On Lattice Normed Space

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ABSTRACT

In the current paper, submitted defined of space as metric of lattice, lattice norm, latticeΔ-norm, Banach lattice, lattice quasi-norm, lattice F-norm, lattice space being modular , lattice semi-normed space, Minkowski functional, sequence of Cauchy , space and we proved some theorms.

1. Introduction

The first to introduce the concept for the standard was the Austrian scientist E. Helly

(1844-1943), but he did not use the name of the standard nor its symbol, it was known as whichever function that fulfills certain conditions. Spaces of Banach are named following after the Stefan Banach (Polish mathematician), who in 1920-1922 familiarized such concept and considered it analytically joining Hans Hahn and Eduard Helly. Al- Mayahi, introduced [3] a description of topological space being linear and lattice metric as invariant. See also [1], [2], [8], [9] and[10], they presented description of lattice normed space. They presented in[5] and [6], define of lattice Frechet space, lattice semenorm, lattice normable, open and ball being closed in lattice norm. In[9], and [7]. They introduced the description of F-normed space, Δ normed space and quasi normed space. Sharma and Vasishtha presented [4] the description of space of Banach, space being modular, lattice functional being convex, lattice symmetric, and Minkowski's functional. our paper, we provide description, proposition, remarks, formula and example for above concept.

2. Basic Concepts

2.1 Space as metric of Lattice being Linear

Metric space being linear is a metrizable topological space being linear. A topology □ be on an X set is supposed as metrizable when a lattice metric d is there on X that is well-matched with □ . In such case, the balls of radius 1/n fixed at x form a resident base at x .

Description (2.1) [1]:

Suppose X is a space being linear over F and suppose d be a function as lattice metric on X. It can be said that d is a lattice metric as invariant on X, when d(x+z,y+z)= d(x,y) for all x,y,z ∈X. Bulleted lists may be included and should look like this:

Remark:

d(-x,0)+d(x,0) ∀x ∈X,

Since d(-x,0)=d(-x+x,0+x)=d(0,x)+d(x,0)

Formula (2.2):

Suppose X be a space being linear over F and suppose d be an metric as invariant on X.

- d(nx,0)≤nd(x,0) for every x∈X and for n=1,2,3,….

When n=2

- d(nx,0)=d(2x,0)=d(x+x,0)=d(3x,0)

2) When{x_n } is an order in X and when x_n→0 as n→∞, after that +ve scalar are there $\boldsymbol{\lambda_n}$→∞ and $\boldsymbol{x}_n\boldsymbol{\lambda_n}$→0, in which n→∞.

Evidence

d(nx,0)≤$\sum_{k=1}^{n} d(kx,(k-1)x)$ =nd(x,0).

Since x_n→0⟹ an increasing +ve integer sequence is there $\{n_k\}$

Thus d(x_n,0)<1/k^2, when n≥ n_k . Place $\lambda_k = \begin{cases} 1 & n<n_k \\ k & n_k \le n \le n_{k+} \end{cases}$

For such "n , we have $d(\lambda_n x_n, 0) = kd(x_n, 0) < \frac{1}{k}$. Therefore $\lambda_n x_n \to 0$ in which $n \to \infty$.

Description (2.3) [1]:

Suppose τ be a Topology say that τ is a Topology on X when it fulfills the axioms as follow:

1. ϕ ,xϵτ
2. When $A_1, A_2, A_3, \dots, A_n \epsilon \tau$, afterthat $\bigcap_{i=1}^{n} A$ iϵ.
3. When $A_\lambda \epsilon \tau$ for all λϵ ∧, afterthat U A_λϵ.

Description(2.4)[6]:

1) A topological space being linear X with topology is named a lattice F-space when its topology is inducing an invariant as complete d .
2) A topological space being linear X is named lattice Fréchet space when X is a locally convex lattice F-space.
3) A space being polish is separable complete space as metric of lattice.

Description(2.5)[6]:

Suppose X be a space being linear over F . A Lattice Δ - norm on X is a function ‖.‖ :X→E taking the properties as follow:

a. ‖x‖>0 for all x∈X ,x≠0.

b. ‖λx‖≤‖x‖ ∀x∈X, and ∀0<|λ |<1 .

c. lim(λ→0)‖λx‖=0 , ∀x∈X .

d. ‖x+y‖≤c max{‖x‖,‖y‖} for whole x,y∈X,where c>0 is independent of x ,y

Remark

1) When . is a lattice Δ - norm on X , afterthat it induces on X a linear ? topology that is metrizable

2) A local base β of 0 is set by the form sets B_n ={ x∈X:‖x‖<1/n},i.e.

β={ $B_n : n \in N^*$},where B_n ={ x∈X:‖x‖<1/n}.

3) An order $\{x_n\}$ in X converges to x ∈ X whenf $||x_n - x|| \to 0$.

4) When τ is a topology on X with a local countable base β ={β_n} Thus n$\beta_n = \{0\}$, every β_n is balanced and $\beta_{n+1} + \beta_{n+1} \subset \beta_n$. Afterthat we can define a Δ - norm on by $||x|| = \sup\{2^{-n}: x \notin \beta_n\}$ and the Δ - norm induces the original topology ; here c= 2.

5) When ||. || is whichever lattice F- norm on X , afterthat d(x,y) = $||x - y||$ is lattice metric as invariant on X.

Description(2.6)[7]:

1) A lattice Δ- norm ‖.‖ on X is named a lattice F-norm when it fulfills ‖x+y‖≤‖x‖+‖y‖ for whole x,y∈X.

2) A lattice Δ - norm ‖.‖ on X is named a lattice quasi-norm when it fulfills ‖λx‖=|λ|‖x‖ for λ∈F and x∈X.

Formula(2.7)

Suppose . be whichever lattice Δ - norm on X .Choose p so that 2^(1/p)=c. After that for whichever x_1,x_2,..,x_n ∈X, we get

‖x_1+x_2+..+x_n ‖≤4^(1/p) ($\sum_{i=1}^{n} ||x_i||^p$)^(1/p)

Evidence:

By induction when n=1 ‖x_1 ‖≤4^(1/p) ‖x_1‖

We assume it is true when n=k

‖x_1+x_2+..+x_k ‖ ≤4^(1/p)($\sum_{i=1}^{n} ||x_i||^p$)^(1/p)

And we proved when n=k+1

‖x_1+x_2+..+x_k+x_{k+1}) ‖≤4^(1/p)($\sum_{i=1}^{k+1} ||x_1 + x_2 + .. + x_k||^p + ||x_{k+1}||^p$ ($\sum_{i=1}^{n} ||x_i||^p$)^(1/p)

Formula(2.8)

Suppose ‖.‖be whichever lattice □ - norm on X , after that when p is chosen so that 2^(1/p)=c , the formal

‖x‖=inf$\|x\| = \inf\{\sum_{i=1}^{n}\|x_i\|^p : \sum_{i=1}^{n} x_i = x\}$, define a lattice F- norm on X given the identical topology .

Evidence: Simply note 1/4 ‖x‖^p≤ |(‖x‖)|≤ ‖x‖^p.

Corollary (2.9)

Suppose X is a Hausdorff topological space being linear with a countable local base of 0 . Afterthat X is amortizable and the topology may be given a metric as invariant.

Remark

Every metrizable topological space being linear X able to be embedded as a lattice linear dense sub-space F- Space Y. The construction of Y is simply to the normal metric space complete of X regarding a metric as invariant and extend the space being linear operations in the clear way. The space Y obtained in this way is unique; it is not depending on the particular metric as invariant choice.

Formula(2.10)

When M is closed sub-space of a lattice F- Space X , after that X\ M is a lattice F-Space .

Evidence :

Suppose M is closed sub-space of a metrizable topological linear spec ,after that M lattice F-space, after that X\ M is also metrizable .

Description (2.11) [11]:

Suppose X is a space being linear over a field $\mathbb{F}$.

A function $M: X \to \overline{E}$ is named a modular on X when staffing the axioms as follow:

1. $M(x) \geq 0$ for whole $x \in X$.
2. $M(x) = 0 \iff x = 0$ for whole $x \in X$.
3. $M(ƛx) = M(x)$ for all $x \in X$ and for whole $ƛ \in \mathbb{F}$ with $|ƛ| = 1$.
4. $M(\alpha x + \beta y) \leq M(x) + M(y)$ when f $\alpha, \beta \geq 0$, for whole x,y ∈ X.
   When (4) replaced by
5. $M(\alpha x + \beta y) \leq \alpha M(x) + \beta M(y)$, for whole x,y ∈ X, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. It can be said M is a convex modular.

This shows M is increasing function

A modular M describes an equivalent space being modular such as, the liner space X_M given by $X_M = \{ x \in X : M(ƛx) \to 0$ whenever $ƛ \to 0 \}$.

Example (2.12):

Suppose $X = E$ along $M(x, y) = |x| + |y|$, for whichever pair (x, y) in X, after that X_M is lattice space being modular . Where x| $= x^+ + x^-$

Solution:

1. Suppose $(x, y) \in X$, since $|x| + |y| \geq 0$,after that $M(x, y) \geq 0$.
2. Suppose $(x, y) \in X$, after that $M(x, y) = 0 \iff |x| + |y| = 0 \iff x = y = 0$.

3. Suppose $(x, y) \in X$ and $\lambda, \in \mathbb{F}$ with $|\lambda| = 1$, after that $\lambda(x, y)$= $(\lambda x, \lambda y)$= $M(\lambda(x, y))$
$= |\lambda x| + |\lambda y| = |\lambda||x| + |\lambda||y| = |x| + |y| = M(x, y) = 0$.
4. Suppose (x_1, y_1), $(x_2, y_2) \in X$ and $\alpha, \beta \geq 0$, after that $\alpha(x_1, y_1) + \beta(x_2, y_2) =$
$(\alpha x_1 + \beta x_2, \alpha y_1, \beta y_2) = M(\alpha(x_1, y_1) + \beta(x_2, y_2) = |\alpha x_1 + \beta x_2| + |\alpha y_1, \beta y_2| \leq |\alpha|(|x_1| + |y_1|) + |\beta|(|x_2| + |y_2|) \leq (|x_1| + |y_1|) + (|x_2| + |y_2|)$.
$M(\alpha(x_1, y_1) + \beta(x_2, y_2)) \leq M(x_1, y_1)$+$M(x_2, y_2)$.

Formula(2.13)

Every lattice space being modular is space as metric of lattice.

Evidence:

Suppose X_M is a space being modular . Describe $d_M: X \times X \rightarrow E$ by d(x,y) = M(x-y) for whole $x, y \in X$.

Suppose $x, y \in X \Longrightarrow x - y \in X$ (Since X is a space being linear) $\Longrightarrow M(x - y) \geq 0$

$\Longrightarrow d_M(x, y) \geq 0$.

1. Suppose $x, y \in X, \Longrightarrow d(x, y) = 0 \Leftrightarrow M(x - y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.
2. Suppose $x, y \in X$, $d(x, y) = M(x - y) = M(-(y - x)) = M(y - x) = d(y, x)$.
3. Suppose $x, y, z \in X$, $M(x - y) = M\big((x - z) + (z - y)\big) \leq M(x - z) + M(z - y)$

$\Longrightarrow d(x, y) = d(x, z) + d(z, y)$.

It is following in which d is a metric on X , and such metric is named the lattice metric induced through the modular. It is clear to show that

When Suppose $x, y, z \in X$, $\lambda \in \mathbb{F}$,after that

1. $d(x + z, y + z) = d(x, y)$.
2. $d(\lambda x, \lambda y) = d(x, y)$, when $|\lambda| = 1$.
3. $M(x) = d(x, 0)$.

Remark

Whichever lattice space being modular is a topological space being linear, Furthermore, it is space of Hausdorff.

Description (2.14)[11]:

Suppose X_M be a space being modular

1. The ball being open along the center $x_0 \in X_M$ and r radius > 0 signified by $\beta_r(x_0)$ and define as $\beta_r(x_0) = \{x \in X_M : M(x - x_0) < r\}$.
2. A sub-set A of X_M is said to be bounded when $daim_M(A) < \infty$, where $daim_M(A) = \sup\{M(x - y): x, y \in A\}$ is named the diameter of A .
3. An order $\{x_n\}$ in X_M is converge to the x point $\in X_M$,

when $\lim_{n\to\infty} M(x_n - x) = 0$,i.e. when for every $\forall\varepsilon > 0, \exists\, k \in Z^+ \ni M(x_n - x) < \varepsilon \;\; \forall\, n \geq k$ and can be written $\lim_{n\to\infty} x_n = x$ or $x_n \to 0$ as $n \to \infty$. It is following that of $x_n \to x$ when f $M(x_n - x) \to 0$.

4. sequence of Cauchy in X, when$\forall\varepsilon > 0, \exists\, k \in Z^+ \ni M(x_n - x_m) < \varepsilon \;\; \forall\, n, m \geq k$.
5. X_M is named complete when every sequence of Cauchy in X_M is converge to X_M point.

Lattice Semi-normed spaces

In mathematics, principally in analysis as functional, a lattice semi-norm is a space being linear norm which require not be +ve definite. Lattice Semi-norms are closely connected with sets as convex, every lattice semi-norm is the functional of Minkowski of few absorbing disk and, contrariwise, the Minkowski functional of whichever set is a lattice semi-norm. A topological space being linear is locally convex when f its topology is induced through a lattice semi-norms family.

Description (2.15) [6]:

A lattice semi-norm on X is a function $p: X \to E$ with the follow

1. P is subadditivity triangle inequality, i.e. $p(x + y) \leq p(x) + p(y)$ for whole $x, y \in X$.
2. P is +vely homogeneous (homogeneity being absolute), i.e. $p(\lambda x) = |\lambda| p(x)$ for whole $x \in X$ and for all $\lambda \in \mathbb{F}$.

Remarks

1) An F family of lattice semi-norms on is thought to be separated when to every $x \neq 0$ Corresponds as a minimum 1 $p \in F$ along $p(x) \neq 0$.

2) When condition (2) of above Description replace by $p(\lambda x) = \lambda p(x)$ for whole $x \in X$ and for all $\lambda \geq 0$ it can be said p is a sub-linear functional on X.

3) A lattice semi-norm p on a space being linear X is a lattice norm when $p^{-1}(\{0\}) = \{0\}$,i.e. when $p(x) = 0$. Implies $x = 0$

4) A sub-linear function f on a real space being linear X is a lattice semi-norm whenf $f(-x) = f(x)$ for all $x \in X$.

5) Every sub-linear function f on a real space being linear X induces a lattice semi-norm $p: X \to E$ defined by $p(x) = \max\{f(x), f(-x)\}$ for all $x \in X$.

6) Whichever finite sum of a lattice semi-norm is a lattice semi-norm.

7) $p: X \to E$ and $q: X \to E$ are 2 lattice semi-norm, after that the function $r: X \times X \to E$ defined by $r(x,y) = p(x) + q(y)$ for whole $(x,y) \in X \times Y$ is a lattice semi-norm on $X \times Y$.

8) $p: X \to E$ and $q: X \to E$ are two lattice semi-norm.

Definition (2.16)[11]

1. A sub-linear $p: X \to E$ is named a lattice functional being convex when $p(x) \geq 0$ for all $x \in X$
2. A functional being convex $p: X \to E$ is said to be lattice symmetric when we have $p(\lambda x) = |\lambda| p(x)$ for whole $x \in X$ and for all $\lambda \in F$.

Example (2.17)

1. The trivial lattice semi-norm on a space being linear X, which refers to the constant 0 function on X, inducing the topology as indiscrete on X.
2. When f is whichever linear on a space being linear X, after that the function $p: X \to \mathbb{R}$ definite via $p(x) = |f|(x)$ for all $x \in X$ be a lattice semi-norm.
3. Suppose X= E . Describe $p: X \to E$ by $p(x) = \sum_{i=1}^{n} |x_i|$ for whole $x = (x_1, \dots, x_n) \in E$.
   After that P is a sub-linear functional on X and functional being convex.
4. Suppose X= E and Suppose M is a sub-space of X .Define $p_M: X \to E$ by

$p_M(x) = \inf\{\|x - y,\|: y \in M\}$.for whole $x \in X$ where$\|.\|$ is the Euclidian .

a) When $d(M) \geq 1$, after that p_M is a lattice semi-norm and not a lattice norm.
b) When M = {0}, after that $p_M(x) = \|x\|$ for whole $x \in X$.

5. Suppose X is a space being linear where it is defined as non-negative sesquilinear Hermitian form $B: X \times X \to F$ After that the function $p_B(x) = \sqrt{B(x,x)}$
   be a lattice semi-norm. p_B is a norm when f B is +ve definite (i.e. $B(x, x) > 0$ for whole $x \neq 0$).

Formula **(2.18)**

Suppose p is a lattice semi-norm on a space being linear X . After that

1. p(x) = 0.
2. p(-x) = p(x) for all $x \in X$.
3. p(y-x) = p(x-y) for whole $x, y \in X$.
4. $|p(x) - p(y)| \leq p(x - y)$ for whole $x, y \in X$.
5. $P(x) \geq 0$ for whole $x \in X$.
6. The set $N(p)=\{ x \in X: p(x) = 0\}$ is sub-space of X.
7. The set $A=\{ x \in X: p(x) < 1\}$ is convex, absorbing and balanced set.
8. p is a norm when it fulfills the case $p(x) \neq 0$ when $x \neq 0$.

Evidence:

(1) ,(2) and (3) straight from description

4) x=(x-y)+y $\Longrightarrow p(x) = p\big((x-y)+y\big) \le p(x-y)+p(y)$

$p(x) - p(y) \le p(x-y)$……(1) Also $-p(x-y) \le p(x) - p(y)$..(2)

From(1) and (2) , we nave $-p(x-y) \le p(x) - p(y) \le p(x-y)$.

$|p(x) - p(y)| \le p(x-y)$.

5. As long $|p(x) - p(y)| \le p(x-y)$for whole $x, y \in X$

Have y = 0 $\Longrightarrow |p(x)| \le p(x)$for all $x \in X$

Since $|p(x)| \ge 0 \Longrightarrow p(x) \ge 0$, for whole $x \in X$.

6. $p(x) = 0 \Longrightarrow 0 \in N(p) \Longrightarrow N(p) \neq \emptyset$

Suppose x,y $\in N(p)$ and $\alpha, \beta \in \mathbb{F} \Longrightarrow p(x) = 0$, $p(y) = 0$

$$p(\alpha x + \beta y) \le p(\alpha x) + p(\beta y) \le |\alpha|p(x) + |\beta|p(y) = 0$$

$\Longrightarrow p(\alpha x + \beta y) \le 0$.Since x,y $\in N(p)$ and $\alpha, \beta \in \mathbb{F}$, and X be space of vector, after that $\alpha x + \beta y \in X \Longrightarrow p(\alpha x + \beta y) \ge 0$, $p(\alpha x + \beta y) = 0$

$\alpha x + \beta y \in N(p) \Longrightarrow N(p)$ is sub-space.

7. (a) Suppose x,y $\in A$ and $0 \le \lambda \le 1$, afterthat $p(x) < 1$, $p(y) < 1$

$P(\lambda x + (1-\lambda)y) \le p(\lambda x) + p\big((1-\lambda)y\big) = |\lambda|\, p(x) + |1-\lambda|p(y)$

$\lambda\, p(x) + (1-\lambda)p(y)$ Since $p(x) < 1$, $p(y) < 1$

$\Longrightarrow \lambda\, p(x) < 1, (1-\lambda)p(y) < 1-\lambda$

$\lambda\, p(x) + (1-\lambda)p(y) < \lambda + (1-\lambda) = 1$

$\Longrightarrow p(\lambda x + (1-\lambda)y) < 1, (\lambda x + (1-\lambda)y) \in A, \Longrightarrow$ A be convex.

(b) Suppose $\lambda \in \mathbb{F}$,with $|\lambda| \le 1$

Suppose x$\in \lambda A \Longrightarrow x = \lambda y$ where $y \in A \Longrightarrow p(y) < 1$

Since $p(x) = p(\lambda y) = |\lambda|p(y)$ and $|\lambda| \le 1, p(y) < 1$

$|\lambda|p(y) < 1 \Longrightarrow p(x) < 1 \Longrightarrow x \in A \Longrightarrow \lambda A \subset A \Longrightarrow$ A be balanced.

(b) Suppose x$\in$ X Suppose $p(x) < \lambda \Longrightarrow \lambda > 0 \Longrightarrow p(\lambda^{-1}x) < 1 \Longrightarrow \lambda^{-1}x \in A$

$x \in \lambda A \Longrightarrow$A is absorbing.

2.3Minkowski' Functional

lattice Semi-norms on space being linear are related strongly to a special functional kind, i.e. functional of Minkowski. Suppose we are investigating in details additional relation. Notice that still we are in the realm of space being linear without topology.

Description (2.19)[11]:

Suppose A be a sub-set of a space being linear X over $\mathbb{F}$ .The functional $\mu_A: X \to \mathbb{R}$ by

$\mu_A(x) = \inf\{\lambda > 0: x \in \lambda A\}$ for whole $x \in X$. is named the Minkowski's functional (or gauge) of A .(where $\mu_A(x) = \infty$ when $\{\lambda > 0: x \in \lambda A\} = \emptyset$).

It is clear to show that following Formula

Formula (2.20)

Suppose A be an absorbing sub-set of a space being linear X over $\mathbb{F}$. Afterthat

1. $\mu_A(x) < \infty$, for whole $x \in X$ Since that A is an absorbing.
2. When $x \in \lambda A$, after that $\mu_A(x) \leq \lambda$ .In distinctive case when $y \in \mu_A(x)A$,afterthat $\mu_A(y) \leq \mu_A(x)$.
3. Whenx $\notin \lambda A$, for few$\lambda > 0$, after that $\mu_A(x) \geq \lambda$.
4. When A is open in topological space being linear X, after that $\lambda A = \{ x \in X: \mu_A(x) < \lambda\}$.

Formula **(2.21)**

Suppose P is a lattice semi-norm on a space being linear X over $\mathbb{F}$ .When $A = \{ x \in X: \}$,

After that $p = \mu_A$.

Evidence:

Since A is convex , absorbing, balanced X set$\in$ X .Since A is absorbing, there exists $\lambda > 0$ Thus $x \in \lambda A \Rightarrow \mu_A(x) \leq \lambda$ and $\lambda^{-1}x \in A \Rightarrow p(\lambda^{-1}x) < 1$

$\Rightarrow p(x) < \lambda$, so that $\mu_A < p$, since P semi-norm, after that $p(x) \geq 0$, there exist

α Thus $0 < \alpha \leq p(x) \Rightarrow p(\alpha^{-1}x) \geq 1 \Rightarrow \alpha^{-1}x \notin A$, therefore $p(x) \leq \mu_A(x)$

$\Rightarrow p \leq \mu_A$. Therefore $p = \mu_A$.

Formula **(2.22)**

Suppose A is a convex absorbing set in a space being linear X over $\mathbb{F}$. Describe

$H_A(x) = \{\lambda > 0: x \in \lambda A\}$ for whole $x \in X$.When $\alpha \in H_A(x)$, afterthat $\beta \in H_A(x)$,

For whole $\beta > \alpha$.

Evidence:

Since$\alpha \in H_A(x) \Rightarrow x \in \alpha A \Rightarrow \alpha^{-1}x \in A$, Since A is a convex and $0, \alpha^{-1}x \in A$, after that $\beta^{-1}x = \beta^{-1}(\beta - \alpha)(0) + \beta^{-1}\alpha(\alpha^{-1}x) \in A \Rightarrow x \in \beta A \Rightarrow \beta \in H_A(x)$.

Formula **(2.23)**

Suppose A is a convex absorbing set in a space being linear X over $\mathbb{F}$. After that

1. $\mu_A$is a sub-linear functional.

2. When B $=\{x \in X: \mu_A(x) < 1\}$ and C $= \{x \in X: \mu_A(x) \leq 1\}$, after that B$\subset$ A$\subset$ C. $\mu_{\mathbf{B}} = \mu_A = \mu_C$.
3. When A is balanced, after that μ_A is a lattice semi-norm.

Evidence:

1. Suppose $x, y \in X$ for whole $\varepsilon > 0$ there exits $\lambda_1 \in H_A(x)$ and $\lambda_2 \in H_A(x)$ such $\lambda_1 < \mu_A(x) + \varepsilon$ and $\lambda_2 < \mu_A(y) + \varepsilon$, after that $(\mu_A(x) + \varepsilon) \in H_A(x)$ and $(\mu_A(y) + \varepsilon) \in H_A(y)$, $x \in (\mu_A(x) + \varepsilon)A$ and $y \in (\mu_A(y) + \varepsilon)A$
$(\mu_A(x) + \varepsilon)^{-1}x \in A$ and $(\mu_A(y) + \varepsilon)^{-1}y \in A$
Place $\lambda = (\mu_A(x) + \varepsilon)(\mu_A(x) + \mu_A(y) + 2\varepsilon)^{-1} \Longrightarrow 0 < \lambda < 1$, since A is convex,
$\lambda(\mu_A(x) + \varepsilon)^{-1}x + (1 - \lambda)(\mu_A(y) + \varepsilon)^{-1}y \in A \Longrightarrow (\mu_A(x) + \mu_A(y) + 2\varepsilon)^{-1}(x + y) \in A$ .It is obvious to show that
$\mu_A(0) = 0$.Suppose $x \in X$ for all $\alpha > 0$, after that $\mu_A(\alpha x) = \inf\{\lambda > 0: \alpha x \in \lambda A\} = \inf\{\lambda > 0: x \in \alpha^{-1}\lambda A\} = \alpha \inf\{\alpha^{-1}\lambda: x \in \alpha^{-1}\lambda A, \lambda > 0\} = \alpha\mu_A(x)$.
3. Since A is a balanced set, after that $\beta^{-1}A = A$ for whole $\beta \in \mathbb{F}$ Thus $|\beta| = 1$
Hence $\{\lambda > 0: \alpha x \in \lambda A\} = \{\lambda > 0: |\alpha| x \in \lambda A\} \Longrightarrow \mu_A(\alpha x) = |\alpha|\mu_A(x) \Longrightarrow \mu_A$ is a lattice semi-norm.

Example (2.24)

Suppose A be an absorbing sub-set of a space being linear X over $\mathbb{F}$. the Minkowski's functional of A is a lattice Δ-norm on X. More than lattice quasi-norm.

Evidence:

Suppose μ_A be the Minkowski's functional of A, after that $\mu_A: X \to E$ defined by

$\mu_A(x) = \inf\{\lambda > 0: x \in \lambda A\}$ for whole $x \in X$.

Description (2.25) [11]:

Suppose X is a space being linear over field $\mathbb{F}$.

1. A function $p: X \to E$ is named a lattice quasi semi-norm when it is (completely) homogeneous and there occurs few b$\leq$ 1 Thus $p(x + y) \leq bp(x) - p(y)$ for all x y$\in$ X. The smallest b value where such holds is named the p multiplier.
2. A lattice quasi semi-norm which is separating points is named a lattice quasi norm on X .
3. A function $p: X \to E$ is named a lattice k -semi-norm when it is sub-additive and there occurs a k. Thus $0 > b \leq 1$ and for whole x y$\in$ X and such scalars λ , $p(\lambda x) = |x|^k p(x)$.
4. A lattice k -semi-norm which is separating points is named k -norm on X.

2.3 Lattice Normed Spaces

The first to introduce the concept for the standard was the Austrian scientist E. Helly (1844 - 1943), but he did not use the name of the standard nor its symbol, it was known as whichever function that fulfills certain conditions (the same conditions of the standard).

Description (2.26) [1]:

A norm on X is a function $\|.\|: X \to E$ of the properties as follow

a. $\|x\| \geq 0$ for whole $x \in X$.
b. $\|x\| = 0$ when and just when $x = 0$.
c. $\|\lambda x\| = |\lambda|\|x\|$ for only $\lambda \in \mathbb{F}$ and $x \in X$.
d. $\|x + y\| \leq \|x\| + \|y\|$ for whole $x, y \in X$.

The X linear over $\mathbb{F}$ collected with $\|.\|$ is named a normed space and is signified via $(X, \|.\|)$ or simply X .

A norm $\|.\|$ on a space being linear X is said to be strictly convex when $\|x+y\| = \|x\| + \|y\|$ only when x and y linearly independent.

Remarks

1. Every sub-space of lattice normed space is as well lattice normed space.
2. $(X, \|.\|_1)$ and $(Y, \|.\|_2)$are lattice normed spaces, after that$(X \times Y, \|.\|)$ is lattice normed space where $\|(x,y)\| = \max\{\|x\|, \|y\|\}$ for whole $(x,y) \in X \times Y$.
3. Every lattice normed space is space as metric of lattice. Therefore every lattice normed space is a metric as invariant space.
4. When $\|.\|_1$ and $\|.\|_2$ are two lattice norms on a space being linear X. Afterthat $\|.\|_1 \sim \|.\|_2$, when f there exists +ve real numbers a and b Thus $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$
   For all $x \in X$.
5. On a dimensional finite space being linear whole lattice norms are equivalent.

Remark

As long each lattice normed space is space as metric of lattice and each space as metric of lattices is a topological space, after that every lattice normed space is topological space. $\beta_r(x_0)$ is x_0 neighborhood. This topology is named a lattice norm topology on X, and the space X is named the lattice normed topological space.

Description **(2.27)[6]**

A topological space being linear X is so called a lattice normable when a norm occurs on X Thus the metric induced by the lattice norm is well-matched with τ .

Description **(2.28)[6]**

Suppose X be lattice normed space

1. The ball being open along center $x_0 \in X$ and r radius > 0 denoted through $\beta_r(x_0)$ and define as $\beta_r(x_0) = \{x \in X : \|x - x_0\| < r\}$ and the ball being closed be $\beta_r(x_0) = \{x \in X : \|x - x_0\| \leq r\}$.
2. A sub-set A of X is said to be an open set when given whichever point $x \in A$, there exists $r > 0$ thus $\beta_r(x) \subseteq A$.
3. A sub-set A of X is so called as bounded when there occurs $k > 0$ Thus$\|x\| \leq k$. For $\forall x \in A$.
4. An order $\{x_n\}$ in X is converge to the point $x \in X$,
   when $\lim_{n\to\infty}\|x_n - x\| = 0$, such as when for every $\forall \varepsilon > 0$, $\exists\, k \in Z^+ \ni \|x_n - x\| < \varepsilon \ \ \forall\, n \geq k$ and we are writing $\lim_{n\to\infty} x_n = x$ or $x_n \to 0$ as $n \to \infty$. It is following that
   $x_n \to x$ when f $\|x_n - x\| \to 0$.
5. sequence of Cauchy in X, when$\forall \varepsilon > 0, \exists\, k \in Z^+ \ni \|x_n - x_m\| < \varepsilon$ ▯ $n, m \geq k$.

Remarks

1. $\beta_r(x_0) = x_0 + \beta_r(0) = x_0 + r\beta_1(0)$.
2. Each open and ball being closed in lattice normed space are convex.
3. The lattice norm $\|.\|$ on X is a continuous function.
4. The addition of vector and multiplication of scalar are together continuous.
5. Every lattice normed space X is topological space being linear.

Example(2.29)

Each lattice normed space is convex locally.

Evidence:

Suppose $(X, \|.\|)$ be a lattice normed space, after that X is topological space being linear.

Suppose $\beta = \{\beta_r(0) : r > 0\}$ in which$\beta_r(x_0) = \{x \in X : \|x\| < r\}$.

Suppose G be an open set in X , after that G is the combination of ball being open s, so $0 \in \beta_r(0) \subset G$.

For few $r > 0$, after that β is a base as local at 0 in X. As long each ball being open is convex set, after that $\beta_r(0)$ is convex set for whole $r > 0$, after that β is a local convex base at 0 in X. Therefore $(X, \|.\|)$ is convex locally space.

2.4 Banach lattice space

Space of Banach is a complete normed space. Therefore, a space of Banach is a space being linear with, a metric which permits the vector distance and length computation between vectors and is complete where vectors sequence of Cauchy converges always to a well-defined boundary" which is within the space. Spaces of Banachs are named following Stefan Banach (Polish mathematician) who familiarized such concept and in 1920–1922 systematically studied it jointly with Hans Hahn and Eduard Helly. Maurice René Fréchet was the 1st to utilizee the space of Banach term and Banach in line after that created the Fréchet space" term. Space of Banachs grew out originally of the function spaces study through Hilbert, Fréchet, and Riesz former in the century. Space of Banachs is of a crucial role in functional analysis. In other analysis areas, the spaces under study are frequently space of Banachs.

Description (2.30) [5]:

A lattice normed space X is named complete when each sequence of Cauchy in X is converging to an X point. A complete lattice normed space is named a space of Banach

Formula (2.31)

Suppose M be a sub-space of Banach lattice space X, after that M is Banach lattice space when f it is closed in X.

Evidence:

Suppose M is Banach lattice space ⟹ M is complete space. Suppose x M, an order $\{x_n\}$ is there in M Thus $x_n \to x$, Therefore $\{x_n\}$ is a sequence of Cauchy in M.
As long M is complete, there is y M Thus $x_n \to y$, nonetheless the converge is exclusive ⟹ y = x ⟹ x M ⟹ M ⊑ M, after that M is closed.
Conversly. Suppose that M is a closed set in X.
Suppose $\{x_n\}$ be a sequence of Cauchy in M.
As long M ⊑ X ⟹ $\{x_n\}$ is a sequence of Cauchy in X.
As long M is complete space, there is x X. Thus $x_n \to x$.
As long x_n M ⟹ x M.
As long M is a closed set in X, after that M= M ⟹x M⟹$\{x_n\}$ is sequence of converge in M, after that M is complete space.

Formula (2.32)

Each dimensional as finite lattice norm space is complete.

Evidence

Suppose X be finite dimensional lattice normed space with dim X = n > 0 and Suppose $\{x_1, x_2 \dots x_n\}$ be abasis for X. Suppose $\{y_n\}$ be whichever sequence of Cauchy in X.

$\|y_m - y_i\| \to 0$ as m,l→ …….(1).

Since $y_m, y_i \in X \Longrightarrow y_m = \sum ƛ_{im} X_i$, $ƛ_{im} \in F$ and $y_i = \sum ƛ_{i1} X_i, ƛ_{i1} \in F \Longrightarrow y_m - y_i = \sum ƛ_{i1}, ƛ_{i1}) x_i$

Since $\{x_{1,\dots} x_n\}$ is independently linear via linear combination of lemma, there is c >0 Thus

$\|y_m - y_i\| = \| \sum ƛ_{i1}, ƛ_{i1}) x_i \| \geq c \sum |ƛ_{im} - ƛ_{i1}|$ ……..(2)

From (1) and (2), we have $\sum |λ_{im}-λ_{i1}| \to 0$ as m, $l \to ια$ for i=1,2…..n for i=1,2….n $\Longrightarrow \{λ_{im}\}$ is sequence of Cauchy in F. As long F is either R or C and every R, C are complete$\Longrightarrow λ_i \in F$ Thus $λ_{im} \to λ_i$. Place $y=\sum λ_i x_i$,$\Longrightarrow y_m \to y$, $y \in X \Longrightarrow$X is complete.

Corollary (2.33)

Each dimensional finite sub-space M of lattice normed space X is closed.

Evidence:

As long M is a dimensional finite sub-space of a lattice normed space X ⟹M be a complete space ⟹M is closed. Notice that, dimensional infinite sub-space of space of Banach required no closing.

Example (2.34)

Suppose X=c[0,1] and Suppose $M=[\{f_0,f_1,…..\}]$ where $f_i(x)=x^i$ so that M is the set of whole polynomials. M is a dimensional infinite sub-space of X nonetheless not closed in X.

Display that X\M is as well space of Banach.

Example (2.34)

Suppose X=c[0,1] and Suppose $M=[\{f_0,f_1,…..\}]$ where $f_i(x)=x^i$ so that M is the set of whole polynomials. M is a dimensional infinite sub-space of X nonetheless not closed in X.

Display that X\M is as well space of Banach.

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