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Response Time Domain of Fractional Water Tank Problem

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ABSTRACT

In this work, linear differential equations for dynamic systems of integer orders were studied by solving them and studying the stability of the system. Stability was studied through the mathematical tool, the transfer function, which represents the relationship between inputs and outputs, and extracting the roots of the denominator that determine the stability of dynamic systems for two types of inputs (unit step function-Dirac function) and programming them, as it was shown that the systems are stable. The work has been developed of study and solution of linear differential equations of fractional orders, where the stability of dynamic systems of fractional orders was studied through the fractional transfer function and for two types of inputs and extracting the roots of the denominator using De movers and other algebraic methods and programming them and comparing them with the correct differential equations, where it was noted through the work that the fractal differential equations and the Dirac input give the system more stability. the response of the water tank problem was studied and modified to (Fractional Order System). to improve the accuracy of stability of the water level. The research focused on stability the response time in the time domain of the water level in the tank when changes in inputs occur. The study showed that the problem system is characterized by higher flexibility and a greater ability to represent the dynamic behavior of the tank. This contributes to improving the control efficiency of hydraulic and industrial systems and reduces operational errors Through comparison, it is shown that fractal differential equations are better at stabilizing dynamic systems, giving a more stable system at a lower cost.

1.Introduction

These findings can be used in a variety of contexts, including industrial systems that need precise level control and water distribution networks, which enhance resource sustainability and cut waste. This work used the Transfer Function to study the input-output linear order differential equation. The fractional water tank problem's fractional transfer function in the response time domain was resolved [1]. Mathematicians and physicists are becoming increasingly interested in fractional calculus. This is because, in contrast to traditional methods, the problem can be discussed in a much more elegant and rigorous manner [2]. Numerous mathematical models in science and engineering have made use of fractional differential equations, a recent development in applied mathematics [3].

In order to create mathematical models of numerous physical phenomena, the class of fractional differential equations of various types is used in engineering, physics, control systems, dynamical systems , and mathematics[4]. Such equations, of course, must be solved. Over the past three decades, numerous studies on fractional calculus and fractional differential equations have been published, involving various operators such as Riemann-Liouville operators, Erdlyi-Kober operators, Weyl-Riesz operators, caputo operators, and Grunwald-Letnikov operators, along with their applications in other

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fields[5]. Furthermore, it is established and posed that holomorphic solutions exist and are unique for nonlinear fractional differential equations, such as diffusion problems in the complex domain and Cauchy problems [6].

Fractional operators (integral and derivative) in the complex z-plane C were defined by Srivastava and Owa in [1]. WTP and FWTP stand for Water Tank Problem and Fractional Water Tank Problem, respectively.

MATLAB was used to program the programs..

1.1 BASIC CONCEPTS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Definition (1) the unit step function ,u(t),is defined as :

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

That is, u is a function of time t ,and u has value zero when time is negative and value one when time is positive.[16][17]

Definition (2): the Dirac Function is defined as the function $\delta_k(t)$ provided that $k \to 0$ and take the formula:

$$.\delta_k(t) = \begin{cases} \frac{1}{k} & 0 \leq t < k \\ 0 & t > k \end{cases}$$

Laplace of Dirac function is

$$l\,(\delta_k(t)) = \int_0^\infty \frac{1}{k} e^{-sk} dx$$

$$= \int_0^k \frac{-1}{s} . \frac{1}{k}\,(-s) e^{-sx} dx$$

$$= \frac{-1}{sk} \int_0^k (-s) e^{-sx} dx$$

$$= \frac{-1}{sk} [e^{-sk}]_0^k$$

$$= \frac{-1}{sk} [e^{-sk} - 1]$$

$$let\ k \to \infty \quad then\ l(\delta_k(t)) = 1$$

Definition (3): The complete gamma function Γ(t) plays a significant role In the theory of fractional calculus .the definition of Γ(t) provided by the Euler limit is comprehensive[12].

$$\Gamma(t) = lim_{N\to\infty} \frac{N!\,Nt}{t(t+1)(t+2)\ldots(t+N+} \ ,t > 0$$

However, the advantageous integral convert form is as follows:

Definition (4) : The Fractional order α differential equations are generalized order differential equations [13].

fractional order (α) where $0 < \alpha < 1$ is defined by

$$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)} D^{m}\left(\int_{0}^{x}(x-t)^{m-\alpha-1}f(t)dt\right); 0<\alpha<1 \quad (1)$$

1.1.1 Rule of fractional order Derivation

1. $D^{\alpha}\left(D^{\beta}y(t)\right) = D^{\beta}\left(D^{\alpha}y(t)\right) = D^{\alpha+\beta}y(t)$
2. $D^{\alpha}\left(\lambda y(t)+\mu g(t)\right) = \lambda D^{\alpha}y(t)+\mu D^{\alpha}g(t), \quad \forall\lambda,\mu \in R$
3. $D^{\alpha}\left(ky(t)\right) = kD^{\alpha}y(t) \ ,0<\alpha<1, k=1,2,3,\dots,n$
4. $D^{\alpha}\left(y(t).g(t)\right) = [D^{\alpha}y(t)].g(t)+y(t)[D^{\alpha}g(t)], \ 0<\alpha<1$
5. $D^{\alpha}\left(y(t)+g(t)\right) = [D^{\alpha}y(t)]+[D^{\alpha}g(t)], 0<\alpha<1$

1.1.2 Basic Rules of Fractional Derivative Equation

1. $D^{\alpha}(t^{n}) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha} \quad ,0<\alpha<1$
2. $D^{\alpha}(sin\ at) = a^{\alpha}sin\left(at+\frac{\pi}{2}\alpha\right), 0<\alpha<1, a\ is\ constant$
3. $D^{\alpha}(cos\ at) = a^{\alpha}cos\left(at+\frac{\pi}{2}\alpha\right), 0<\alpha<1, a\ is\ constant$
4. $D^{\alpha}(e^{kt}) = k^{\alpha}e^{kt} \quad , k\ is\ constant$
5. $\boldsymbol{D^{\alpha}(c)} = \frac{\boldsymbol{ct^{-\alpha}}}{\boldsymbol{\Gamma(1-\alpha)}}\boldsymbol{t^{n-\alpha}} \quad \boldsymbol{, c\ is\ constant}$.

2. PROPOSED METHOD FOR SOLVING FRACTIONAL WATER TANK PROBLEM USING DE MOIVRE'S

2.1 *De Moivre's theorem*

De Moivre's theorem is aformula used for finding powers of complex numbers. De Moivre's states that for any complex number [17],

$$Z^{n} = r(\cos(\theta)+i\sin(\theta))^{n}, \quad n\in N, \theta\in R \quad (2)$$

$$Z^{n} = cos\ (n\theta)+i\ sin\ (n\theta) \quad (3)$$

$$(cos\theta - i\ sin\theta)^{n} = \cos(n\theta) - i\ sin(n\theta) \quad (4)$$

the problem $Z^{n}+c=0$, c = 1

the roots of c is founded using De Moivre's theorem as follows [18]:

$\forall\ n\in Z^{+}, \theta\in R \qquad then$

$$\sqrt[n]{Z} = r^{n}\left[cos\frac{\theta+2\pi k}{n}+i\ sin\ \frac{\theta+2\pi k}{n}\right] \ ,k=1,2,\dots,n-1 \quad (5)$$

2.2 *Stability of Physical Problem*

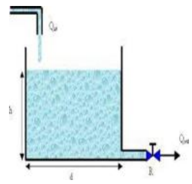
we discuss the treatment unit step function and Dirac function by finding the roots of the equation using De Movires, and taking the inverse Laplace of the function for all values of and drawing the figure and comparison of both cases.

2.2.1 *FWTP using unit step function*

The ODE of water tank is the from ,

$$f(t) = A\frac{\partial h(t)}{\partial t} + \frac{pg}{R}\,h(t) \qquad (6)$$

$(A{:}\,Tank\ space\ , p{:}\,preser, g{:}\,Earth\ acceleration\ constant\ , R{:}\,Tank\ width)$



FIGURE(1):WATER

Using Laplace Transform

$F(S) = \left(AS + \frac{pg}{R}\right)H(S)$

$G(S) = \frac{H(S)}{F(S)} = \frac{input}{output}$

G(S) is transfer function of input-output

$H(S) = G(S)F(S)\ ,\ \ H(S) = \frac{1}{S}\ unit\ step\ function$

$$G(S) = \frac{1}{(S+1)}.\frac{1}{S}$$

$$if\ G_{\alpha}(S) = \frac{1}{(S^{\alpha}+1)}.\ \frac{1}{S} \quad 0 < \alpha < 1\ , \alpha\ is\ order\ transfer\ function$$

with $\alpha = \frac{1}{2}$

Using a result of De Moivre's theorem

$S_1 = -i\ , S_2 = i$

$$G(S) = \frac{1}{(S+i)(S-i)(S-0)}$$

$$g(t) = \frac{-1}{2}e^{-it} + \frac{1}{2}\,e^{it} + 1 \qquad (7)$$

2.2.2 *FWTP using Dirac function*

$$G_{\alpha}(S) = \frac{1}{(S^{\alpha}+1)}.\ 1 \quad 0 < \alpha < 1\ , \alpha\ order\ of\ transfer\ function$$

$$G_{\frac{1}{2}}(S) = \frac{1}{(S^{\frac{1}{2}}+1)}$$

Using a result of De Moivre's theorem

$S_1 = -i \quad , \quad S_2 = i$

$$G(S) = \frac{1}{(S+i)(S-i)}$$

$$g(t) = \frac{-1}{2}e^{-it} + \frac{1}{2}\,e^{it} \qquad (8)$$

The following table shows state roots for α

state	Value of α	Roots of $s^{\alpha} + 1 = 0$
1	1	-1

2	1/2	i , -i
3	1/3	$\frac{1}{2}+\frac{\sqrt{3}}{2}i$, -1 , $\frac{1}{2}-\frac{\sqrt{3}}{2}i$
4	1/4	$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $\frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$, $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$
5	1/5	0.8+0.6 i , -0.3+0.1i , -1 , -0.3-0.1 i , 0.8-0.6 i

Table 1- state roots for α

The following table shows other state of α of case I and case II :

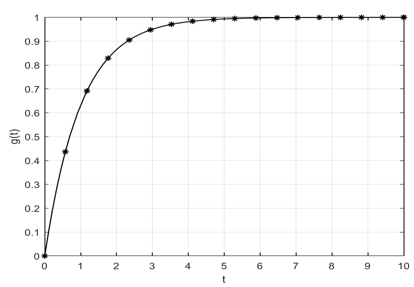
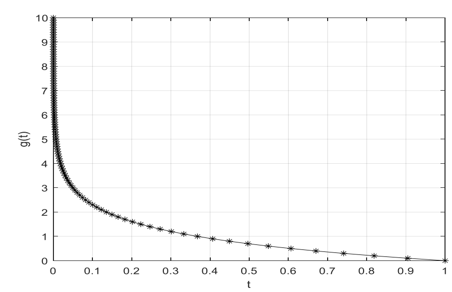
α	**Fig** . *case I*	***Fig*** . *case II*
1		

Figure (2) Response of system for $\alpha = 1$

for the figure (2) above,In the case (I)and case (II) with $\alpha = 1$ is stable in both cases so that it is more stable in the Dirac function.

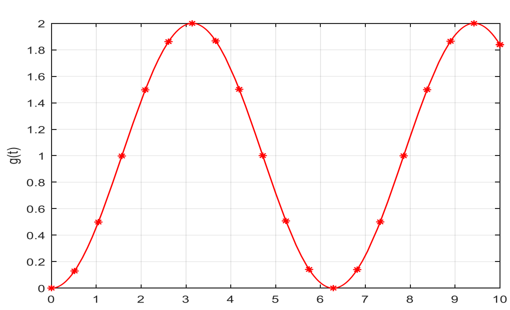
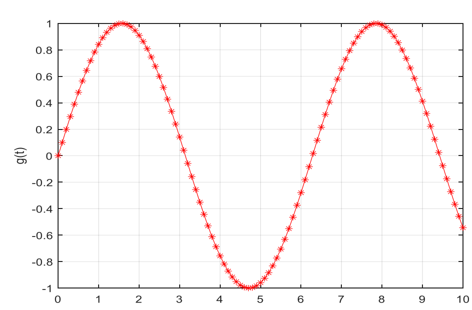
α	**Fig** . *case I*	***Fig*** . *case II*
1/2		

Figure (3) Response of system for $\boldsymbol{\alpha = \frac{1}{2}}$

For the figure (3)above ,In the case (I)and case (II) with $\alpha = 1/2$ is stable in both cases ,but faster stable in Dirac function ,where the curve reaches1 ,while in the unit step function the curve reaches 2.

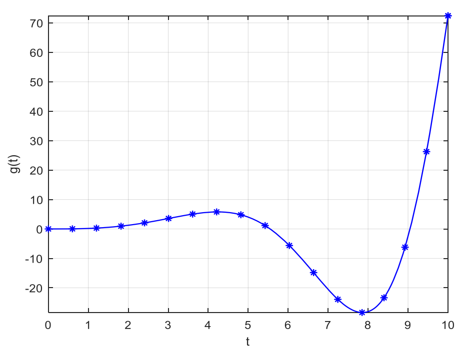
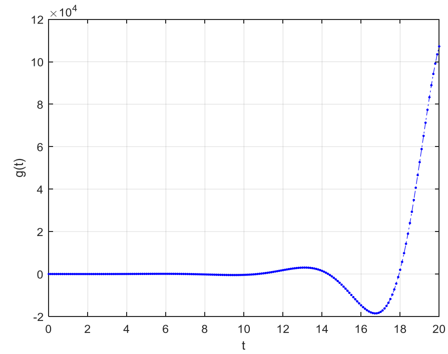
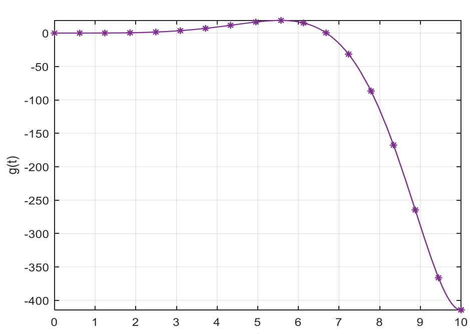
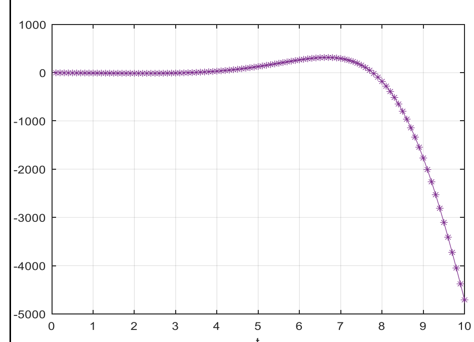
α	**Fig** . *case I*	***Fig*** . *case II*
1/3		

Figure (4) Response of system for $\boldsymbol{\alpha = \frac{1}{3}}$

For the figure (4) above,In the case (I)and case (II) with $\alpha = 1/3$ in the case one The curve changes in period [2,8] after which it becomes unstable, but in the case two The curve changes in period [12,17] after which it becomes unstable.

α	**Fig** . *case I*	***Fig*** . *case II*
1/4		

Figure(5) Response of system for $\boldsymbol{\alpha = \frac{1}{4}}$

For the figure (5) above ,In the case (I)and case (II) with $\alpha = 1/4$ in the case one The curve changes in period [3,6] after which it becomes stable, but in the case two The curve changes in period [5,6]after which it becomes stable.

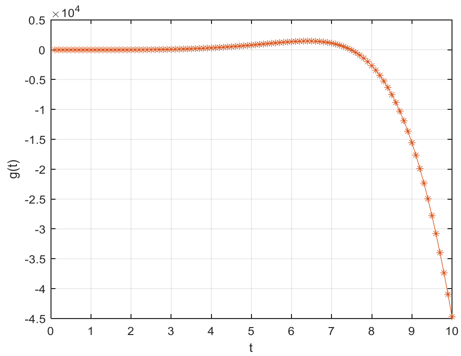
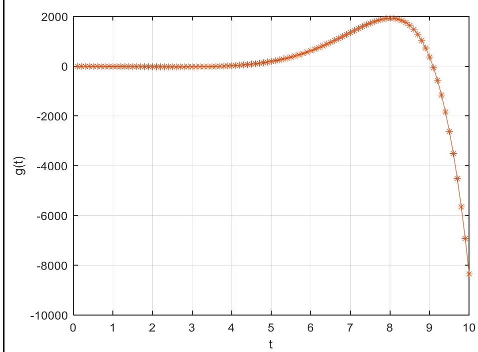
<table>
<tr><th>α</th><th>Fig . case I</th><th>Fig . case II</th></tr>
<tr><td>1/5</td><td>
</td><td>
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Figure (6) Response of system for α=1/5.

For figure (6) above ,In the case (I)and case (II) with $\alpha = 1/5$ in the case one The curve changes in period [5,7] after which it becomes stable, where the curve it reaches less than 0.5 ,while in the unit step function the curve up to whom 2000.

3. IMPLEMENTATIONS AND COMPARISONS OF PROPOSED METHOD

As shown in Figures (3,4), a problem involving multiple fractional derivatives was examined in order to clear the suggested approach. De Moivre's method is used to solve the fractional equations and find the roots in order to investigate the stability of water falling into the tank. The inputs were compared to determine which one produced more consistent results.

4.CONCLUSIONS AND DISCUSSION

In this research, a new approach was used effectively to solve the response of the tank to the descent of water using a fractional model, and this gives the most appropriate solution. Comparison between the inputs was also studied. It was found that the best method for maintaining the stability of the descent of water is the Dirac function, as the method of finding the roots was an acceptable and encouraging method to obtain high work efficiency and the lowest cost.

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