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Bipolar Theorems for Graded Fuzzy Topological Linear Space

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Abstract In this work, we presented the concept of graded fuzzy topological linear spaces and proved some of the properties related to them. We also presented the concept of polar and bipolar sets in this spaces, and we proved the properties of these sets. Finally, we proved the bipolar theorem and the result associated with it.

1- Introduction

Many authors like Felbin [1], Cheng and mordeson [4], Bag and Samanta [5], Sadeqi and Yaqub Azari [7], and so on started to later, Xiao and Zhu [9], Fang [6], Daraby et.a ([1],[8]) redefined, the idea of Felbin's [2] studied various properties of it is graded fuzzy topology structure.In this paper, we introduce the concept of graded weak topological space as a generalization of usual weak topology we prove that the graded weak fuzzy topology is not equivalent with the fuzzy topology. A consequence of the Hahn-Banach theorem is the classical bipolar theorem which states that the bipolar of a subset of a locally convex space equals its closed convex hull.

The bipolar theorem is a theorem in functional analysis that characterizes the bipolar (i.e. the polar of the polar) of a set. In convex analysis, the bipolar theorem refers to a necessary and sufficient conditions for a cone to be equal to its bipolar. The bipolar theorem can be seen as a special case of the Fenchel–Moreau theorem.

In section 3.intrsction Ming 1991 and puri and Rales 1983 ,Zhang initiated the notion of yin yang Bipolar fuzzy set as any extension of fuzzy set ,2011 introduce the concept of bipolar fuzzy graphs[11] ,Al gham detail 2018 multi-criteria decision – making methods in bipolar fuzzy environment[10] .some properties in bipolar in graded fuzzy .

2. Weak Topology on Graded Fuzzy Topological Linear Spaces

Definition (2.1)

Let X be any set . A function $\tau : \mathrm{I}^X \to \mathrm{I}$ is called a (gradation)fuzzy topology if

1. $\tau(0) = \tau(1) = 1$
2. $\tau(A \cap B) \geq \min\{\tau(A), \tau(B)\}$ for all $A, B \in \mathrm{I}^X$
3. $\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \min_{\lambda\in\Lambda} \tau(A_\lambda)$ for all $A_\lambda \in \mathrm{I}^X$

- The real number $\tau(A)$ will be called the degree of openness of the fuzzy set A.
- The set X together with the fuzzy topology τ is called a gradation fuzzy topological space and is denoted by (X, τ) or simply X .

Definition (2.2)

Let τ_1 and τ_2 be two gradation fuzzy topologies on the same set X ,we say that τ_1 is stronger than τ_2 if $\tau_1(\mu) \geq \tau_2(\mu)$ for every $\mu \in \mathrm{I}^X$ and denoted by $\tau_1 \leq \tau_2$. i.e.

Let τ_1 and τ_2 be fuzzy topologies on a set X . Then τ_1 is finer than τ_2 denoted by $\tau_1 \leq \tau_2$ if $\tau_2(A) \leq \tau_1(A)$ for all $A \in \mathrm{I}^X$

Remark (2.3)

If T is an ordinary topology on a set X, then the induced gradation fuzzy topology on X is given by $\omega(T) = \{A \in \mathrm{I}^X : A_{\alpha^+} \in T \ \forall \alpha \in \mathrm{I}_1\}$.

Definition (2.4)

Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces . A function $f : X \to Y$ is called fuzzy continuous if $\tau_1(f^{-1}(A)) \geq \tau_2(A)$ for every $A \in \mathrm{I}^Y$.

Theorem (2.5)

Let X be a set, (Y,τ_1) a graded fuzzy topological space. Then there exists a graded topology τ_1 on X such that the function $f:(X,\tau_1)\to(Y,\tau_2)$ is a fuzzy continuous.

Proof :

Let $G=\{f^{-1}(B):B\in \mathrm{I}^Y\}$. then G is a family of fuzzy subsets of X .

For a given $A\in G$, let $P_A=\{B\in \mathrm{I}^Y : A=f^{-1}(B)\}$ and define $\tau_1(A)=\sup\{\tau_2(B):B\in P_A\}$

It is obvious that $\cup\{P_A : A\in \mathrm{I}^X\}=\mathrm{I}^Y$ and $\tau_1(f^{-1}(B))\ge\tau_2(B)$ for every $\nu\in \mathrm{I}^Y$

Now to prove $\tau_1:G\to \mathrm{I}$ satisfies the axioms of definition (2.1).

1. it is obvious that $0=f^{-1}(0)\in G$, $1==f^{-1}(1Y)\in G$ and $\tau(0)=\tau(1)=1$

2.Let $A_1,A_2\in G$, then $A=A_1\cap A_2\in G$ and moreover, $P_A=\{B_1\cap B_2 : B_1\in P_{A_1}, B_2\in P_{A_2}\}$.

Therefore $\tau_1(A)=\sup\{\tau_2(B):B\in P_A\}\ge\sup\{\tau_2(B_1\cap B_2):B_1\in P_{A_1},B_2\in P_{A_2}\}$

$\ge\sup\{\min\{\tau_2(B_1),\tau_2(B_2)\}:B_1\in P_{A_1},B_2\in P_{A_2}\}=\min\{\sup\{\tau_2(B_1)\}:B_1\in P_{A_1}\},\sup\{\tau_2(B_2)\}:B_2\in P_{A_2}\}$

$=\min\{\tau_1(A_1),\tau_1(A_2)\}$

$\tau_1(A_1\cap A_2)\ge\min\{\tau_1(A_1),\tau_1(A_2)\}$.

3. let $A_\lambda\in G$. In a similar way we can show that $\tau_1(\bigcup_{\lambda\in\Lambda}A_\lambda)\ge\min_{\lambda\in\Lambda}\{\sup\tau_2(B_\lambda):B_\lambda\in P_{A_\lambda}\}=\min_{\lambda\in\Lambda}\tau_1(A_\lambda)$

Thus $\tau_1:G\to \mathrm{I}$ satisfies the axioms of definition (2.1).

Now we extend τ_1 to a mapping $\tau_1:\mathrm{I}^X\to \mathrm{I}$ by letting $\tau_1(A)=0$ for all $A\notin G$.

It is easy to check that the function τ_1 thus defined is indeed a graded fuzzy topology.

Moreover, from the construction it is clear that τ_1 is the weakest fuzzy topology on X making the mapping $f:(X,\tau_1)\to(Y,\tau_2)$ fuzzy continuous.

Theorem (2.6)

Let X be any nonempty set and let $\{(Y_a,\sigma_a):a\in \mathrm{A}\}$ be a family of graded fuzzy spaces and consider for each $a\in\mathrm{A}$, a function $f_a:X\to Y_a$. Let $\tau_a:\mathrm{I}^X\to \mathrm{I}$ be the graded fuzzy topology on X for f_a, and let the function $\tau:\mathrm{I}^X\to \mathrm{I}$ be defined by $\tau(A)=\inf\{\tau_\lambda(A):\lambda\in\Lambda\}$ where $A\in \mathrm{I}^X$. Then τ is graded fuzzy topology on X .

Proof :

Let $A_1,A_2\in \mathrm{I}^X$

Since $\tau(A_1\cap A_2)=\inf\{\tau_a(A_1\cap A_2):a\in\mathrm{A}\}\ge\inf\{\min\{\tau_a(A_1)\tau_\lambda(A_2)\}:a\in\mathrm{A}\}$

$\ge\min\{\inf\{\tau_a(A_1):a\in\mathrm{A}\},\inf\{\tau_a(A_2):a\in\mathrm{A}\}\}=\min\{\tau(A_1),\tau(A_2)\}$

2. let $A_\lambda\in \mathrm{I}^X$

Since $\tau(\bigcup_{\lambda\in\Lambda}A_\lambda)=\inf\{\tau_a(\bigcup_{\lambda\in\Lambda}A_\lambda):a\in\mathrm{A}\}\ge\inf\{\min\tau_a(A_\lambda):\lambda\in\Lambda\}:a\in\mathrm{A}\}=\min\{\inf\{\tau_a(A_\lambda):a\in\mathrm{A}\}:\lambda\in\Lambda\}$

$=\min\{\tau(A_\lambda):\lambda\in\Lambda\}$

Hence τ is a graded fuzzy topology on X .

Remark (2.7)

It is clear from theorem (2.5) and from the construction of τ in theorem (2.6) that it is the weakest fuzzy topology on X for which all function $f_\lambda:(X,\tau)\to(X_\lambda,\tau_\lambda)$ are fuzzy continuous. This fuzzy topology τ will be called the initial fuzzy topology for the family of mappings $\{f_\lambda:X\to X_\lambda,\lambda\in\Lambda\}$. It is called the weak topology on X generated by the $\{f_\lambda\}_{\lambda\in\Lambda}$'s and we denote it by $\sigma(X,\{f_\lambda\}_{\lambda\in\Lambda})$ or denoted by $\sigma(X,\mathrm{F})$ where $\mathrm{F}=\{f_\lambda:X\to X_\lambda,\ \ \lambda\in\Lambda\}$

Theorem (2.8)

Let $\{\tau_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of graded fuzzy topologies on a set X and (Y,τ^*) any anther graded fuzzy topological space. If the function $f:(X,\tau_\lambda)\to(Y,\tau^*)$ is fuzzy continuous for every $\lambda\in\Lambda$,then $f:(X,\tau)\to(Y,\tau^*)$ is fuzzy continuous function where $\tau=\inf_{\lambda\in\Lambda}\tau_\lambda$.

Proof :

Let $A\in I^Y$, since $f:(X,\tau_\lambda)\to(Y,\tau^*)$ is fuzzy continuous, then $\tau_\lambda(f^{-1}(A))\geq\tau_2(A)$ for $\lambda\in\Lambda$, so $\inf\{\tau_\lambda(f^{-1}(A)):\lambda\in\Lambda\}\geq\tau_2(A)$

Since $\tau=\inf_{\lambda\in\Lambda}\tau_\lambda$, then $\tau(f^{-1}(A))\geq\tau_2(A)$ $\lambda\in\Lambda$,then $f:(X,\tau)\to(Y,\tau^*)$ is fuzzy continuous .

Theorem (2.9)

If has the weak topology induced by a collection $\{f_\lambda:\lambda\in\Lambda\}$ of functions $f_\lambda:X\to X_\lambda$, then $f:Y\to X$ is fuzzy continuous iff $f_\lambda\circ f$ is fuzzy continuous for each $\lambda\in\Lambda$

Proof :

Suppose that $f:Y\to X$ is fuzzy continuous

Since $f_\lambda:X\to X_\lambda$ is fuzzy continuous for each $\lambda\in\Lambda$, then $f_\lambda\circ f:Y\to X_\lambda$ is fuzzy continuous for each $\lambda\in\Lambda$.

Remark (2.10)

Let F_1 and F_2 be family of functions from X into X_λ , $\lambda\in\Lambda$ such that $F_1\subseteq F_2$ .Then $\sigma(X,F_1)$ is weaker that $\sigma(X,F_2)$.

Theorem (2.11)

Let (X,τ) be a topological space and $F=\{f_\lambda:X\to X_\lambda,\quad \lambda\in\Lambda\}$ be a family of continuous functions from X into the topological space (X_λ,τ_λ). Then $\sigma(X,F)$, the weak topology generated by F is weaker than τ .

3. Bipolar Sets

Definition (3.1)

Let X,Y and Z be linear spaces over F and a function $G:X\times Y\to Z$ associate to each $a\in X$ and to each $b\in Y$ the functions $G_b:X\to Z$ and $G_a:Y\to Z$ by defining $G_a(b)=G(a,b)=G_b(a)$. We say that G is a bilinear function if every G_a and every G_b are linear.

Theorem(3.2)

Let X and Y be linear spaces over F and a function $G:X\times Y\to F$ is a bilinear functional. Put $N_X=\{x\in X:G(x,y)=0,\quad \forall y\in Y\}$ and $N_Y=\{y\in Y:G(x,y)=0,\quad \forall x\in X\}$

Then N_X is a subspace of X and N_Y is a subspace of Y. N_X and N_Y are called null spaces.

Proof :

Since $G(0,0)=G_\circ(0)=0 \Rightarrow 0\in N_X \Rightarrow N_X\neq\phi$

Let $x_1,x_2\in N_X$ and $\alpha,\beta\in F$. For all $y\in Y$

$$G(\alpha x_1+\beta x_2,y)=G_y(\alpha x_1+\beta x_2)=\alpha G_y(x_1)+\beta G_y(x_2)=\alpha G(x_1,y)+\beta G(x_2,y)=\alpha(0)+\beta(0)=0$$

$\alpha x_1+\beta x_2\in N_X \Rightarrow N_X$ is a subspace of X. Similarly to prove N_Y is a subspace of Y.

Definition (3.3)

Let X and Y be linear spaces over F. A bilinear functional $G:X\times Y\to F$ is called a non-degenerate if $N_X=\{0\}$ and $N_Y=\{0\}$.

Remarks (3.4)

1. $N_X=\{0\}$, ,means that : $\forall\ x\neq 0,\ x\in X$, $\exists y\in Y$ such that $G(x,y)\neq 0$,

i.e. if $G(x,y)=0$ for all $y\in Y$, then $x=0$

2. A non-degenerate bilinear functional $G:X\times Y\to F$ will be denoted by $\langle\,,\rangle$,

i.e. $G(x,y)=\langle x,y\rangle$

Definition (3.5)

Tow linear spaces X,Y over F are said to be dual spaces, if there is a non-degenerate bilinear functional $G:X\times Y\to F$.

Example (3.6)

Let X be a linear space over F. Show that X,X' are dual spaces.

Proof :

$X'=\{\psi:X\to F\ ,\ \ \psi$ is linear functional $\}$. Define $G:X'\times X\to F$ by $G(\psi,x)=\langle\psi,x\rangle=\psi(x)$ for all $\psi\in X'$ and for all $x\in X$.

It is clear to show that G is a bilinear functional

$N_X=\{x\in X:G(\psi,x)=0,\ \ \forall\psi\in X'\}=\{x\in X:\psi(x)=0,\ \ \forall\psi\in X'\}$

It is clear to show that N_X is a subspace of X and $N_X=\{0\}$

$N_{X'}=\{\psi\in X':\psi(x)=0,\ \ \forall x\in X\}\ \Rightarrow\ N_{X'}=\{0\}\ \Rightarrow\ G$ is a non -degenerate bilinear functional , then X,X' are dual spaces.

Theorem (3.7)

Let X and Y be dual linear spaces over F. For any element $y\in Y$, define the functional $\phi_y:X\to F$ by $\phi_y(x)=\langle x,y\rangle$ for all $x\in X$.

1. ϕ_y is linear functional
2. $F_Y=\{\phi_y:y\in Y\}$ is a subspace of X'
3. $Y\approx F_Y$

Proof :

1. Let $x_1,x_2\in X$ and $\alpha,\beta\in F$

$\phi_y(\alpha x_1+\beta x_2)=\langle\alpha x_1+\beta x_2,y\rangle=\alpha\langle x_1,y\rangle+\beta\langle x_2,y\rangle=\alpha\phi_y(x_1)+\beta\phi_y(x_2)\Rightarrow\ \ \phi_y$ is linear

2. since $0\in Y\ \Rightarrow\ \phi_y=0\ \Rightarrow\ 0\in F_Y\ \Rightarrow\ F_Y\neq\phi$

Let $\phi_{y_1},\phi_{y_2}\in F_Y$ and $\alpha,\beta\in F$

$$\begin{aligned}(\alpha\phi_{y_1}+\beta\phi_{y_2})(\alpha_1x_1+\alpha_2x_2)&=(\alpha\phi_{y_1})(\alpha_1x_1+\alpha_2x_2)+(\beta\phi_{y_2})(\alpha_1x_1+\alpha_2x_2)\\&=\alpha[\alpha_1\phi_{y_1}(x_1)+\alpha_2\phi_{y_1}(x_2)]+\beta[\alpha_1\phi_{y_2}(x_1)+\alpha_2\phi_{y_2}(x_2)]\\&=\alpha_1[\alpha\phi_{y_1}(x_1)+\beta\phi_{y_2}(x_1)]+\alpha_2[\alpha\phi_{y_1}(x_2)+\beta\phi_{y_2}(x_2)]\\&=\alpha_1(\alpha\phi_{y_1}+\beta\phi_{y_2})(x_1)+\alpha_2(\alpha\phi_{y_1}+\beta\phi_{y_2})(x_2)\end{aligned}$$

$\Rightarrow\ \ \alpha\phi_{y_1}+\beta\phi_{y_2}\in F_Y\ \ \Rightarrow\ \ F_Y$ is a subspace of X'

3. Define $H:Y\to F_Y$ by $H(y)=\phi_y$ for all $y\in Y$

(i) let $y_1,y_2\in Y$ and $\alpha,\beta\in F$. For all $x\in X$

$$\begin{aligned}H(\alpha y_1+\beta y_2)(x)&=\phi_{\alpha y_1+\beta y_2}(x)=\langle x,\alpha y_1+\beta y_2\rangle=\alpha\langle x,y_1\rangle+\beta\langle x,y_2\rangle\\&=\phi_{y_1}(x)+\beta\phi_{y_2}(x)=(\alpha\phi_{y_1}+\beta\phi_{y_2})(x)=(\alpha H(y_1)+\beta H(y_2))(x)\end{aligned}$$

$\Rightarrow\ \ H(\alpha y_1+\beta y_2)=\alpha H(y_1)+\beta H(y_2)\ \ \Rightarrow\ \ H$ is linear

(ii) let $y_1,y_2\in Y$ such that $H(y_1)=H(y_2)$

$\Rightarrow\ \ \phi_{y_1}=\phi_{y_2}\ \ \Rightarrow\ \ \phi_{y_1}(x)=\phi_{y_2}(x)$ for $x\in X\ \ \Rightarrow\ \ \langle x,y_1\rangle=\langle x,y_2\rangle$ for $x\in X$

$\Rightarrow\ \ G(x,y_1)=G(x,y_2)$ for $x\in X\ \ \Rightarrow\ \ G(x,y_1-y_2)=0$ for $x\in X$

$\Rightarrow \quad y_1 - y_2 \in N_Y$

Since $N_Y = \{0\} \quad \Rightarrow \quad y_1 - y_2 = 0 \quad \Rightarrow \quad y_1 = y_2 \quad \Rightarrow \quad H$ is one to one

(iii) Let $\varphi_y \in F_Y \quad \Rightarrow \quad y \in Y \quad \Rightarrow \quad H(y) = \varphi_y \quad \Rightarrow \quad H$ is onto.

So that $Y \approx F_Y$

Definition (3.8)

Lea A subset A of a linear space X over F . We say that A is

1. Symmetric if $-A = A$,
   so that $A \cap (-A)$ is symmetric for any subset A of X
2. Balanced if $\lambda A \subset A$ for every $\lambda \in F$ with $|\lambda| \leq 1$
3. Absorbing if for every $x \in X$, there exists $\lambda > 0$ such that $x \in \lambda A$.
4. Convex if $\lambda x + (1-\lambda) y \in A$ for every $x, y \in A$, $0 \leq \lambda \leq 1$ .Or equivalently if $\lambda A + (1-\lambda) A \subset A$
   for all $0 \leq \lambda \leq 1$.

Definition (3.9)

Let A be a subset of a linear space X over F . The smallest convex set in X which contains A is called the convex hull (or generated) by A and denoted by $conv(A)$.

It is clear to show that

1. $A \subseteq conv(A)$
2. $conv(A)$ = intersection of all convex sets of X which containing A
3. A is a convex iff $A = conv(A)$

Definition (3.10)

A topological linear space X is called a locally convex if there is a convex local base,
i.e. there is a local base β at 0 in X such that every members of β are convex sets.
(every open set in X is a union of convex open sets)

Definition (3.11)

Let X be a real linear space. A partial order relation $\leq$ on X is call linear order if the following axioms are satisfied

1. $x \leq y \quad \Rightarrow \quad x + z \leq y + z$ for all $x, y, z \in X$
2. $x \leq y \quad \Rightarrow \quad \lambda x \leq \lambda y$ for all $x, y \in X$ and $\lambda \in \square$, $\lambda \geq 0$.

- A real linear space endowed with a linear order is called an ordered linear space.
- An element x of an ordered linear space X is said to be positive if $x \geq 0$, and negative if $x \leq 0$.
- The set of all positive elements of an ordered linear space X with be denoted by K, i.e.
  $K = X_+ = \{x \in X : x \geq 0\}$, , where 0 denotes the zero element in X .
  K is called the positive cone (or simply the cone) of X .
- We write (X, K) to denote an order linear space X with positive cone K.

Theorem (3.12)

1. Let X be an ordered linear space with positive cone K. Then

a. K is a convex cone (or wedge) of X , i.e. $K + K \subseteq K$ and $\lambda K \subseteq K$

b. $K \cap (-K) = \{0\}$

2. Let K be a convex cone of a real space X such $K \cap (-K) = \{0\}$, then $x \leq y$ if $y - x \in K$ define an order relation $\leq$ on X for which X become an ordered linear space with K as positive cone.

Definition (3.13)

A subset A of an ordered linear space X over an ordered field F is called

1. A convex cone if A is closed under linear combinations with positive coefficients.
2. A cone (or sometimes called a linear cone) if for each $x \in A$ and positive scalar $\lambda \in F$,
   then $\lambda x \in A$.

Note that : A cone A is convex cone if $\alpha x + \beta y \in A$, for any positive scalars α, β and $x, y \in A$. i.e. A cone A is convex cone if $A + A \subseteq A$.

Definition (3.14)

A graded fuzzy topology τ on a linear space X over F is called a graded fuzzy linear topology if the functions
$f : X \times X \to X$,which defined by $f(x, y) = x + y$ for all $x, y \in X$ and
$g : F \times X \to X$, which defined by $g(\lambda, x) = \lambda x$ for all $x \in X$ and for all $\lambda \in F$.
are continuous, when F is equipped with the graded fuzzy topology induced by the usual topology, $X \times X$ and $F \times X$ are the corresponding product graded fuzzy topologies.

A linear space X with a graded fuzzy linear topology τ , denoted by the pair (X, τ) is called graded fuzzy topological linear space (abbreviated to DFTLS).

Definition (3.15)

Let X and Y be dual graded fuzzy topological linear spaces over F . The weakest topology on X , for which all functional ϕ_y are continuous ,is called the weak topology

on X and it is denoted by $\sigma(X, Y)$, the member of this topology is weakly open sets. Similarly: one may define the weak topology $\sigma(Y, X)$ on Y .

Remark (3.16)

$\sigma(X, Y)$ is a locally convex topology as it is defined by the family $\{P_y\}_{y\in Y}$ of all seminorms $P_y(x) = \left|\langle x, y\rangle\right|$

It is clear to show that : A subset of X is weakly open if for every $x_0 \in A$, there is an $\varepsilon > 0$ and there are $y_1, y_2, \cdots, y_n \in Y$ such that

$$\bigcap_{i=1}^{n}\{x \in X : \mathrm{Re} < x - x_0, y_i > \ \leq \varepsilon\} \subseteq A$$

Theorem (3.17)

Let X and Y be dual graded fuzzy topological linear spaces over F . Any weakly continuous linear functional f on X has a unique representation of the form $f(x) = \langle x, y\rangle$ for all $x \in X$

Proof :

There exists $y_i \in Y$, $i = 1,2,\cdots,n$ such that

$$|f(x)| \leq \max\{\left|\langle x, y_i\rangle\right| : i = 1,2,\cdots,n\}$$

Denoting $f_i(x) = \langle x, y_i\rangle, \quad i = 1,2,\cdots,n$. We have $f(x) = 0$ whenever $f_i(x) = 0 \quad i = 1,2,\cdots,n$

Hence f is a linear combination of $f_i \quad i = 1,2,\cdots,n$

Definition (3.18)

Let X and Y be dual graded fuzzy topological linear spaces over F , and let $A \subseteq X$. The polar set of A is denoted by A° and defined as :

$$A^{\circ} = \{y \in Y : \left|< x, y >\right| \leq 1 , \forall x \in A\} = \{y \in Y : \sup_{x\in A}\left|< x, y >\right| \leq 1\}$$

while the premolar of subset $B \subseteq Y$ is ${}^{\circ}B = \{x \in X : \left|< x, y >\right| \leq 1 , \forall y \in B\} = \{x \in X : \sup_{y\in B}\left|< x, y >\right| \leq 1\}$.

The bipolar set of subset $A \subseteq X$, often denoted by $A^{\circ\circ}$ is the set

$$A^{\circ\circ} = {}^{\circ}(A^{0}) = \{x \in X : \left|\langle x, y\rangle\right| \leq 1 , \forall y \in A^{\circ}\} = \{x \in X : \sup_{y\in A^{\circ}}\{\left|\langle x, y\rangle\right| \leq 1\}$$

It is clear to show that $A \subset A^{\infty}$.

In particular, if X is a Hausdorff locally convex space and $Y = X^{*}$, then

$$A^{\circ} = \{ f \in X^{*} : \left|f(x)\right| \leq 1 , \forall x \in A\} \qquad A^{\circ\circ} = \{x \in x : \left|f(x)\right| \leq 1 , \forall f \in A^{\circ}\}$$

Theorem (3.19)

Let X be a Hausdorff locally convex space over F , and let $A, B \subseteq X$

1. If $A \subset B$, then $B^{\circ} \subset A^{\circ}$

2. If $\lambda \in F$ and $\lambda \neq 0$, then $(\lambda A)^\circ = \lambda^{-1} A^\circ$

3. If A is a subspace of X, then A° is a subspace of Y and $A^\circ = \{y \in Y : \langle x, y \rangle = 0 \quad \forall x \in A\}$

4. $A^\circ = \bigcap_{x \in A} \{ f \in X^* : |f(x)| \leq 1\}$

5. A° is convex

6. A° is weakly closed

Proof :

1. Let $f \in B^\circ \Rightarrow f \in X^*$ and $|f(x)| \leq 1$ for all $x \in B$

Since $A \subseteq B \Rightarrow |f(x)| \leq 1$ for all $x \in A \Rightarrow f \in A^\circ \Rightarrow B^\circ \subseteq A^\circ$

2. Since $f(\lambda x) = \lambda f(x) \Rightarrow |f(\lambda x)| = |\lambda f(x)| = \lambda |f(x)| \Rightarrow |\lambda f(x)| \leq 1$ for all $x \in A \Rightarrow \lambda y \in A^\circ$

Since $\lambda \neq 0 \Rightarrow \lambda^{-1}(\lambda y) \in \lambda^{-1} A^\circ \Rightarrow y \in \lambda^{-1} A^\circ \Rightarrow (\lambda A)^\circ \subseteq \lambda^{-1} A^\circ$

Similarly $\lambda^{-1} A^\circ \subseteq (\lambda A)^\circ$ so that $(\lambda A)^\circ = \lambda^{-1} A^\circ$

3. Since A is absorbs $B \Rightarrow \lambda_0 \in F$ such that $B \subset \lambda A_0$ whenever $|\lambda| \geq |\lambda_0|$

Since $B \subset \lambda A \Rightarrow (\lambda A)^\circ \subset B^\circ \Rightarrow \lambda^{-1} A^\circ \subset B^\circ \Rightarrow A^\circ \subset \lambda B^\circ \Rightarrow B^\circ$ absorbs A°

4. Take $D = \bigcap_{x \in A} \{ f \in X^* : |f(x)| \leq 1\}$

Let $f \in A^\circ \Rightarrow f \in X^*$ and $|f(x)| \leq 1$ for all $x \in A \Rightarrow f \in D \Rightarrow A^\circ \subseteq D$.

Similarly to prove $D \subseteq A^\circ \Rightarrow D = A^\circ$

5. Let $f, g \in A^\circ$ and $0 \leq \lambda \leq 1$, then $|f(x)| \leq 1$ for all $x \in A$ and $|g(x)| \leq 1$ for all $x \in A$

Since $(\lambda f + (1-\lambda) g)(x) = \lambda f(x) + (1-\lambda) g(x)$

$|(\lambda f + (1-\lambda) g)(x)| = |\lambda f(x) + (1-\lambda) g(x)| \leq \lambda |f(x)| + (1-\lambda)|g(x)| \leq \lambda + 1 - \lambda = 1$

$\Rightarrow \lambda f + (1-\lambda) g \in A^\circ \Rightarrow A^\circ$ is convex set.

6. Since $\{ f \in X^* : |f(x)| \leq 1\}$ is weak closed, then $A^\circ = \bigcap_{x \in A} \{ f \in X^* : |f(x)| \leq 1\}$ is weak closed

Theorem (3.20)

Let X be a Hausdorff locally convex space over F, and let $A \subseteq X$, then $A^{\circ\circ} = \overline{co}(A \cup \{0\})$ where the closure "$-$" taken in the weak topology.

Proof :

Let $B = co(A \cup \{0\}) \Rightarrow \overline{B}$ is smallest closed convex set contained A, Since A° is closed convex set in Y $\Rightarrow A^{\circ\circ}$ is closed convex s set in X contained $A \Rightarrow \overline{B} \subset A^{\circ\circ}$. We have to show that $A^{\circ\circ} \subset \overline{B}$

Since $A^\circ = B^\circ \Rightarrow A^{\circ\circ} = B^{\circ\circ}$. To show that $B^{\circ\circ} \subset \overline{B}$

Let us assume that there exists an $x_0 \in B^{\circ\circ}$ such that $x_0 \notin \overline{B}$. By second separation theorem, there exists $f_0 \in X^*$ such that $|f_0(x)| > 1$ and $|f_0(x)| < 1$ for all $x \in B$

Since $|f_0(x)| < 1$ for all $x \in B$, then $f_0 \in B^\circ$, but $|f_0(x_0)| > 1$, then $x_\circ \notin B^{\circ\circ}$. This contradiction $B^{\circ\circ} \subset \overline{B} \Rightarrow A^{\circ\circ} \subset \overline{B} \Rightarrow A^{\circ\circ} = \overline{B}$

Corollary (3.21)

Let X be a Hausdorff locally convex space over F,

1. If M is a subspace of X, then $M^{\circ\circ} = \overline{M}$

2. If $A \subseteq X$, then $A^{\circ\circ\circ} = A^\circ$

Proof :

1. Since $M \subseteq X \quad \Rightarrow \quad M^{\circ\circ} = \overline{co}(M \cup \{0\})$

Since M is a subspace , then $0 \in M \quad \Rightarrow \quad M \cup \{0\} = M \quad \Rightarrow \quad M^{\circ\circ} = \overline{co}(M)$

Since M is a subspace , then M is convex set $\Rightarrow \quad co(M) = M \quad \Rightarrow \quad M^{\circ\circ} = \overline{M}$

2. Let $B = A^0 \quad \Rightarrow \quad A^{000} = (A^0)^{00} = B^{00} = \overline{Co}B$

Since A^0 is convex set $\Rightarrow$ B is convex set $\Rightarrow$ $CoB = B \quad \Rightarrow \quad A^{000} = \overline{B}$ is convex set

Since A^0 is weak closed $\Rightarrow$ B is weak closed set $\Rightarrow$ $\overline{B} = B \Rightarrow \quad A^{000} = B = A^0$

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