{ and } N \neq \emptyset \end{cases}$$

$= \beta(N, M)$

b) let $M, N \in M_2$: $(M N)(M N)^{-1} = M N N^{-1} M^{-1} = I$

Hence $(M N)^{-1}$ which makes (M, N) element of M_2, so M_2 is closed under matrix multiplication.

It is clear (M_2, ʘ) is a group

c) for all M, N, P ∈ M_2

$d\,(M,N) \;=\; d\,(M^{-1},N^{-1})$ and for all $(M,N,Z,W)\;\in\; M_2$

$d\,(M\,N,ZW) \;\leq\; d\,(MN,ZN) \;+\; d\,(ZN,ZW) =\; d\,(M,Z) \;+\; d\,(N,W)$

So ʘ is β −contractive.

d) $d\,(M^{-1},X^{-1}) \;=\; d\,(M,X)$, invention is contractive. Then (M_2, β(d), ʘ) is approach group.

Definition 5.5: Let (X, β, *) be a β –app. group and Y⊂X. Then, $(Y,\beta,*)$ is called β –app. sub- group, if satisfy:

1) (Y,β_Y) is β −app space.
2) $(Y,*)$ is sub- group.
3) £: $Y\times Y\;\rightarrow Y$ with £ $(x,y) \;= x*y^{-1}$ is β-contraction.

Example 5.6: Let Ƶ be the set of all integer numbers and sub set of R with usual distance β ,

$$\beta(M,N)=\begin{cases}\infty & \text{if } M=\emptyset \text{ or } N=\emptyset\\ \inf_{x\in M}\ \inf_{y\in N}\ |x-y| & \text{if } M\neq\emptyset \text{ and } N\neq\emptyset\end{cases}$$

Then (Ƶ, β, +) is β −app. sub- group.

Definition 5.7: Let X be a non-empty set with binary operations: addition and scalar multiplication, β is distance on X. We said (X, β, *, ʘ) to be β −approach vector space if satisfy:

1- (X, β, *) is approach group.
2- ɤ. x ∈ X
3- ɤ (x+ y) = ɤ x + ɤ y for all ɤ ∈ Ƒ for all x, y ∈ X
4- (x+ y) ɤ = x ɤ + y ɤ for all ɤ ∈ Ƒ for all x, y ∈ X
5- $(\lambda$. ɤ).x= λ (ɤ. x), for all x ∈ X and λ, ɤ ∈ Ƒ
6- ʘ: Ƒ × X → X, ʘ (ɤ, x) = ɤ. x is β -contraction
7- 1.x = x, x ∈ X

Example 5.8: Let R be set of all real numbers. (R, β, +, ⊙) with usual distance β, addition and scalar multiplication ⊙ is β − approach vector space.

Proposition 5.9: If Ɣ is β −app- vector space, then Ɣ is vector space.

Proof: The proof is straight forward. According to definition of β −App vector space, Ɣ satisfy the condition of vector space.

Remark 5.10: The convers of Proposition (5.9) is not true, the following example is show this.

Example 5.11: Let (S, +,.) be vector space of real numbers with usual distance β and scalar multiplication, such that

$$\beta(M,N)=\begin{cases}0 & \text{if } x\in M \text{ and } x\in N\\ 2 & \text{if } x\notin M \text{ or } x\notin N\end{cases}$$

$\beta\,(M,N)$ is not β –app. space since β(M, ∅) $\neq\infty$ thus (S, β, +, .) is not App – vector space.

Definition 5.12 (β -Approach sub – space): A subset Y of approach vector space over the field Ƒ is called approach subspace if satisfy the following

1) Y subspace of vector space (X, +,.)
2) (Y, β_y) β −app. space

Theorem 5.13: β –App. topological space is a topological space (X, T) that associated with natural β -approach space, we define a function

$\beta_T : 2^X \times 2^X \rightarrow [0,\infty]$ by: $\beta_T\ (M, N) = \begin{cases} 0 & \text{if } x \in CL(M) \text{and } x \in CL(N) \\ \infty & \text{if } x \notin CL(M) \text{ or } x \notin CL(N) \end{cases}$

for all $x \in X$, $M, N \in 2^X$, (X, T, β_T) for topology T on X is called a topological β –app. space, and β_T is called topological β -distance.

Definition 5.14: Let (X, β) be app- space. For $x \in X$ the center at x and of radius $r > 0$ is the set

$D_r(x)=\{ t \in X ,\beta(\{t\} ,\{x\}) < r \}$ the set D_r is called β – open ball.

Definition 5.15: Let $\mathbb{V}$ β –app. vector space on field Ғ. Then a topological β –app. vector space $T_{\mathbb{V}}$ be satisfy:

1) The map + : $\mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$,(x, y) → x+ y is β –contraction .
2) The map: $Ғ \times \mathbb{V} \rightarrow \mathbb{V}$ is β –contraction.

   When it is written as $(\mathbb{V}, T_{\mathbb{V}})$.

Proposition 5.16: Let (X, T) be a topological space, then the function
$\beta_T: 2^X \times 2^X \rightarrow [0,\infty]$

Defined by : $\beta_T (M , N) = \begin{cases} 0 & \text{if } x \in CL(M) \text{and } x \in CL(N) \\ \infty & \text{if } x \notin CL(M) \text{ or } x \notin CL(N) \end{cases}$

is β -distance on X.
Proof:

1) $Since\ M \subset CL(M),\ M \cap N \neq \emptyset, x \in CL(M)\ and\ x \in CL(N)$
$, N \subset CL(N) \Rightarrow \beta(M,N) = 0$
2) $X \in CL(M) \cap CL(N), CL(M \cap N) \subset CL(M) \cap CL(N)$
$If\ M = \emptyset\ or\ N = \emptyset\ and\ CL(\emptyset) = \emptyset, x \notin CL(M) \Rightarrow \beta(\emptyset, N) = \infty$
$If\ N = \emptyset \Rightarrow x \notin CL(N) \Rightarrow \beta(M,\emptyset) = \infty$
3) $For\ all\ M,N \in 2^x, since\ CL(N \cup P) = CL(N) \cup CL(P) = min\{\beta(M,N), \beta(M,P)\}$
$= min\{CL(M), CL(N) \cup CL(P)\} = min\{CL(M), CL(N \cup P)\} = min\{\beta(M,N), \beta(M,P)\}$.
4) $For\ all\ M, N \in 2^x\ and\ for\ all\ \gamma, \varepsilon \in [0,\infty]$
$since\ M\varepsilon = \{x \in X : \beta(\{x\}, N) \leq \varepsilon\}, x \in M$

And $x \in M^\varepsilon$ since $M^\varepsilon \subset cl(M^\varepsilon) \Rightarrow x \in CL(M^\varepsilon)$ and if $x \in N^\varepsilon$, , $N^\varepsilon \subset CL(N^\varepsilon)$

$That\ is\ x \in CL(N^\varepsilon)\ then\ \beta(M,N) \leq \beta(M^\varepsilon, N^\gamma) + \varepsilon + \gamma$
$If\ x \notin M\ or\ x \notin N\ then\ \beta(M,N) = \infty$
$If\ x \notin CL(M\varepsilon)\ or\ x \notin CL(N^\varepsilon) \Rightarrow \beta(M^\varepsilon, N^\gamma) = \infty$
$\beta(M,N) \leq \beta(M^\varepsilon, N^\gamma) + \varepsilon + \gamma = \infty. If\ x \in CL(M\varepsilon) \Rightarrow \beta(M^\varepsilon, N^\gamma) = 0$
$\beta(M,N) \leq \beta(M^\varepsilon, N^\gamma) + \varepsilon + \gamma = 0. Hence \beta_T(M,N)\ is\ \beta - distance\ on\ X.$

Theorem 5.17: Let $(\mathbb{V}, \beta_{\mathbb{V}})$ be β −app. vector space , M is Closed approach sub space of X . then $(\mathbb{V}/M, \beta`_{\mathbb{V}/M})$ is $\beta$ -app-vector space , we define $\beta`_{\mathbb{V}/M} : 2^{\mathbb{V}/M} \times 2^{\mathbb{V}/M} \rightarrow [0,\infty]$ as follows :
$\beta`_{\mathbb{V}/M}(D, N) = \beta`(D + M, N + M) = \beta(D,N)$
Proof: We will prove β` satisfy distance condition:

1) $\beta`(D + M, N + M) = \beta(D, N)$
2) $If\ D = \emptyset\ or\ N = \emptyset \Rightarrow if\ N = \emptyset\ , \beta`(D + M, \emptyset) = \beta(D, \emptyset) = \infty$
$If\ D = \emptyset \Rightarrow \beta`(\emptyset, N + M) = \beta(\emptyset, N) = \infty$
3) $If\ D \neq \emptyset\ and\ N \neq \emptyset \Rightarrow N \cap D \neq \emptyset\ then\ \beta(D, N) = 0, x \in D\ and\ x \in N$
$\beta`(D + M, N + M) = \beta`(\{x + M\}, \{x + M\}) = \beta(D,N) = 0$
4) $\beta`(D + M, N + M \cup P + M) = \beta(D, N \cup P)$
$\leq min\{\beta(D,N), \beta(D,P)\}$
5) $\beta`(D + M, N + M) = \beta(D, N)$
$\leq \beta(D^\varepsilon, N^\varepsilon) + \varepsilon + \gamma$
$= \beta(D^\varepsilon + M, N^\varepsilon + M) + \varepsilon + \gamma$

Definition 5.18: Let (X , β) be β -app-space a sequence and $\{A_n\}$ is the convergent sequence in the β -app .space to $A \in X$ if $\lim_{n\rightarrow\infty} \inf_{A \subseteq M} \beta(\{A_n\},M)=0$ and $\lim_{n\rightarrow\infty} \operatorname{Sup}_{A \subseteq M} \beta(\{A_n\},M)=0$

Definition 5.19: Let (X , β) and (Y , β`) are β -app . spaces. The function £ : X→ Y is called sequentially contraction if $\lim_{n\rightarrow\infty} \beta(\{£(x_n)\}, £(M)) = 0$ Whenever $\lim_{n\rightarrow\infty} \beta(\{x_n\},M)=0$

Definition 5.20: Let $\mathcal{V}$ and ω be two β –app. vector spaces on app-space over the same field Ғ, a mapping: $\Omega: \mathcal{V} \to \omega$ is said to be approach linear transformation if the following hold :

1) $\Omega(a+b) = \Omega(a) * \Omega(b)$
2) $\Omega(\gamma a) = \gamma\Omega(a)$ $for\ all\ \gamma \in$ Ғ, $for\ all\ a, b \in \mathcal{V}$

Definition 5.21: Let $\Omega: \mathcal{V} \to \omega$ be a β –app. linear transformation. Then the set $\beta - ker\ (\Omega) = \{A \subseteq \mathcal{V} : £(A) = \{0\}\} = £^{-1}(\{0\})$ is called the β –app. kernel of Ω.

Theorem 5.22: Let $(\mathcal{V}, T_{\mathcal{V}}, \beta)$ and $(\omega, T_{\omega}, \beta')$ be a topological β –app. vector spaces and the β –app. linear transformation £ : $\mathcal{V} \to \omega$ is contraction. Then ker(£) is closed .

Proof: Suppose £ is the β-contraction.

To prove Ker(£) is closed set, let $\{A_n\}$ be a disjoint sequence that convergent to A in Ker(£) such that $\lim_{n\to\infty} \inf_{x\epsilon ker£} \beta(\{A_n\}, M) = 0$ and $\lim_{n\to\infty} \sup_{x\epsilon ker£} \beta(\{A_n\}, M) = 0$
Since £ is β- contraction, that is $\beta`(£(\{A_n\}), £(M)) \leq \beta(\{A_n\}, M)$.
Then, $0 = \lim_{n\to\infty} \inf_{x\epsilon M} \beta(\{A_n\}, M) \leq \lim_{n\to\infty} \inf_{x\epsilon M} \beta`(£(\{A_n\}), £(M))$
$\leq \lim_{n\to\infty} \sup_{x\epsilon M} \beta`(£(\{A_n\}), £(M)) \leq \lim_{n\to\infty} \sup_{x\epsilon M} \beta(\{A_n\}, M) = 0$
$\lim_{n\to\infty} \sup_{x\epsilon M} \beta`(£(\{A_n\}), £(M)) = 0$ and $\lim_{n\to\infty} \inf_{x\epsilon M} \beta`(£(\{A_n\}), £(M)) = 0$
$(£(\{A_n\}) = 0$, $\lim_{n\to\infty} \beta`(£(\{A_n\}), £(M)) = 0$ then $£(\{A\}) = 0$, $A \subseteq$ Ker (£)
Conversely, suppose Ker (£) is closed set, let $\{A_n\}$ be disjoint sequence convergent to A in β -Ker (£), to prove $£(\{A_n\})$ convergent to $£(\{A\})$, since β -Ker (£) is closed, x∈ β -Ker (£) , assume that $£(\{A_n\})$ is not convergent to $£(\{0\})$ in M, that is £ is not β − contraction.
Then $\lim_{n\to\infty} \sup_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) \neq 0$ or $\lim_{n\to\infty} \inf_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) \neq 0$
If $\lim_{n\to\infty} \sup_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) \neq 0$ or $\lim_{n\to\infty} \inf_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) = 0$
$\lim_{n\to\infty} \inf_{A\subseteq M} \beta(\{A_n\}, M) > \lim_{n\to\infty} \inf_{A\subseteq M} \beta`(£(\{A_n\}), £(M))$
$\Rightarrow \beta`(£(\{A_n\}), £(M)) < 0$,this impossible
If $\lim_{n\to\infty} \sup_{x\in M} \beta`(£(\{A_n\}), £(M)) = 0$ or $\lim_{n\to\infty} \inf_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) \neq 0$
$0 = \lim_{n\to\infty} \sup_{A\subseteq M} \beta(\{A_n\}, M) < \lim_{n\to\infty} \sup_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) = 0$
this impossible.
If $\lim_{n\to\infty} \sup_{A\subseteq M} \beta`(£(\{A_n\}), £(M)) \neq 0$ and $\lim_{n\to\infty} \inf_{x\epsilon M} \beta`(£(\{A_n\}), £(M)) \neq 0$
But, $£(\{x_n\}) \in$ β -Ker(£) then $\lim_{n\to\infty} \sup_{x\epsilon M} \beta`(0, £(x)) \neq 0$ and
$\lim_{n\to\infty} \inf_{x\epsilon M} \beta`(0, £(x))$ ,that is $\beta`(0, £(x)) \neq 0$, $£(x) \neq 0$
So A ⊈ β -Ker (£) this impossible. Hence £ sequentially contraction, then £ is β- contraction.

6. Conclusion

We investigated several problems in the theory of approach spaces: a topological space called topological approach structure and generalization of metric spaces, which means we need to define some concepts in approach spaces as approach vector spaces, approach topological vector spaces, an approach subspace, and we solved some examples in an approach space, an approach vector spaces, and an approach group. We clarify that every approach space is metric space, but the converse is not true, as demonstrated by an example. We established some new properties of contractions, defined convergent and sequentially convergent in approach spaces, and demonstrated that contraction is a necessary and sufficient condition for obtaining a linear sequentially convergent.

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