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Nearly Endo Semi T–ABSO Submodules and Related Concepts

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ABSTRACT

The interactions of the Endo semi prime submodule, the Endo semil-T-ABSO submodule, and the Jacobson radical are compared in this paper. In order to address the relationship between these notions, we present two concepts: the Nearly Endo semi prime submodule and the Nearly Endo semi T-ABSO submodule. The examination reveals a number of traits that back up the creative notions. The Nearly Endo semi T-ABSO submodule structure is also looked at, and it is demonstrated that the Nearly Endo T-ABSO submodule and the Nearly Endo semi T-ABSO submodule are connected using basic algebraic methods. The Nearly Endo semi T-ABSO submodules compatibility with other submodule kinds is another issue that is covered. The study's findings are critical to the development of a new Nearly Endo semi TABSO submodule.

1. Introduction

In this work, G has the identity and would be a commutative ring, while W is a unitary G-module. Research on the concept of a prime submodule of modules was done by Lu [1] in 1983 .presented the concept of prime submodule as a generalization of concept of prime ideal, Eman A. A. in [9] presented the concept of semiprime submodule as a generalization of concept of semiprime ideal, Ahmad Y. D. and Fatemeh S. in [2], introduced the concept of 2-Absorbing submodule as a generalization of concept of prime submodule while Abdulrahman A. H. in [5], presented the concept of semi 2-Absorbing submodule as a generalization of concept of semiprime submodule. The terms Nearly Endo T-ABSO submodule and Nearly Endo prime submodule were introduced by Abd Ali and Hanoon [6].

The concepts of Endo semi prime submodule and Endo semi T-ABSO submodule are respectively generalized in this article as Nearly Endo semi prime submodule and Nearly Endo semi T-ABSO submodule. This article consists of two parts. We provide several basic concepts and traits that are essential in the first section. The Nearly Endo semi T-ABSO submodule is examined in Section 2 along with its many important characteristics, findings, and outputs.

2. Perliminnares

This section discusses the several fundamental concepts as well as any prerequisites they may have for the following section.

Definition 2.1 [7]: A submodule $P \leq W$ is referred to as minimal (respectively maximal) submodule of W if $P \neq 0$, $\forall\ B \leq W$, $B \subsetneq P \Longrightarrow B=(0)$ [respectively $P \subsetneqq W$, $\forall\ B \leq W$, $P \subset B \Longrightarrow B=W$]

Definition 2.2 [7]: A $G-$ module W is referred to as $\boldsymbol{a\ cyclic}$ if $m \in\ W$ such that W = < m> = {rm : r ∈ G}.

Definition 2.3 [9]: If a módulè W has a finite generating set, it is said to be finitely generated., say X, that is $W = \langle X \rangle$.

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Definition 2.4 [8]: A submod P of a G − module W is referred to as a direct summand of W, for short $P \leq^{\oplus} W$ if, there exists a submódulè K of W such that $P + K = W$ and $P \cap K = 0$.

Definition 2.5 [1]: Let W as G-Módulè and $P \subset W$. P is referred to as a prime submódulè if $g \in G$, $s \in W$, *with* $gs \in P$ implies that $s \in P$ or $g \in (P :_G W)$.

Definition 2.6 [3]: Let W as G-Módulè and $P \subset W$. P is referred to as Endo Prime submódulè if $L \in End(W)$, L(g)∈ □, g ∈ W implies that g∈ P or $L(W) \subseteq P$.

Definition 2.7 [2]: Let W as G-Módulè and $P \subset W$. P is referred to as T-ABSO submódulè if whenever □,□∈ G, □∈ W, with □□□∈ P implies that □□∈ P or □□∈ P or □□∈ P or ab ∈ $(P:_G W)$.

Definition 2.8 [5]: Let W as G-Módulè and $P \subset W$. P is referred to as Endo T-ABSO submod if for each $f, g \in End(W)$, $m \in W$ With $(f \circ g)(m) \in P$ implies that $f(m) \in P$ or $g(m) \in P$ or $(f \circ g)(W) \subseteq P$.

Definition 2. 9[7]: Let W be a G-module. The Jacóbsón radical of W is denoted by J(W), and defined as the intersection of all maximal submódulè of W, and denoted by sum of all small submod of W. If W has no maximal submod, then we set $J(W) = W$.

Theorem 2.10 [7]: If ∅ :W⟶W' is a G- hómómórphism, then ∅ (J W)⊆ J(W') , If ∅ :W⟶W′ is a G-epimórphism and ker∅ ≪W,then ∅ (J (W)) = J(W′) , and J(W) .R⊆□(W),where G is a ring , if W is projective module then J(W).R=J(W).

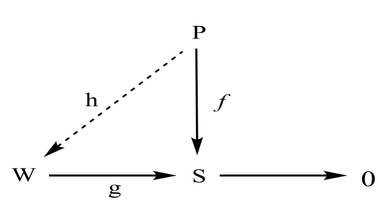
Definition 2. 11[6]: Let W as G-module and $P \subset W$, P is refered to as Nearly Endo prime submódulè (in short N-E- prime submod) if for each $L \in End$W , $x \in$ W such that $L(x) \in P$ *implies that* $x \in P + J(W)$ or $L(W) \subseteq P + J(W)$.

Definition 2.12 [6]: Let W as G-module and P ⊂ W, P is refered to as Nearly Endo T-ABSO submódulè (in short N-E- T-ABSO submod) if for each $L, h \in End W$, $x \in W$ such that $(L \circ h)(x) \in P$ implies that either $L(x) \in P + J(W)$ or $h(x) \in P + J(W)$ or $(L \circ h)(W) \subseteq P + J(W)$.

Definition 2.13 [9]: A G- module W is refered to as a scalar module if for each $f \in End(W)$, there exists $r \in G$ such that $f(x) = rx$, for $x \in W$

Corollary 2.14 [9]: Every finitely generated multiplication G-modul W is scalar módulè.

Definition 2.15 [7] : A G-module P is referred to as W-Projective
modul if each pattern diagram :



$g \, o f = f$

with exactrow can be extended commutatively via homomorphism $h: P \to W$ that is $goh = f$.

Definition 2.16 [4]: A submod P of a module W is refered to as fully invariant if, $f(P) \subseteq P$ for all $L \in End_G(W)$.

Definition 2.17 [10]: A nonzero G-module W is called coquasi – Dedekind if Hom(W,P) = 0 for all proper submodule P of W. Equivalently "A nonzero G-module W is coquasi-Dedekind if for each nonzero f ∈ EndM , f is an epimorphism

3. Nearly Endo Semi T–ABSO Submodule

Definition 3.1 : A proper submodule P of a G- module W is called Nearly Endo Semi Prime submodule (by briefly N-E-Semi Prime submodule), if whenever $K \in End(W)$, $m \in W$ such that $K^2(m) \in P$, implies that $K(m) \in P + J(W)$.

Example 3.2: Let Z_{15} as $Z -$ module, $P = (\bar{0})$is $N - E -$ Semi Prime submodule since if $k(x) = 5x$, $\forall x \in Z_{15}$, $k \in Endo(Z_{15})$ and $J(Z_{15}) = (\bar{3}) \cap (\bar{5}) = (\bar{0})$, $k^2(3) = k(k(3)) = k(15) = k(0) = 0 \in P = (\bar{0})$, implies that $k(3) = 15 = 0 \in (\bar{0}) + J(Z_{15})$.

Definition 3.3 : A proper submodule. P of G-module W is called Nearly Endo semi T-ABSO (by briefly N-E-Semi PT-ABSO) submodule of W, if for each $k \in EndW$, $m \in W$ *such that* $k^2(m) \in P$ *implies either* $k(m) \in P + J(W)$ *or* $k^2(W) \subseteq P + J(W)$.

Example 3.4: Consider Z_6 as Z-module, P=$(\bar{2}) = \{0,2,4\}$ is N-E-Semi T-ABSO submodule, since if $k \in Endo(Z_6)$, $m \in Z_6$, $k(x) = 2x$ $\forall x \in Z_6$ $J(Z_6) = (\bar{2}) \cap (\bar{3}) = (\bar{0})$, $1 \in Z_6$ $k^2(1) = k(k(1)) = k(2) = 4 \in (\bar{2}) = P$, implies that either

$k(1) = 2 \in (\bar{2}) + J(Z_6) = (\bar{2})$ or $k(Z_6) \subseteq (\bar{2}) + J(Z_6) = (\bar{2})$

$$k(Z_6) = \left.\begin{array}{l} k(0) = 0 \\ k(1) = 2 \\ k(2) = 4 \\ k(3) = 0 \\ k(5) = 4 \\ k(4) = 2 \end{array}\right] \subseteq P + J(Z_6) = (\bar{2})$$

Remark and Examples 3.5: 1) Every E- semi Prime submodule of an G-module W is N-E-semi Prime submodule of W.

Proof : Let P be E-semi Prime submodule of and $k \in End(W)$, $m \in W$ such that $k^2(m) = (k \circ k)(m) \in P$, to prove $k(m) \in P + J(W)$, Since P is E-semi Prime submodule of W, then $k(m) \in P$, hence $k(m) \in P + J(W)$, since $P \subseteq P + J(W)$, Thus P is N-E-semi Prime submodule of W. But contrariwise isn't true, for example: Consider Z_{20} as Z-module , $P = (\bar{4})$ is N-E-semi Prime submodule of Z_{20} since if $k \in End(Z_{20})$, $k(x) = x - 2$,$\forall x \in Z_{20}$ where $J(Z_{20}) = (\bar{2}) \cap (\bar{5}) = (\overline{10})$, $k^2(8) = (k \circ k)(8) = k(k(8)) = k(6) = 4 \in P = (\bar{4})$, then either $k(8) = 6 \in P + J(Z_{20}) = (\bar{2})$, but P is not E-semi Prime submodule of Z_{20}, since $k^2(8) = (k \circ k)(8) = 4 \in P = (\bar{4})$, then $6 \notin P = (\bar{4})$.

2) Every E-semi T-ABSO submodule of an G-module W is N-E-semi T-ABSO submodule.

Proof: Let P be E-semi T-ABSO submodule of an G-modul W and $k \in End(W), m \in W$ such that $k^2(m) = (k \circ k)(m) \in P$, but P is E-semi T-ABSO submodule of W, then $k(m) \in P$ or $k^2(W) \subseteq P$, hence $k(m) \in P + J(W)$ or $k^2(W) = (k \circ k)(W) \subseteq P + J(W)$ since $P \subseteq P + J(W)$, Thus P is N-E-semi T-ABSO submodule. But contrariwise isn't true, for example: consider Z_{24} as Z-module, $P = (\bar{8})$ is N-E-Semi T-ABSO submodule of Z_{24}, since if $k \in End(Z_{24})$, $k(x) = x - 2$, $\forall x \in Z_{24}$, where $J(Z_{24}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$ such that $k^2(12) = (k \circ k)(12) = k(k(12)) = k(10) = 8 \in P = (\bar{8})$, then $k(12) = 10 \in P + J(Z_{24}) = (\bar{2})$, but $k(12) = 10 \notin P = (\bar{8})$ or $k^2(Z_{24}) = (k \circ k)(Z_{24}) \nsubseteq P$ where $k^2(5) = k(k(5)) = k(3) = 1 \notin P, then\ k^2(Z_{24}) \nsubseteq P$. Thus P is not E-semi T- ABSO submodule of W.

3) Let P, S be two submodules of an G-module W, and $P \subset S$. If P is N-E- Semi T-ABSO submodule of W, then P is N-E-Semi T-ABSO submodule of S with $J(W) \subseteq J(S)$.

Proof: Let $k^2(m) = k(k(m)) \in P$, $\forall m \in S$ since $S < W, so\ m \in W, k \in End(W)$, Since P is N-E-Semi TABSO submodule of W, then either $k(m) \in P + J(W)$ or $k^2(W) = (k \circ k)(W) \subseteq P + J(W)$, since $J(W) \subseteq J(S)$, hence $k(m) \in P + J(S)$ or $k^2(W) \subseteq P + J(S)$, but $S < W$, so that $k^2(S) \subseteq k^2(W)$, hence $k^2(S) \subseteq P + J(S)$. Thus P is N-E- Semi T-ABSO submodule of S.

4) The intersection of two N- E-Semi T-ABSO submodule of an G-module W is not N-E-Semi T-ABSO submodule of W, the following example explains that: consider Z as Z-module take $P = 2Z, S = 9Z$ are N-E-Semi T-ABSO submodules of Z, since $\forall k \in Endo(Z)$, $k(x) = 3x$, $\forall x \in Z$ where $J(Z) = (\bar{0})$ such that $k^2(2) = k(k(2)) = k(6) = 18 \in P = 2Z$, then $k(2) = 6 \in 2Z + J(Z) = 2Z$, also $k^2(1) = k(k(1)) = k(3) = 9 \in S = 9Z$, then $k^2(Z) \subseteq 9Z + J(Z) = 9Z$. But $(2Z \cap 9Z) = 18Z$ is not N-E-Semi T-ABSO submodules of Z, since $k^2(2) = k(k(2)) = k(6) = 18 \in 18Z$, then $k(2) = 6 \notin 18Z + J(Z) = (\overline{18})$ and $k^2(Z) \nsubseteq 18Z + J(Z)$, Where $k^2(1) = k(k(1)) = k(3) = 9 \notin 18Z + J(Z)$.

5) Every N-E-Semi prime Submodule is N-E-Semi T-ABSO submodule of W

6) Every E- Semi Prime submodule is N-E-Semi T-ABSO submodule , but the converse incorrcet in general, for example: Consider Z_{20} $as\ Z - module$, $(\bar{4})$ is a submodule of Z_{20}, $P = (\bar{4})$ is N-E-Semi T-ABSO submodule , since if $k \in End(Z_{20}), k(x) = 3x - 2, \forall x \in Z_{20}$, $J(Z_{20}) = (\bar{2}) \cap (\bar{5}) = (\overline{10})$ $k^2(4) = k(k(4)) = k(10) = 28 = 8 \in (\bar{4})$, implies that $k(4) = 10 \in (\bar{4}) + J(Z_6) = (\bar{2})$, But it is not E -Semi prime submodule of Z_{20}, $k^2(4) = k(k(4)) = k(10) = 28 = 8 \in (\bar{4})$, implies that $k(4) = 10 \notin P = (\bar{4})$.

7) Let P, S be two submodules of an G-module W and $P \subset S$. If S is N- E- Semi T-ABSO submodule of W, then P is not necessary that N-E-Semi T-ABSO submodule of W, for example: consider Z as Z-module take $S = 2Z$ is N-E-SemiT-ABSO submodules of Z, since $\forall k \in Endo(Z)$, $k(x) = 3x$, $\forall x \in Z$ where $J(Z) = (\bar{0})$ such that $k^2(2) = k(k(2)) = k(6) = 18 \in P = 2Z$, then $k(2) = 6 \in 2Z + J(Z) = 2Z$. But $P = 18Z$ is not N-E-Semi T-ABSO submodules of Z, since $k^2(2) = k(k(2)) = k(6) = 18 \in 18Z$, then $k(2) = 6 \notin 18Z + J(Z) = 18Z$ and $k^2(Z) \nsubseteq 18Z + J(Z)$, Where $k^2(1) = k(k(1)) = k(3) = 9 \notin 18Z + J(Z)$.

Proposition 3.6: Let L and K be N-E-semi-T-ABSO proper submodules of an G-module W, with $K \nsubseteq L$ and either $J(w) \subseteq L$ or $K(w) \subseteq L$ then $K \cap L$ is N-E-semi-T-ABSO of W.

Proof: Since $K \nsubseteq L$, then $L \cap K \subset L \subset W$ is follows that $K \cap L \subset W$, let $f^2(w) \subseteq L \cap K$, $f \in End(W), m \in W\ then\ f^2(m) \in K$ and $f^2(m) \subseteq L$ but both K and L, $N - E - semi - T -$

ABSO submodules of W, then either $f(m) \in K + J(W)$ or $f^2(W) \in K + J(W)$. It follows that $f(m) \in L + J(W)$ or $f^2(W) \in L + J(W)$ *Then* $f(m) \in (K + J(W)) \cap (L + J(w))$ or $f^2(m) \subseteq (K + J(w)) \cap (L + J(w))$.

Case1: if $J(w) \subseteq L$, then $f(m) \in (K + J(w)) \cap L$ it follows that by lemma(2.15) $f(m) \in (K \cap L) + J(w)$

Case2: $J(w) \subseteq K$ then $f(m) \in (L + J(w)) \cap K$ if follows that by lemma(2.15) $f(m) \in (L \cap K) + J(w)$. Also $f^2(m) \subseteq (K + J(w)) \cap (L + J(w)$ by case1 and case 2, $f^2(m) \subseteq (L \cap K) + J(w)$. Hence $(L \cap K)$ is $N - E - semi - T - ABSO$ submodules of W .

Proposition 3.7: Let P be N-E-semi-T-ABSO submodule of an G-module W is Scalar module and W and k is a proper submodule of W with $K \nsubseteq P$ *and* $J(K) = J(W)$, *and* $J(K)$ is distributive submodule. then $P \cap K$ *is* N-E-semi-T-ABSO submodule in K

Proof: Since $K \nsubseteq P$, then $P \cap K < K$, let $f^2(m) \in P \cap K$ where $f \in End(W), m \in K$. since P is N-E-semi-T-ABSO submodule and $f^2(m) \subseteq P$, *implies that* $f(m) \in P + J(w)$ *or* $f^2(w) \subseteq P + J(w)$, *but* $J(k) = J(W)$, *then* $f(m) \in P + J(k)$ *or* $f^2(w) \subseteq P + J(K)$. *since* $m \in k$, *then* $f(m) \in K$, *hence* $f(m) \in (P + J(K) \cap K$ *since* $K \subseteq (K + J(K)$, *so that* $f(m) \in (P + J(K)) \cap (K + J(K))$ hence $f(m) \in (P \cap K) + J(K)$, *since* $J(K)$ *is* distributive submodule or $f^2(W) \subseteq P + J(K)$ implies that $f^2(W) \subseteq P + J(K)$ for all $m \in W$, Since $f(m) \in K$, *then* $f^2(m) \in K$. then $f^2(m) \in (P + J(K) \cap K$. Hence $f^2(m) \in (P \cap J(K)) \cap (K + J(K)$, so that $f^2(m) \in (P \cap K) + J(K)$ since $J(K)$ is distributive submodule then $f^2(m) \in (P \cap K) + J(K)$ Thus , $P \cap K$ is N-E-semi-T-ABSO submodule in K.

Proposition 3.8: Let P a submod of a scalar G-module W. Then P is semi T-ABSO submod if and only if P is N-E-Semi T-ABSO submod and $J(W) \subseteq P$.

Proof: ($\Longrightarrow$) Let $L^2(x) \in P$ *where* $L \in \mathrm{Endo}(W)$, $\forall x \in W$ *since* W is Scalar module , then there exist $a \in G$ such that $ax = L(x)$, $\forall x \in W$, but P is Semi T-ABSO submodule of W, then $L^2(x) = a^2x \in P$, implies that either $ax = L(x) \in P$ *or* $a^2 \in (P:_G W)$. Since $J(W) \subseteq P$, hence $L(x) \in P + J(W)$ or $L^2(W) \subseteq P + J(W)$.Then P is N-E-Semi T-ABSO submodule of W.

($\Longleftarrow$) Let $L \in \mathrm{Endo}(W), \forall x \in W$ such that $a^2x \in P$, *and* $L(x) = ax$, $L^2(W) = a^2W \subseteq P$, *since* W is Scalar module , but P is N-E-Semi T-ABSO submodule of W, then either $ax = L(x) \in P + J(W)$ *or* $L^2(W) = a^2W \subseteq P + J(W)$, hence $ax \in P$ or $a^2W \subseteq P$, so that $a^2 \in (P:_G W)$ since $J(W) \subseteq P$. Then P is Semi T-ABSO submodule of W.

Remark 3.9: If delete the condition of Scalar module the converse of Proposition 3.8 is not true in general, for example: Consider Z_{12} as Z-module, $P = (\bar{6})$ is semi T- ABSO submodule of Z, $6^2 \cdot (\bar{1}) = 36 = 0 \in P = (\bar{6})$, implies that $6 \cdot (\bar{1}) = 6 \in P = (\bar{6})$ or $6^2 = 36 = 0 \in (P:_Z Z_{12}) = 6Z$. So that P is semi T-ABSO submodule of Z_{12}. But P is not N-E-semi T-ABSO submodule of Z_{12} since if $K \in Endo(Z_{12})$, $K(x) = 2x - 2$ such that $K^2(3) = K(4) = 6 \in P$, where $J(Z_{12}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$, then $K(3) = 4 \notin P + J(Z_{12}) = (\bar{6})$ and $K^2(Z_{12}) \nsubseteq P + J(Z_{12})$ Where $K^2(2) = K(2) = 2 \in P + J(Z_{12})$.

Corollary 3.10: Let P a proper submodule of a finitely generated multiplication G-module W, $J(W) \subseteq P$. Then P is Semi T-ABSO submodule if and only if P is N-E-Semi T-ABSO submodule

Proof : By Proposition 3.8 and Corollary 2.14 we get the result.

Proposition 3.11: Let $P\ be$ a proper submodule of an G-module W, $f: W \to W$ be a G-homomorphism. If P is N- E- Semi T- ABSO submodule of W fully invariant and EndW is commutative then $f^{-1}(P)$ is N-E- Semi T-ABSO submodule of W.

Proof: Since P is a proper submod of W, So $f^{-1}(P)$ is a proper submodule of W. Let $k \in \mathrm{End}(W)$ such that $k^2(m) \in f^{-1}(P), for\ some\ m \in P. Then\ f(k^2(m)) \in P$ since $\mathrm{End}(W)$ commutative , then $f(k^2(m)) = k^2(f(m)) \in P.\ But$ P is N- E-Semi T-ABSO submodule of W, so either $k(f(m)) \in P\ or\ k^2(W) \in P.$

Case 1: If $k(f(m)) \in P$, we have $k(f(m)) = f\big(k(m)\big) \in P$ since EndW is commutative, so $k(m) \in f^{-1}(P)$

Case 2: If $k^2(W) \subseteq P$, then $k^2(W) \subseteq f^{-1}(P)$, since $f(P) \subseteq P$ implies $P \subseteq f^{-1}(P)$. Hence $f^{-1}(P)$ is N-E- Semi T-ABSO submodule of W.

Proposition 3.12: Let P be a proper submodule of G-module of W, Let S be a fully invariant submodule of G-module W and S ⊆ P. If $\frac{P}{S}$ is N-E- Semi T- ABSO submodule of $\frac{W}{S}$, then P is N-E-Semi T-ABSO submodule of W.

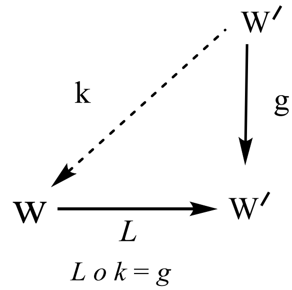
Proof: Let $K \in EndW$, $K^2(m) \subseteq P,\ m \in W$. Define $L: \frac{W}{S} \to \frac{W}{S}$ by

$L(m+S) = K(m) + S\ ,\ for\ each\ m \in W.$ It is clear that L are well-define, since S is a fully invariant.

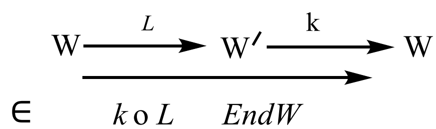
Now $L^2(m+S) = L\big(L(m+S)\big) = L\big(K(m+S)\big) = K^2(m) + S \in \frac{P}{S}$, but $\frac{P}{S}$ is N-E- Semi T-ABSO submodule of $\frac{W}{S}$. Then either $L(m+S) = K(m) + S \in \frac{P}{S} or L^2\left(\frac{W}{S}\right) \subseteq \frac{P}{S}$,then $K(m) \in P\ or\ L^2(W) \subseteq P,\ K(m) \in P + J(W)\ or\ K^2(W) \subseteq P + J(W)$. Thus P is N-E- Semi T-ABSO submodule of W.

Proposition 3.13: Let P be a fully invariant N-E-semi- T-ABSO submodule of G-module W. let $\mathrm{L}: W \to W'$ be an epimorphism such that kerL ≪W and kerL $\subseteq P. Then$ $\mathrm{L}(m)$ is N-E- semi- T-ABSO submodule of W', where W' is W'-projective module.

Proof: Let $g \in EndW'$, $m' \in W'$ such that $(gog)(m') \in L(P)$ since L is epimorphism then $m' = L(m)$ for some $m \in P$ since W' is W-projective Module , there exist $K: W' \to W$, such that $LoK = g$



Now consider the following digrams



$g^2(m') = (gog)(m') = (LoK)o(LoK)(m') \in L(P) = L[(KoLo\,K)(m')] \in L(P)$Then $(KoLo\,K)(m') \in P + KerL$, since $KerL \subseteq P, and\ m' = L(m)\ , so\ (KoLo\,K)(L(m)) \in P$. That is $(KoL)o(\,KoL\,)(m)) \in P$, since P is N- Endo semi-T-ABSO submodule of W , then either $(KoL)(m) \in P + J(W)$ ***or*** $(\,KoL)o(KoL)(W) \subseteq P + J(W)$

If $(KoL)(m) \in P + J(W)$, then $(K\big(L(m)\big) \in P + J(W), i.e$

$K(m') \in P + J(W)$ so $L(K(m')) \in L(P) + L(J(W))$ thus $g(m') \in L(P) + L(J(W))$.

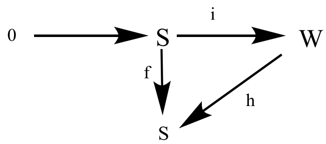
If $(KoL)o(KoL)(W) \subseteq P + J(W)$, then ($(\boldsymbol{K}oLoK)\,L(W)) \subseteq P + J(W)$, Since $L(W) = W'\ , (KoLoK)(W') \subseteq P + J(W)$ So $L[(KoLoK)(W')] \subseteq L(P) + L\,\big(J(W)\big)\ , i.e\ (\,LoK)o(LoK)(W') \subseteq L(P) + J(L(W)) = (g\,)o(g)(W') \subseteq L(P) + J(W') = g^2(W') \subseteq L(P) + J(W')\ , thus\ L(m)\ is\ N - E -\ semi\ T - ABSO\ submodule\ of\ W'$.

Corollary 3.14: Let P be an N-E- semi- T-ABSO submodule of G-module W. If K is submodule of W such that $K \subseteq P$, then $\frac{P}{K}$ is N-E- semi- T-ABSO submodule of $\frac{W}{K}$, Provided$\frac{W}{K}$ is an W-projective module.

Proof: Define $\pi: W \to \frac{W}{K}$ epimorphism, then by Proposition 3.13 , we get result.

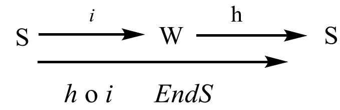
Proposition 3.15: Let P be N-E- semi- T-ABSO submodule of G-module W. and P is submodule of W which is W-injectiv submodule, $J(W) \subseteq J(S)$ and $J(S)$ is distributive submodule. Then either $S \subseteq P$, or $S \cap P$ is N-E- semi- T-ABSO submodule of S .

Proof: Suppose that $S \nsubseteq P\ , then\ S \cap P \subsetneq S\ , let\ f \in EndS\ , x \in S$ such that $fof(x) \in S \cap P$, since $J(W) \subseteq J(S)$, to prove $f(x) \in S \cap P + J(S)\ or\ fof(S) \subseteq S \cap P + J(S)$, since S is W-injective submodule, then there exist $h : W \to S$ as in the figure :



$h \, o \, i = f$

Now consider the following diagrams ∈



Where I is the inclusion mapping and $hoi = f$, clearly that h $\in End\, S$, But $fof(x) = [(hoi)_0(hoi)](x) \in P$, since p is N-E-semi-TABSO submodule of W. implies that $(hoi)(x) \in P + J(W)$ or $(hoi)^2(W) \subseteq P + J(W)$.

If $(hoi)(x) \in P + J(W), (hoi)(x) \in S \cap P + J(W), then\ h(x) \in S \cap P + J(W)$ since $h(x) \in S$, so $h(x) \in S + J(W)$ but $J(W) \subseteq J(S),\ hence\ h(x) \in S \cap P + J(S)$. Thus $f(x) \in S \cap P + J(S)$. Now, if $(hoi)(W) \subseteq P + J(W)$. As $fof(S) = [(hoi)_0(hoi)](S) = [(hoioh)](S) = (hoi)[\,h(S)] = (hoi)(S) \subseteq P + J(W), then\ fof(S) \subseteq P + J(S),$

$since\ J(W) \subseteq J(S)$. Also $(fof)(S) \subseteq S$, hence $fof(S) \subseteq S + J(S)$.

Thus $(fof)(S) \subseteq \big(P + J(S)\big) \cap \big(S + J(S)\big)$, so that $(fof)(S) \subseteq (P \cap S) + J(S)$, since $J(S)$ is distributive submodule . Therefore $S \cap P$ is N-E-semi-TABSO submodule of S.

Proposition 3.16: Let W be a coquasi – Dedekind G-module, then P is N-E -semi- T-ABSO submodule of W, if and only if P is N- E-prime submodule.

Proof: (⟹) let $f \in EndW$ with $f \neq 0, f(y) \in P \quad for\ \ y \in W$. Since W is coquasi – Dedekind, f is onto, so there exists $x \in W\ such\ that\ y = f(x)$, hence $f\ (f(x)) \in P$. Since P is N- E- semi- T-ABSO, then either $f(x) \in P + J(W)\ or\ f^2(W) \subseteq P + J(W).\ But\ f^2(W) = W \nsubseteq P, since\ W\ is$ coquasi – Dedekind. So $y = f(x) \in P + J(W)$ thus P is N-E- prime submodule.

(⟸) It is clear.

Corollary 3.17: Let W is a coquasi – Dedekind G-module W , P is N- E- T-ABSO – submodule of W, then P is N-E- prime submodule of W.

Proof: It is clear

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