On i-subspace of i- topological Proximity Space

Authors Names	ABSTRACT
[a]*Ali Khalid Hasan* [b]*Luay A. Al-Swidi* ***Publication data***: 18 /12 /2023 ***Keywords***: i −topological space, Focal set, proximity space	In this paper, we will define the subspace of i −topological space and how the forms of sets and relationships will become in it, and we will discuss the notation of the Focal set with its closure in subspace i − topological space.

1. Introduction

The ideal theory plays an important role in solving topological problems and has been studied since the 20th century. Kuratowski, K. [7] in 1966 and Vidyanathaswamy, R. [9] in 1960 were the first batch of mathematicians who proposed the idea of ideal topological spaces, and then conducted extensive research in different fields and were widely used. Many researchers worked on ideal topological space and study different concepts with it like Jankovic, D. and T. R. Hamlet [6] in 1990, they defined the notion of an ideal as a nonempty collection closed by inherited properties and bounded unions. And Ekici and Noiri [5] in 2009 studied the connectedness in ideal topological space. In 2012 Modak studied new topology on ideal topological space [8]. Al-Omari and Noiri introduced the local closure function in an ideal topological space in 2013 [2,3]. The i-topological spaces represent another form of these spaces, which are the compensation of the family T and the ideal I defined on the space X. This was proposed by researcher Irina Zvina [13] in 2006 and is considered to be a special case of ideal topological spaces, and study more generalization of this space in 2011 [11,12].

2. Background

Definition (2.1) [13]: Let I be an ideal defined on a set X, and let $U,\mathcal{K}$ are subsets of X, the relation α defined on X by: $U\alpha\mathcal{K}$, if and only if $U-\mathcal{K}\in I$. Also, a relation ≈ defined on X by: $U\approx\mathcal{K}$, if and only if $U-\mathcal{K}\cup\mathcal{K}-U\in I$.

Definition (2.2) [13]: Let I be ideal on X, an i – topological space on X is a family T of subsets of X satisfies:

1. $X,\emptyset\in T$.
2. For any $U\subseteq T$, there exist $W\in T$ such that $\cup U\approx W$.
3. For any $U,W\in T$, there exist $H\in T$ such that $U\cap W\approx H$.
4. $I\cap T=\{\emptyset\}$.

Then (X,T,I) is called i – topological space, and the elements of T is called i – open set.

Definition (2.3) [13]: In i −topological space (X,T,I) and for $Y\subseteq X$ then (Y,T_Y,I_Y) is called i −subspace of (X,T,I) such that $T_Y=\{Y\cap U\notin I,U\in T\}\cup\{\emptyset,Y\}$, $I_Y=\{Y\cap U,U\in I\}$.

Since $T\cap I=\emptyset$ and $T_Y\cap I_Y\subseteq T\cap I=\emptyset$, then $T_Y\cap I_Y=\emptyset$

Definition (2.4): [10] Let (X,T,I) bean i– topological space. A point $x\in U$ is called i– interior point of $U\subseteq X$,if and only if there exist i – open set H such that $x\in H\subseteq U$ and the set of all i –interior point of U is denoted by $i\text{–}int\,(U)$.

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Definition (2.5): [10] Let (X, T, I) be i–topological space and let U subset of X ,the i–closure of U is the intersection of all $i-$ closed sets consist of U ,and is denoted by $i-cl(U)$, i.e, $i-cl(U) = \cap \{H: H\ is\ i\text{–}closed\ set, U \subseteq H\}$.

Definition (2.6) [9]: A binary relation δ defined on the power set of X is called proximity relation on X , if and only if it satisfies the following axioms:

1. $U\delta\mathcal{K}$ implies $\mathcal{K}\delta U$
2. $(U \cup \mathcal{K})\delta C$, if and only if $U\delta C$ or $\mathcal{K}\delta C$
3. $U\delta\mathcal{K}$ implies $U \neq \emptyset$ and $\mathcal{K} \neq \emptyset$
4. $U \cap \mathcal{K} \neq \emptyset$ implies $U\delta\mathcal{K}$
5. $U\bar{\delta}\mathcal{K}$ implies there exists a subset E of X such that $U\bar{\delta}E$ and $X - E\bar{\delta}\mathcal{K}$.

   The pair (X, δ) is called a proximity space. And it will be denoted by X^{δ}.

Definition (2.7)[4] :Let X^{δ_1} and X^{δ_2} be a proximity spaces such that $\delta_1 > \delta_2$, and let $U, \mathcal{K}$ are subsets of X then If $U\delta_1\mathcal{K}$ implies $U\delta_2\mathcal{K}$.

Definition (2.8):[9] Let X^{δ} be proximity space and Y subset of X . Now for subsets $U, \mathcal{K}$ of Y then $U\delta_Y\mathcal{K}$ if and only if $U\delta\mathcal{K}$ then Y^{δ} is a subspace proximity on Y.

Definition (2.9) : [10] The quadruple (X, T, I, δ) is called $i-$ topological proximity space, where (X, T, I) is $i-$ topological space and (X, δ) is a proximity space. And we will denote it by $i - TPS$

For this paper we will use the nation X_{TI}^{δ} for any $i - TPS$, (X, T, I, δ)

Definition (2.10) : [1] Let X_{TI}^{δ} be an $i - TPS$, then a subset U is named a focal set of a point $x \in X$ if we have $U \in T(x)$ such that $U\ \alpha$ U.The system of all focal sets of a point x is denoted by $I_{\oint}(x) = \{U \subseteq X: \exists U \in T(x), U\alpha U\}$ .Noted that X is a focal set for each $x \in X$. where $T(x) = \{U \in T, x \in U\}$. Also, we define the set of all focal set for some $x \in X$, by $\oint_I(x) = \{U \in I_{\oint}(x), x \in U\}$.

Proposition (2.11): [1] Let $X_{TI_i}^{\delta}$ $i = 1,2$ be an $i - TPS$, such that $I_1 \subseteq I_2$, then $I_{1\oint}(x) \subseteq I_{2\oint}(x)$.

Proposition (2.12): [1] Let $X_{T_iI}^{\delta}$ $i = 1,2$ be an $i - TPS's$,such that T_2 is finer than T_1 and $I_2$is finer than I_1 ,then :

1) $I_{\oint T_1}(x) \subseteq I_{\oint T_2}(x)$

2) $I_{1\oint T_1}(x) \subseteq I_{2\oint T_2}(x)$

Definition (2.13) : [1] Let X_{TI}^{δ} bean $i - TPS$, and $AU \subseteq X$, $x \in X$,then x is called a focal limit point of U, if and only if for each $U \in \oint_I(x), U_x \cap U/_{\{x\}} \neq \emptyset$, and the set of all focal limit points is called the focal derived set and denoted by $Fd(U)$, also the focal closure of the set U denoted by $Fcl(U)$ and defined by $Fcl(U) = U \cup Fd(U)$. Note that $Fcl(U)$ is not necessary $i-$ closed set.

3. Main Result

Now we will study the concept of subspace in i-topological proximity space and we will investigate some properties on these relations.

Definition (3.1): Let X be a set and Y be a subset of X, and let Y_I be an ideal define on a subset Y such that $I_Y = I \cap \{Y\}$, then we can define the relation α_Y on Y as following $U\alpha_Y B$ if and only if $U \cap (Y - \mathcal{K}) \in I_Y$. Also, we can define the relation $\approx_Y$ on Y as the following $U \approx_Y \mathcal{K}$ if and only if $(U \cap (Y - \mathcal{K})) \cup \big(\mathcal{K} \cap (Y - U)\big) \in I_Y$

Example (3.2): Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, with the ideal $I = \{\emptyset, \{\mathcal{k}\}\}$, If $Y = \{\mathcal{k}, \mathcal{g}\}$ and $U = \{\mathcal{k}\}$, $\mathcal{K} = \emptyset$, then $I_Y = I \cap Y = \{\emptyset, \{\mathcal{k}\}\}$ Then we have:

$U - \mathcal{K} = \{\mathcal{k}\} - \emptyset = \{\mathcal{k}\} \in I_Y$, so, we get that $U\alpha_Y\mathcal{K}$. Also, we have $(U - \mathcal{K}) \cup (\mathcal{K} - U) = (\{\mathcal{k}\} - \emptyset) \cup (\emptyset - \{\mathcal{k}\}) = \{\mathcal{k}\} \in I_Y$. Thus $U \approx_Y C$.

The following proposition we will study some of this relation property in i-subspace which most of its point is obvious and easy to proof

Proposition (3.3): Let I_Y be any ideal defined on a subset Y of X and $U, \mathcal{K}, C$ are subsets of Y, then:

1. $U\alpha_Y Y$, for each subset U of Y .

2. $\text{U}\alpha_Y\emptyset$, if and only if $\text{U} \in I_Y$.
3. If $\text{U} \in I_Y$,then $\text{U}\ \alpha_Y\ \mathcal{K}$ for each subset $\mathcal{K}$ of Y.
4. If $C \subseteq \text{U}$,such that $\text{U}\ \alpha_Y\ \mathcal{K}$ then $C\ \alpha_Y\ \mathcal{K}$.
5. If $\mathcal{K} \subseteq D$,such that $\text{U}\ \alpha_Y\ \mathcal{K}$ then $\text{U}\ \alpha_Y\ D$.
6. If $\text{U}\ \alpha_Y\mathcal{K}_\lambda$,for each $\lambda \in \Lambda$,where Λ is any index, then $\text{U}\alpha_Y \cup_{\lambda\in\Lambda} \mathcal{K}_\lambda$
7. If $\text{U}_\lambda\alpha_Y\mathcal{K}$,for each $\lambda \in \Lambda$,where Λ is any index, then $\cap_{\lambda\in\Lambda} \text{U}_\lambda\alpha_Y\mathcal{K}$
8. $\text{U}\alpha_Y\text{U}$, for each subset U of Y .
9. If $\text{U}\alpha_Y\mathcal{K}$,and $\mathcal{K}\alpha_Y\text{C}$ then $\text{U}\alpha_Y C$.
10. If $\text{U}\alpha_X\mathcal{K}$ then $\text{U}\alpha_Y\mathcal{K}$

Proof:

9- Since $[\text{U} \cap (Y - \mathcal{K})] \cup [\mathcal{K} \cap (Y - C)]$
$= \left[\left(\text{U} \cap (Y - \mathcal{K})\right) \cup \mathcal{K}\right] \cap \left[\left(\text{U} \cap (Y - \mathcal{K})\right) \cup (Y - C)\right]$
$= [\text{U} \cup \mathcal{K}] \cap [(Y - \mathcal{K}) \cup \mathcal{K}] \cap [\text{U} \cup (Y - C) \cap (Y - \mathcal{K}) \cup (Y - C)]$
$= (\text{U} \cup \mathcal{K})\ \cap \left(\text{U} \cup (Y - C)\right) \cap [(Y - \mathcal{K}) \cup (Y - C)]$
But, $\text{U} \cap (Y - C) \subseteq \left[(\text{U} \cup \mathcal{K})\ \cap \left(\text{U} \cup (Y - C)\right)\right] \cap [(Y - \mathcal{K}) \cup (Y - C)]$
so, $\text{U} \cap (Y - C) \in I_Y$, thus $\text{U}\alpha_Y C$.

10- If $\text{U}\alpha_X\mathcal{K}$, then $\text{U} \cap (X - \mathcal{K}) \in I$, so $\text{U} \cap (X - \mathcal{K}) \cap Y \in I_Y$, but $(X - \mathcal{K}) \cap Y = Y - \mathcal{K}$, thus $\text{U} \cap (Y - \mathcal{K}) \in I_Y$. Hence $\text{U}\alpha_Y\mathcal{K}$.

From this proposition specially by part (6) and (7), we can inclusion directly the following corollary.

Corollary (3.4): Let I_y be an ideal define on subset $Y$of a set X such that for $i = 1,2, \dots, n$ $\text{U}_i\alpha_Y\mathcal{K}_i$, then $\bigcap_{i=1}^{n} \text{U}_\text{i}\ \alpha_Y \bigcup_{i=1}^{n} \mathcal{K}$.

Proposition (3.5): Let I_Y be an ideal defined on a subset Y of a set X, and let $\text{U}, \mathcal{K}, C$ are subset of Y then:

1. $\text{U} \approx_Y \text{U}$ for each subset U of Y .
2. $\text{U} \approx_Y \emptyset$ for each $\text{U} \in I_Y$.
3. $\text{U} \approx_Y Y$ for each $\text{U} \subseteq Y$ such that $Y - \text{U} \in I_Y$.
4. If $\text{U} \approx_Y \mathcal{K}$, then $\mathcal{K} \approx_Y \text{U}$.
5. If $\text{U} \approx_Y C$, and $\mathcal{K} \approx_Y C$,then $\text{U} \cup \mathcal{K} \approx_Y C$.

Proof: It is clearly that (1,2,3,4) is obvious so we will proof point (5)

5- Since $[\text{U} \cap (Y - C)] \cup [C \cap (Y - \text{U})] \in I_Y$, and
$[\mathcal{K} \cap (Y - C)] \cup [C \cap (Y - \mathcal{K})] \in I_Y$ then
$[(\text{U} \cap (Y - C)) \cup (C \cap (Y - \text{U})]$
$\cup [(\mathcal{K} \cap (Y - C)) \cup (C \cap (Y - \mathcal{K})] \in I_Y \dots (*)$

But, $\left(\left(\text{U} \cap (Y - C)\right) \cup \left(\mathcal{K} \cap (Y - C)\right)\right) \cup \left((C \cap (Y - \text{U}) \cup (C \cap (Y - \text{U})\right)$
$= \left((\text{U} \cup \mathcal{K}) \cap (Y - C)\right) \cup \left(C \cap \left((Y - \text{U}) \cup (Y - \mathcal{K})\right)\right) \in I_Y \dots (**)$

And, $\left((\text{U} \cup \mathcal{K}) \cap (Y - C)\right) \cup \left(C \cap \left((Y - \text{U}) \cap (Y - \mathcal{K})\right)\right) \subseteq \left((\text{U} \cup \mathcal{K}) \cap (Y - C)\right) \cup \left(C \cap \left((Y - \text{U}) \cup (Y - \mathcal{K})\right)\right)$

So $\left((\text{U} \cup \mathcal{K}) \cap (Y - C)\right) \cup \left(C \cap \left((Y - \text{U}) \cap (Y - \mathcal{K})\right)\right) \in I_Y$,

Then $\left((\text{U} \cup \mathcal{K}) \cap (Y - C)\right) \cup \left(C \cap (Y - (\text{U} \cup \mathcal{K}))\right) \in I_Y$. Thus $\text{U} \cup \mathcal{K} \approx_Y C$.

Definition (3.6): Let (X, T, I) be an $i - TS$ and for Y be i −subspasce of X and for a subset U of Y, a point $\hbar \in \text{U}$ is called $i -$ interior point of U with respect to i −subspace Y if and only if there existe $i_Y - open$ set H_Y such that $\hbar \in H_Y \subseteq \text{U}$. And the set of all i_Y −inteioer of U denoted by $i - int_Y(\text{U})$.

Example (3.7): Let $X = \{\hbar, g, f\}$, $T = \{\emptyset, X, \{\hbar\}, \{g\}\}$, with the ideal $I = \{\emptyset,\ \{f\}\}$. Now let $Y = \{\ \hbar, f\}$ then $T_Y = \{Y, \emptyset, \{\hbar\}\}$, $I_Y = \{\emptyset,\ \{f\}\}$.

Then for $\mathcal{K} = \{k\}$ we have $i - int_Y(\mathcal{K}) = \{k\}$

Proposition (3.8): Let Y be an i −subspace of an $i\_TS\ X$,
then $i - int(U) \cap Y \subseteq i - int_Y(U)$ for any $U \subseteq Y$

Proof: let $k \in i - int(U) \cap Y$ so $k \in i - int(U)$ and $k \in Y$, so there exist $H \in T_X(k)$ such that $H \subseteq U$, but $H \cap Y = H_Y$ so $k \in H_Y \subseteq U$. Hence $k \in i - int_Y(U)$, when $T_X(k) = \{H \in T_X, s.t. k \in H\}$

Example (3.9): Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k, g\}\}$, with the ideal $I = \{\emptyset,\ \{f\}\}$, and let $Y = \{k, f\}$ then $T_Y = \{Y, \emptyset, \{k\}\}$, $I_Y = \{\emptyset,\ \{f\}\}$. Then for $U = \{k\} \subseteq Y$ we have $i - int_Y(U) = \{k\}$. But $i - int(U) = \emptyset$, so we included that the converse of proposition (3.8) not true in general which mean that $i - int_Y(U) \nsubseteq i - int(U) \cap Y$

Definition (3.10): Let (X, T, I) be an $i - TS$ and Y be i −subspasce of X and for a subset U of Y, the $i -$ closure of U with respect to i −subspace Y is the intersection of all $i_Y - closed$sets D_Y consisting U, and denoted by $i - cl_Y(U)$, i.e., $i - cl_Y(U) = \{D_Y: D\ is\ i_Y - closed\ set, U \subseteq D_Y\}$

Example (3.11): Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k\}, \{g\}\}$, with the ideal $I = \{\emptyset,\ \{f\}\}$. Now let $Y = \{k, f\}$ then $T_Y = \{Y, \emptyset, \{k\}\}$, $I_Y = \{\emptyset,\ \{f\}\}$. Then for $\mathcal{K} = \{k\}$, $i - cl_Y(\mathcal{K}) = Y$

Definition (3.12) : The quadruple $Y^{\delta_Y}_{T_Y I_Y}$ is called i– $subspace$ of X^{δ}_{TI}topological proximity space, where (Y, T_Y, I_Y) is i– topological space and (Y, δ) is a proximity subspace.

Example (3.13): Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k\}, \{g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D . And let $Y = \{g, f\}$ so $T_Y = \{\emptyset, X, \{g\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with δ_{Y_D} . Thus $Y^{\delta_Y}_{T_Y I_Y}$ is i– $subspace$ of X^{δ}_{TI}.

Our aim now is to introduce the definition for the notion of focal set with respect to i − subspace of i − TPS . with some of the properties and relations

Definition (3.14): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $U \subseteq Y$, $y \in Y$, then U is called focal set w. r. t. $i - subspace\ Y$ if there is $U_Y \in T_Y(y)$ s.t. $U_Y \alpha_Y U$ i.e. $\exists\ V \in T_X(y), (V \cap Y)\alpha_Y U$then $V \cap Y \cap (Y - U) \in I_Y$ when $V \cap (Y - U) \in I$ and we denoted by $I_{Y_\oint}(y) = \{U \subseteq Y: \exists U_Y \in T_Y(x), U_Y \alpha_Y U\}$ for some $y \in Y$. Also, we can define the set of all focal set w. r. t. $i - subspace\ Y^{\delta_Y}_{T_Y I_Y}$ by $\oint_{I_Y}(y) = \{U \in I_{Y_\oint}(y), y \in U\}$.

Proposition (3.15): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, then $I_{Y_\oint}(y) = I_{X_\oint}(y) \cap \{Y\}$.

Proof: Let $U \in I_{Y_\oint}(y)$ then $\exists\ U_Y \in T_Y(x)\ and\ U_Y \alpha_Y U$ which means that $U_Y \cap (Y - U) \in I_Y$. So $\exists\ V \in T_X(y)$ s.t. $U_Y \cap (Y - U) = V \cap (X - U) \cap Y \in I_Y$so $V \cap (X - U) \in I$. Thus $V\alpha U$ and hence $U \in I_{X_\oint}(y)$, but $U \subseteq Y$, so $U \in I_{X_\oint}(y) \cap \{Y\}$. Now let $U \in I_{X_\oint}(y) \cap \{Y\}$, then $U \in I_{X_\oint}(y)$ and $U \subseteq Y$. Since $U \in I_{X_\oint}(y)$, then $\exists\ u \in T_X(y)$ s.t. $U\alpha U$ then we have $(U \cap Y)\alpha U$ hence $U \in I_{Y_\oint}(y)$.

Example (3.16): Let $X = \{k, g, f\}$, $T = \{\emptyset, X, \{k, g\}\}$, $I = \{\emptyset, \{f\}\}$ with δ_D . And let $Y = \{k, f\}$ so $T_Y = \{\emptyset, X, \{k\}\}$, $I_Y = \{\emptyset, \{f\}\}$ with δ_{Y_D} be i–$subspace Y^{\delta_Y}_{T_Y I_Y}$ of X^{δ}_{TI}. Then $I_{Y_\oint}(k) = Y.\{k\} = I_{Y_\oint}(g) = I_{Y_\oint}(f)$

Theorem (3.17): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, then for the subsets $U, \mathcal{K}$ of Y , the following statement are true for some $k \in Y$

1) If $U \in T_Y(k)$, then $U \in I_{Y_\oint}(k)$ and $\emptyset \notin I_{Y_\oint}(k)$
2) If $\mathcal{K} \in I_{Y_\oint}(k)$ and $\mathcal{K} \subseteq U$, then $U \in I_{Y_\oint}(k)$
3) If $U, \mathcal{K} \in I_{Y_\oint}(k)$, then $U \cap \mathcal{K} \in I_{Y_\oint}(k)$
4) $\forall\ \mathcal{K} \in I_{Y_\oint}(k)$, then $\exists U \in Y$ s.t. $\mathcal{K}\alpha_Y U$ and $U \in I_{Y_\oint}(x)\ \forall x \in U$.
5) $\forall U \in I_Y$, then $U \notin I_{Y_\oint}(k)$ for some $k \in Y$.
6) If $U \in I_{Y_\oint}(k)$, then $Y - U \notin I_{Y_\oint}(k)$.
7) If $U \in I_Y$, then $Y - U \in I_{Y_\oint}(k)$

8) If $\text{Ⴎ}, \mathcal{K} \in I_{Y\oint}(\mathscr{k})$, then $\text{Ⴎ} \cup \mathcal{K} \in I_{Y\oint}(\mathscr{k})$

Proof: It is easy to proof (1,2,4,8) so we will proof the other points below :

3) Since $\text{Ⴎ}, \mathcal{K} \in I_{Y\oint}(\mathscr{k})$, then $\text{Ⴎ} = U_1 \cap Y$, $U_1 \in I_{\oint}(\mathscr{k})$ and $\mathcal{K} = U_2 \cap Y$, $U_2 \in I_{\oint}(\mathscr{k})$. Now $\text{Ⴎ} \cap \mathcal{K} = U_1 \cap Y \cap U_2 \cap Y = (U_1 \cap U_2) \cap Y$, since $U_1, U_2 \in I_{\oint}(\mathscr{k})$ so we get $U_1 \cap U_2 \in I_{\oint}(\mathscr{k})$. Then $\text{Ⴎ} \cap \mathcal{K} = U \cap Y$ s. t. $U = U_1 \cap U_2 \in I_{\oint}(\mathscr{k})$. Thus $\text{Ⴎ} \cap \mathcal{K} \in I_{Y\oint}(\mathscr{k})$. Conversely, let $\text{Ⴎ} \cap \mathcal{K} \in I_{Y\oint}(\mathscr{k})$, but $\text{Ⴎ} \cap \mathcal{K} \subseteq \text{Ⴎ}$ and $\text{Ⴎ} \cap \mathcal{K} \subseteq \mathcal{K}$, so we have that $\text{Ⴎ}, \mathcal{K} \in I_{Y\oint}(\mathscr{k})$.

5) Suppose that $\text{Ⴎ} \in I_Y$ and $\text{Ⴎ} \in I_{Y\oint}(\mathscr{k})$, so $\exists\, U_Y \in T_Y(\mathscr{k})$ s. t. $U \cap (Y - \text{Ⴎ}) \in I_Y$, but $\text{Ⴎ} \in I_Y$, so $\left(U \cap (Y - \text{Ⴎ})\right) \cup \text{Ⴎ} \in I_Y$, which means that $U \cup \text{Ⴎ} \in I_Y$, then $U \in I_Y$ and that is a contradiction with the definition (2.3) .

6) Since $\text{Ⴎ} \in I_{Y\oint}(\mathscr{k})$ then $\text{Ⴎ} \in I_{\oint}(\mathscr{k}) \cap \{Y\}$ so $\text{Ⴎ} \in I_{\oint}(\mathscr{k})$which I mplies, $X - \text{Ⴎ} \notin I_{\oint}(\mathscr{k})$ hence $(X - \text{Ⴎ}) \cap \{Y\} \notin I_{\oint}(\mathscr{k}) \cap \{Y\}$. which means $(Y - \text{Ⴎ}) \notin I_{\oint}(\mathscr{k}) \cap \{Y\}$. Thus $Y - \text{Ⴎ} \notin I_{Y\oint}(\mathscr{k})$.

7) If possible, that $(Y - \text{Ⴎ}) \notin I_{Y\oint}(\mathscr{k})$, then $\forall\, U_Y \in T_Y(\mathscr{k})$, that mean $U_Y \cap \left(Y - (Y - \text{Ⴎ})\right) \notin I_Y$, so $U_Y \cap \text{Ⴎ} \notin I_Y$, but $U_Y \cap \text{Ⴎ} \subseteq \text{Ⴎ} \in I_Y$, which a contradiction.

Proposition (3.18): Let $X^{\delta}_{T_j I_j}$ $j = 1,2$ be an $i - TPS's$,such that T_2 is finer than T_1 and $I_2$is finer than I_1 ,then for $i - subspace\ Y^{\delta_Y}_{T_{j_Y} I_{j_Y}}$ of $i - TPS\ X^{\delta}_{T_j I_j}$:

1) $I_{Y\oint\, T_{1_Y}}(x) \subseteq I_{Y\oint\, T_{2_Y}}(x)$

2) $I_{1_{Y\oint T_{1_Y}}}(x) \subseteq I_{2_{Y\oint T_{2_Y}}}(x)$

Proof:

1) Let $\text{Ⴎ} \in I_{Y\oint\, T_{1_Y}}(x)$, then $\exists\, U_Y \in T_Y(x)$ s. t. $U_Y \alpha_Y \text{Ⴎ}$, so we have $U_Y \cap (Y - \text{Ⴎ}) \in I_Y$, since $U_Y \in T_{2_Y}(x)$, then $U_Y \cap (Y - \text{Ⴎ}) \in I_Y$ w. r. t. T_2 . Thus $\text{Ⴎ} \in I_{Y\oint\, T_{2_Y}}(x)$.

2) Since $I_{1_Y} \subseteq I_{2_Y}$ so we get the result immediately by part (1).

Our aim now is to introduce the definition for the notion of focal limit point and focal derivative set with respect to i − subspace of i − TPS . with some of the properties and relations.

Definition (3.19) : Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $\text{Ⴎ} \subseteq Y$, then the focal limit point of Ⴎ w. r. t. $i - subspace\, Y$ can be defined as the following every $y \in Y$ s. t. for each $U_Y \in \oint_{I_Y}(y), U_Y \cap \text{Ⴎ}/_{\{y\}} \neq \emptyset$, and the set of all focal limit points w. r. t. $i - subspace\, Y$ is called the focal derived set in $i - subspace\, Y$ and define by $Fd_Y(\text{Ⴎ}) = \cup\{\, y \in Y,\ \forall\, U_Y \in \oint_{I_Y}(y),\ \exists\, z \neq y, s.t. z \in U_Y\ and\ \ z \in \text{Ⴎ}\}$, also the focal closure of the set Ⴎ w. r. t. $i - subspace\, Y$denoted by $Fcl_Y(\text{Ⴎ})$

Example (3.20): Let $X = \{\mathscr{k}, \mathscr{g}, \mathscr{f}\}$, $T = \{\emptyset, X, \{\mathscr{k}, \mathscr{g}\}\}$, $I = \{\emptyset, \{\mathscr{f}\}\}$ with δ_D . And let $Y = \{\mathscr{k}, \mathscr{f}\}$ so $T_Y = \{\emptyset, X, \{\mathscr{k}\}\}$, $I_Y = \{\emptyset, \{\mathscr{f}\}\}$ with δ_{Y_D} be $i\text{–}subspace\ Y$ of X^{δ}_{TI}. Then $I_{Y\oint}(\mathscr{k}) = \left\{Y, \{\mathscr{k}\}\right\} = I_{Y\oint}(\mathscr{g})$, but $I_{Y\oint}(\mathscr{f}) = Y$. Now let $\text{Ⴎ} = \{\mathscr{k}, \mathscr{f}\}$, then $Fd(\text{Ⴎ}) = \{\mathscr{g}, \mathscr{f}\}$, and $Fd_Y(\text{Ⴎ}) = \{\mathscr{f}\}$.

Proposition (3.21): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i - subspace$ of $i - TPS\ X^{\delta}_{TI}$, and let $\text{Ⴎ} \subseteq Y$, Then $Fd_Y(\text{Ⴎ}) = Fd_X(\text{Ⴎ}) \cap \{Y\}$

Proof: Let $y \in Fd_Y(\text{Ⴎ})$, so $\forall\, U_Y \in \oint_{I_Y}(y)\ \exists\, z \neq y, s.t. z \in U_Y$ and $z \in \text{Ⴎ}$ but $U_Y = U \cap \text{Ⴎ}$ s.t. $U \in \oint_I(y)$ so $z \in U \cap \text{Ⴎ}$. Therefore $y \in Fd_X(\text{Ⴎ})$, and $y \in Y$ so, $y \in Fd_X(\text{Ⴎ}) \cap \{Y\}$. Thus $Fd_Y(\text{Ⴎ}) \subseteq Fd_X(\text{Ⴎ}) \cap \{Y\}$.

Conversely, Let $y \in Fd_X(Ա) \cap \{Y\}$, then $y \in Fd_X(Ա)$, and $y \in Y$, and $\forall\, U \in \oint_I(y),\ \exists\, z \neq y, s.t. z \in U \cap Ա$, but $Ա \subseteq Y$, then $z \in Y$ which implies $z \in U \cap Y$, but $U_Y = U \cap Y$, therefore by definition (319) we have $y \in \oint_{I_Y}(y)$, so, $y \in Fd_Y(Ա)$. Thus $Fd_X(Ա) \cap \{Y\} \subseteq Fd_Y(Ա)$. And this complete the proof.

Proposition (3.22): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, and let $Ա \subseteq Y$, Then $Fcl_Y(Ա) = Fd_Y(Ա) \cup Ա$

Proof: Since $Fcl_X(Ա) = Fd_X(Ա) \cup Ա$, so $Fcl_X(Ա) \cap \{Y\} = (Fd_X(Ա) \cup Ա) \cap \{Y\}$
$= (Fd_X(Ա) \cap \{Y\}) \cup Ա = Fd_Y(Ա) \cup Ա$. Thus $Fcl_Y(Ա) = Fd_Y(Ա) \cup Ա$.

Proposition (3.23): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, and let $Ա \subseteq Y$, Then $Fcl_Y(Ա) = \bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\}$

Proof: Let $y \in Fcl_Y(Ա)$, then $y \in Fd_Y(Ա)$ or $y \in Ա$. If $y \in Fd_Y(Ա)$ then $\forall\, U_Y \in \oint_{I_Y}(y)\ \exists\, z \neq y, s.t. z \in U_Y\ and\ z \in Ա$, then $y \in \bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\}$.

If $y \in Ա$ and for any $U_Y \in \oint_{I_Y}(y), y \in U_Y$, so we get $y \in U_Y \cap Ա$ then $U_Y \cap Ա \neq \emptyset$, Hence $y \in \bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\}$. Thus $Fcl_Y(Ա) \subseteq \bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\}$

Conversely, let $y \in \bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\}$, then $\forall U_Y \in \oint_{I_Y}(y), y \in U_Y$, $\exists\, z \neq y, z \in U_Y, z \in Ա$. So $y \in Fd_Y(Ա)$, then $y \in Fcl_Y(Ա)$, Thus $\bigcup \left\{y \in Y, \forall\, U_Y \in \oint_{I_Y}(y), \exists\, z \in U_Y\ and\ z \in Ա\right\} \subseteq Fcl_Y(Ա)$.

Proposition (3.24): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, and let $Ա \subseteq Y$, Then $Fcl_Y(Ա) = Fcl_X(Ա) \cap Y$

Proof: $Fcl_X(Ա) \cap Y = [Fd_X(Ա) \cup Ա] \cap Y = (Fd_X(Ա) \cap Y) \cup Ա = Fd_Y(Ա) \cup Ա = Fcl_Y(Ա)$.

Definition (3.25): Let $Y^{\delta_Y}_{T_Y I_Y}$ be $i-subspace$ of $i-TPS\ X^{\delta}_{TI}$, and let $Ա$ be a subset of Y, then we say that $Ա$ is focal dense w. r. t. $i-subspace$ if and only if $Fcl_Y(Ա) = X$, and it is denoted by FO_Ydense.

Example (3.26): Let $X = \{\mathcal{k}, \mathcal{g}, \mathcal{f}\}$, $T = \{\emptyset, X, \{\mathcal{k}, \mathcal{g}\}\}$, $I = \{\emptyset, \{\mathcal{f}\}\}$ with δ_D . And let $Y = \{\mathcal{k}, \mathcal{f}\}$ so $T_Y = \{\emptyset, X, \{\mathcal{k}\}\}$, $I_Y = \{\emptyset, \{\mathcal{f}\}\}$ with δ_{Y_D} be $i-subspace\ Y$ of X^{δ}_{TI}. Now let $Ա \subseteq Y$ s.t. $Ա = \{\mathcal{k}\}$, then $Fd_Y(Ա) = \{\mathcal{f}\}$, then we get $Fcl_Y(Ա) = Ա \cup Fd_Y(Ա) = \{\mathcal{k}\} \cup \{\mathcal{f}\} = Y$. Thus $Ա$ is FO_Ydense.

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