# Subordination Result for new subclass of Regular Univalent Functions

| Authors Names | ABSTRACT |
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| *Mohammed H. Saloomi*<br>***Publication data***: 18 /12 /2023<br>***Keywords:*** Regular function, univalent Regular function, convolution, subordination, subordination factor sequence | The objective of present work is to introduce new classes $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$ and discuss the sufficient conditions for $\hbar(t)$ which are given by using coefficient inequality. Several new consequences of subordination for this subclass are pointed out, which are new and not yet studied. |

## 1. Introduction and Standard Result

Let Ǻ be symbolize the class of functions in the shape

$$\hbar(t) = t + \sum_{n=2}^{\infty} \mathcal{E}_n t^n . \qquad \dots (1.1)$$

and regular in the disk $\hat{E} = \{t: |t| < 1\}$ and normalized by $\hbar(0) = 0$ and $\hbar'(0) = 1$ .

Furthermore, by Ş we shall symbolize the class of functions in Ǻ which are univalent in Ê. Let $C(\xi)$ and $\mathcal{S}^*(\xi)$be class of convex and starlike functions of order $\xi$ respectively where$0 \le \xi < 1$. If we put $\xi = 0$, then $C$ and$\mathcal{S}^*$ represent the class of convex, starlike functions respectively.

Given two functions ħ, ƙ in Ǻ such that,

$$ƙ(t) = t + \sum_{n=2}^{\infty} \nu_n t^n ,$$

the convolution $\hbar(t) * ƙ(t)$ is defined by

$$\hbar(t) * ƙ(t) = t + \sum_{2}^{\infty} \mathcal{E}_n \nu_n t^n, \quad t \in \hat{E}.$$

A regular function $\hbar(t)$ is subordinate to regular function $ƙ(t)$ if there exists a regular function $\psi(t)$ in Ê satisfying $\psi(0) = 0$ and $|\psi(t)| < 1$ $(t \in \hat{E})$ such that

$$\hbar(t) = ƙ(\psi(t)) \text{ for all } t \in \hat{E},$$

we, denote this subordination by

$$\hbar(t) \prec ƙ(t)$$

In special, when $ƙ(t)$ is univalent in Ê,

$$\hbar(t) \prec ƙ(t) \leftrightarrow \hbar(0) = ƙ(0) \; and \; \hbar(\hat{E}) \subset ƙ(\hat{E}).$$

The concept of subordination can be found in [1]. Also several authors have defined different subclasses of univalent regular functions by using the concept of subordination such as [2,3,4,].

In [5] for sequence of complex numbers $\{\mathcal{E}_k\}$. This sequence is said to be subordination factor sequence if whenever ħ(t) is convex and regular univalent in Ê, then

$$\sum_{1}^{\infty} \theta_k \mathcal{E}_k t^k \prec \hbar(t) \text{ where } (t \in \hat{E}, \mathcal{E}_1 = 1) \qquad (1.2)$$

To discuss fundamental results, we shall introduce the following

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Lemma (1.1)[6]: Let $\{\theta_k\}$ be sequence in $\mathbb{C}$. Then this sequence $\{\theta_k\}$ is subordinating factor sequence if and only if

$$\mathcal{R}e\{1+2\sum_{k=2}^{\infty}\theta_k t^k\}>0, (\ t\in \hat{E}) \qquad (1.3)$$

Many authors have investigated subordination results and obtained sufficient conditions for functions in some subclasses of(see [7,8,9]). So it will be the aim of this work to get a several conditions and Interesting properties for functions related to this subclass $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$.

## 2. Sufficient Conditions for the Function Class $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$

Motivated by earlier works on differential subordination [7,8,9] we, introduce the next definition:

**Definition (2.1):** For $0\le\alpha\le 1$, $0<\beta\le 1$, $-1\le\mu\le 1$, $0\le\lambda\le 1$ and $0\le\gamma\le 1$

A function in (1.1) belong to family $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$if it satisfies

$$\left|\frac{\hbar'(t)+t\hbar''(t)}{\alpha[\hbar'(t)+t\hbar''(t)]-\mu\hbar'(t)-(1-\lambda)(1-\mu)\gamma\hbar'(t)}\right|<\beta \text{ , for all } t\in\hat{E}.$$

In this part of our work, we will prove a sufficient condition for regular functions in $\hat{E}$ to be in $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$.Also we shall prove some results of subordination for this class $(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$.

Theorem 2.2: Let $\hbar(t)\in\mathring{A}$ given in (1.1) and satisfies the following relation

$$\sum_{n=2}^{\infty}[1+n^2-\beta(\alpha n^2+\{\mu+(1-\lambda)(1-\mu)\gamma\})]|\mathcal{E}_n|\le\beta[\alpha-\mu-(1-\lambda)(1-\mu)\gamma], \text{ where}$$

$$0\le\alpha\le 1, 0<\beta\le 1, -1\le\mu\le 1, 0\le\lambda\le 1, \text{and } 0\le\gamma\le 1.$$

Then $\hbar$ in the class$(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$.

Proof: Assume the next inequality hold

$$\sum_{n=2}^{\infty}[1+n^2-\eta(\alpha n^2+\{\mu+(1-\lambda)(1-\mu)\gamma\})]|a_n|\le\eta[\alpha-\mu-(1-\lambda)(1-\mu)\gamma].$$

its suffices to prove that

$$\left|\frac{\hbar'(t)+t\hbar''(t)}{\alpha[\hbar'(t)+t\hbar''(t)]-\mu\hbar'(t)-(1-\lambda)(1-\mu)\gamma\hbar'(t)}\right|<\beta \text{for all } t\in\hat{E}$$

$$\left|\frac{\hbar'(t)+t\hbar''(t)}{\alpha[\hbar'(t)+t\hbar''(t)]-\mu\hbar'(t)-(1-\lambda)(1-\mu)\gamma\hbar'(t)}\right|=$$

$$\frac{1+\sum_2^{\infty}n^2\mathcal{E}_n t^{n-1}}{\alpha[1+\sum_2^{\infty}n^2\mathcal{E}_n t^{n-1}]-\mu-\mu\sum_2^{\infty}n\mathcal{E}_n t^{n-1}-(1-\lambda)(1-\mu)\gamma-(1-\lambda)(1-\mu)\gamma\sum_2^{\infty}n\mathcal{E}_n t^{n-1}}$$

$$<\frac{1+\sum_2^{\infty}n^2|\mathcal{E}_n|}{\alpha[1+\sum_2^{\infty}n^2|\mathcal{E}_n|]-\mu-\mu\sum_2^{\infty}n|\mathcal{E}_n|-(1-\lambda)(1-\mu)\gamma-(1-\lambda)(1-\mu)\gamma\sum_2^{\infty}n|\mathcal{E}_n|}\le\beta$$

$$1+\sum_2^{\infty}n^2|\mathcal{E}_n|\le\beta\{[\alpha+\alpha\sum_2^{\infty}n^2|\mathcal{E}_n|]-\mu-\mu\sum_2^{\infty}n|\mathcal{E}_n|-(1-\lambda)(1-\mu)\gamma-(1-\lambda)(1-\mu)\gamma\sum_2^{\infty}n|\mathcal{E}_n|\}$$

$$1+\sum_2^{\infty}n^2|\mathcal{E}_n|-\beta[\alpha\sum_2^{\infty}n^2|\mathcal{E}_n|+\mu\sum_2^{\infty}n|\mathcal{E}_n|+(1-\lambda)(1-\mu)\gamma\sum_2^{\infty}n|\mathcal{E}_n|]\le\beta(\alpha-\mu-(1-\lambda)(1-\mu)\gamma$$

$$\sum[1+n^2-\beta\{\alpha n^2+n(\mu+(1-\lambda)(1-\mu)\gamma)\}]|\mathcal{E}_n|\leq\beta[\alpha-\mu-(1-\lambda)(1-\mu)\gamma].$$

Thus, the last step is equivalent to our condition in this theorem. Thus, the proof is complete.
Another important result provides subordination result involving the function class$(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$.
Theorem (2.3): If the function $\hbar(t)$ of the form (1.1) is in the class$(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$ and the following increasing sequence

$$\{|n-\beta+1|+|n-\beta-1|-2\alpha(n-1)\}_{n=2}^{\infty}\text{, for all } n\geq 2,$$

then for any univalent function $\Bbbk(t)\in C$ and $t\in\hat{E}$

$$\frac{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}{2[7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma|]}(\hbar*\Bbbk)(t)\prec\Bbbk(t)\ldots \qquad (2.1)$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $-1\leq\mu\leq 1$, $0\leq\lambda\leq 1$ and $0\leq\gamma\leq 1$

and

$$\mathcal{R}e\hbar(t)>-\frac{7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma}{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|} \qquad \ldots (2.2)$$

The factor which is constant in the subordination (2.1).

$$\frac{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}{2[7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma|]}$$

cannot be changed by another greater than it. It is the best chosen.
Proof: Let $\hbar(t)\in(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$and suppose that

$$\Bbbk(t)=t+\sum\nu_n t^n\in C.$$

Assume

$$\varrho_n=1+n^2-\beta\,|\alpha n^2+[\mu+(1-\lambda)(1-\mu)\gamma]|.$$

The assertion (2.1) become

$$\frac{\varrho_2}{2[\varrho_2+\varrho_1]}(\hbar*\Bbbk)(t)\prec\Bbbk(t). \qquad \ldots(2.3)$$

Then we readily have

$$\frac{\varrho_2}{2[\varrho_2+\varrho_1]}(\hbar*\Bbbk)(t)=\frac{\varrho_2}{2[\varrho_2+\varrho_1]}\left[t+\sum_2^{\infty}\mathcal{E}_n\nu_n t^n\right]. \qquad \ldots(2.4)$$

Accordingly, and through (1.2), the confirmation of the subordination result (2.3) is true if $\{\frac{\varrho_2\mathcal{E}_n}{2[\varrho_2+\varrho_1]}\}_1^{\infty}$ is a subordination factor sequence, with $\mathcal{E}_1$=1.

By using Lemma (1.1) this is equivalent to the condition

$$\mathcal{R}e\left[1+\sum_1^{\infty}\frac{\varrho_2\mathcal{E}_n}{\varrho_2+\varrho_1}t^n\right]>0, t\in\hat{E}, \qquad \ldots(2.5)$$

$$\mathcal{R}e\left[1+\sum_1^{\infty}\frac{\varrho_2\mathcal{E}_n}{\varrho_2+\varrho_1}t^n\right]=\mathcal{R}e\left\{1+\frac{\varrho_2}{\varrho_2+\varrho_1}t+\frac{1}{\varrho_2+\varrho_1}\sum_2^{\infty}\varrho_2\mathcal{E}_n t^n\right\}$$

$$=1+\mathcal{R}e\left\{\frac{\varrho_2}{\varrho_2+\varrho_1}t+\frac{1}{\varrho_2+\varrho_1}\sum_2^{\infty}\varrho_2\mathcal{E}_n t^n\right\}$$

$$\geq 1-\left|\frac{\varrho_2}{\varrho_2+\varrho_1}\mathrm{t}+\frac{1}{\varrho_2+\varrho_1}\sum_2^\infty \varrho_2\mathcal{E}_n\, t^n\right|$$

$$\geq 1-\left\{\frac{\varrho_2}{\varrho_2+\varrho_1}\rho+\frac{1}{\varrho_2+\varrho_1}\sum_2^\infty \varrho_2|\mathcal{E}_n|\,\rho^n\right\}, \qquad \ldots (2.6)$$

Where $\varrho_1, \varrho_2 > 0$.

Since

$$\varrho_n = \{1+n^2-\beta|\alpha n^2+[\mu+(1-\lambda)(1-\mu)\gamma]|\}_2^\infty$$

is an increasing of $n(n \geq 2)$ we get

$$\varrho_2\sum_2^\infty|\mathcal{E}_n| \leq \sum_2^\infty \varrho_n\,|\mathcal{E}_n| \leq \beta|\alpha-\mu-(1-\lambda)(1-\mu)\gamma|. \quad \ldots(2.7)$$

Applying (2.7) in (2.6) we get

$$1-\left\{\frac{\varrho_2}{\varrho_2+\varrho_1}\rho+\frac{1}{\varrho_2+\varrho_1}\sum_2^\infty \varrho_2|\mathcal{E}_n|\,\rho^n\right\} > 1-\left\{\frac{\varrho_2}{\varrho_2+\varrho_1}\rho+\frac{\varrho_1}{\varrho_2+\varrho_1}\rho\right\}>0, \text{ (since } |\mathrm{t}|=\rho<1).$$

Thus (2.5) is realized in | t | »1. Consequently the subordination (2.1) is established.

By taking a convex function in (2.1)

$$g(t)=\frac{t}{1-t}=t+\sum_{n=2}^\infty t^n$$

The inequality (2.2) follows.

Now, we consider function $g(t) \in (MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$given by

$$g(t)=t-\frac{\beta|\alpha-\mu-(1-\lambda)(1-\mu)\gamma|}{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}t^2, \qquad (2.8)$$

where $0 \leq \alpha \leq 1$ , $0 < \beta \leq 1$ , $-1 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \gamma \leq 1$.

By using the result (2.1), we get

$$\frac{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}{2[7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma|]}g(t) \prec \frac{t}{1-t}\ . \qquad (2.9)$$

Also, we can prove the following for the function$g(t)$

$$\min\left[\mathcal{Re}\left\{\frac{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}{2[7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma|]}g(t)\right\}\right]=-½, t \in \hat{E}\ .$$

Then the value of next constant

$$\frac{5-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|}{2[7-2\beta|2\alpha+\mu+(1-\lambda)(1-\mu)\gamma|-\beta|\alpha+\mu+(1-\lambda)(1-\mu)\gamma|]}$$

cannot be changed by another greater than it. The proof is complete.

Letting$\mu=-1 and\ \lambda={}^1/_2$, in Theorem (2.2), we have the next result for the class$(MO)_{-1,{}^1/_2}^{\alpha,\gamma}(\beta)$).

Corollary (2.4): If $ћ(t)$ of the form (1.1) is in the class$(MO)_{\mu,\lambda}^{\alpha,\gamma}(\beta)$, then for each univalent function $ƙ(t) \in$ C , $t \in \hat{E}$

$$\frac{5-2\beta|2\alpha+\gamma-1|}{2[7-2\beta|2\alpha+\gamma-1|-\beta|\alpha+\gamma-1|]}(\hbar * k)(t) \prec k(t) \;,$$

and

$$\mathcal{R}e(\hbar(t)) > -\frac{7-2\beta|2\alpha+\gamma-1|-\beta|\alpha+\gamma-1|}{5-2\beta|2\alpha+\gamma-1|}.$$

The next constant factor

$$\frac{5-2\beta|2\alpha+\gamma-1|}{2[7-2\beta|2\alpha+\gamma-1|-\beta|\alpha+\gamma-1|]}$$

cannot be changed by another greater than it..

Putting $\mu = -1$, $\lambda=0$ and $\beta = 1$ in Theorem (2.3), we get the next result for the class $(MO)_{-1,0}^{\alpha,\gamma}(1)$.

Corollary (2.5): If $\hbar(t)$ of the form (1.1) belong to the class $(MO)_{-1,0}^{\alpha,\gamma}(1)$,

then for each univalent function $k(t) \in C$, $t \in \hat{E}$

$$\frac{5-2|2\alpha+2\gamma-1|}{2[7-2|2\alpha+2\gamma-1|-|\alpha+2\gamma-1|]}(\hbar * k)(t) \prec k(t) \;,$$

and

$$\mathcal{R}e(\hbar(t)) > -\frac{7-2|2\alpha+2\gamma-1|-|\alpha+2\gamma-1|}{5-2|2\alpha+2\gamma-1|}.$$

The constant factor

$$\frac{5-2|2\alpha+2\gamma-1|}{2[7-2|2\alpha+2\gamma-1|-|\alpha+2\gamma-1|]}$$

cannot be changed by another greater than it.

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