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Convergence On Daniell Space With Some Of Their Properties

Authors Names

Noor Ramzi Hameed[a],

Noori F. Al – Mayahi[b]

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ABSTRACT

In this work, we introduce the concept of three types of convergence (convergence almost everywhere, almost uniformly convergence, convergence in norm) in relation to Daniell integration with the properties of each type, and then discuss the relationship between these three types, and finally we present the basic theorems of convergence, such as Dominated convergence theorem, monotone convergence theorem and Fatou's lemma

1. Fundumental Concept

Recall that, A sequence $\{x_n\}$ of real numbers is said to be Converge to the point $x \in \mathbb{R}$, if for each $\varepsilon > 0$, there is $k \in z^+$ such that $|x_n - x| < \varepsilon$ for all $n \geq k$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$. Given a sequence $\{f_n\}$ of real valued functions defined on Ω, for $x \in \Omega$, we have real sequence $\{f_n(x)\}$. If $\{f_n(x)\}$ is converge for all point of Ω. we can define the function $f: \Omega \to \mathbb{R}$ by, for any $x \in \Omega$, then $f(x)$ is limit point of $\{f_n(x)\}$; that is $f(x) = \lim_{n\to\infty} f_n(x)$ or $f_n(x) \to f(x)$.

Definition 1.1

let $\{f_n\}$ be a sequence of real valued functions defined on Ω and a function $f: \Omega \to \mathbb{R}$, $A \subseteq \Omega$. We say that

(1) $\{f_n\}$ converges to f (pointwise) on A, if for every $x \in A$, then $f_n(x) \to f(x)$, i.e. if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that

$|f_n(x) - f(x)| < \varepsilon$ for all $n > k$. We write $\lim_{n\to\infty} f_n(x) = f(x)$

or $f_n \to f$ on A.

(2) $\{f_n\}$ is a Cauchy sequence (pointwise) on A, if for every $x \in A$, then $\{f_n(x)\}$ is a Cauchy sequence, i.e., for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ suh that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$.

- Note that: In the above definition when $A = \Omega$ we can omit “on A” from the statements i.e. $f_n \to f$, if for every $x \in \Omega$ and for $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such $|f_n(x) - f(x))| < \varepsilon$ for all $n > k$.

- This has meaning only if $f_n: \Omega \to \mathbb{R}$ is finite valued. Because $\mathbb{R}$ is complete it is clear that if $\{f_n\}$ is a Cauchy sequence pointwise on Ω, there must be an $f: \Omega \to \mathbb{R}$ such that $f_n \to f$ on Ω.

Department Of Mathematics, College Of Computer Science and Mathematics, AL-Qadisiyah University \Iraq.
Ma20.post22@qu.edu.iq[1.a], nafm60@yahoo.comi[2.b]

Definition 1.2

let $\{f_n\}$ be a sequence of real valued functions defined on Ω and a function $f: \Omega \to \mathbb{R}$, we say that

(1) $\{f_n\}$ converges uniformly to f on A, if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$

such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and all $x \in \Omega$, we write

$f_n \overset{u}{\rightarrow} f$ on A

(2) $\{f_n\}$ is a Cauchy sequence uniformly on A, if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$

Such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$ and all $x \in A$.

- It is clear that every converges uniformly sequence are convergent pointwise, but the converse is not true.

Remark

Let $a, b \in \mathbb{R}$ and let $f: \Omega \to \overline{\mathbb{R}}$, $g: \Omega \to \overline{\mathbb{R}}$ be functions

1.$(f \wedge g)(x) = f(x) \wedge g(x) = \min\{f(x), g(x)\}$

2.$(f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\}$

3.$\{f = g\} = \{f \leq g\} \cap \{f \geq g\}$

4.$\{f > g\} = \bigcup_{n=1}^{\infty}(\bigcup_{m=1}^{\infty}\left\{f > \frac{m}{n}\right\} \cap \left\{g < \frac{m}{n}\right\})$

5. $\{\min\{f, g\} < a\} = \{f < a\} \cup \{g < a\}$

6. $\{\min\{f, g\} > a\} = \{f > a\} \cap \{g > a\}$

7. $\{\max\{f, g\} < a\} = \{f < a\} \cap \{g < a\}$

8. $\{\max\{f, g\} > a\} = \{f > a\} \cup \{g > a\}$

Let $a \in \mathbb{R}$ and $f_n: \Omega \to \overline{\mathbb{R}}$ be a function for all n

1.$\{\sup f_n \leq a\} = \bigcap_{n=1}^{\infty}\{f_n \leq a\}$

2. $\{\sup f_n > a\} = \bigcup_{n=1}^{\infty}\{f_n > a\}$

3.$\{\sup f_n < a\} = \bigcap_{n=1}^{\infty}\{f_n < a\}$

4.$\{\inf f_n > a\} = \bigcap_{n=1}^{\infty}\{f_n \leq a\}$

5.$\{\inf f_n < a\} = \bigcup_{n=1}^{\infty}\{f_n < a\}$

Definition 1.3

A function $f \in L$ is called a null function if $D(|f|) = 0$.

Example 1.4

the function $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ be an example of a null function.

Remark:

if f is a null function and $|g| \leq |f|$ then g is a null function.

Since $0 \leq D(|g|) \leq D(|f|) = 0$.

Definition 1.5

A set $A \subseteq \Omega$ is called a null set if the characteristic function of A is a null function,

i.e., $D(|I_A|) = 0$.

Theorem 1.6

If A be a null set and $B \subseteq A$ then B is a null set

Proof

Let A be a null set then $D(|I_A|) = 0$, since $B \subseteq A$ then $I_B \leq I_A \Longrightarrow |I_B| \leq |I_A| \Longrightarrow D(I_B) \leq D(I_A) = 0$ and since $|I_B| \geq 0 \Longrightarrow D|I_B| \geq D(0) = 0$ implies that $D(|I_B|) = 0$. Therefore B is a null set.

Theorem 1.7

Let A_i be a sequence of null set in L for all $i = 1,2, \ldots$ then $\bigcup_{i=1}^{n} A_i$ is a null set in L

Proof

Let A_i be a sequence of null set, $A_i \subset \Omega$ for all $i = 1,2, \ldots$, since $I_{\bigcup_{i=1}^{n} A_i} = I_{A_1} + I_{A_2} + \cdots + I_{A_n} - I_{\bigcap_{i=1}^{n} A_i} \Longrightarrow \left|I_{\bigcup_{i=1}^{n} A_i}\right| = \left|I_{A_1} + I_{A_2} + \cdots + I_{A_n} - I_{\bigcap_{i=1}^{n} A_i}\right| \leq |I_{A_1}| + |I_{A_2}| + \cdots + |I_{A_n}| - \left|I_{\bigcap_{i=1}^{n} A_i}\right| = |I_{A_1}| + |I_{A_2}| + \cdots + |I_{A_n}| - |I_{A_1 \cdot A_2 \cdot \ldots \cdot A_n}| \Longrightarrow D\left(\left|I_{\bigcup_{i=1}^{n} A_i}\right|\right) = D(|I_{A_1}|) + D(|I_{A_2}|) + \cdots + D(|I_{A_n}|) - D(|I_{A_1 \cdot A_2 \cdot \ldots \cdot A_n}|) = 0 \Longrightarrow D\left(\left|I_{\bigcup_{i=1}^{n} A_i}\right|\right) \leq 0$ and $\left|I_{\bigcup_{i=1}^{n} A_i}\right| \geq 0 \Longrightarrow D\left(\left|I_{\bigcup_{i=1}^{n} A_i}\right|\right) \geq 0 \Longrightarrow D\left(\left|I_{\bigcup_{i=1}^{n} A_i}\right|\right) = 0$ there fore $\bigcup_{i=1}^{n} A_i$ be a null set.

Definition 1.8

A function f, g in L are called equivalence if $f - g$ is a null function,

i.e., $f \sim g \; if \; D(|f - g\,|) = 0$.

We will denoted to the space of equivalent class in L by $\mathcal{L}$ and $[f]$ be the equivalence class of $f \in L$ such that $[f] = \{g \in L: D(|f - g|) = 0\}$.

To prove that $\sim$ be an equivalent relation on $\mathcal{L}$ we must show that $\sim$ is

(1) Reflexive: Let $f \in L,\ |f - f| = |0| = 0 \Longrightarrow D(|f - f|) = D(0) = 0 \Longrightarrow D(|f - f|) = 0 \Longrightarrow f \sim f$.
(2) Symmetric: let $f, g \in L$ and $f \sim g$ then $D(|f - g\ |) = 0 = D(|g - f\ |)$ hence $g \sim f$.
(3) Transitive: let $f, g, h \in L$ with $f \sim g$ and $g \sim z$ then $|f - z| = |f - h + g - g| \le |f - g| + |g - h| \Longrightarrow |f - h| \le |f - g| + |g - h| \Longrightarrow D(|f - h|) \le D(|f - g| + |g - h|) = D(|f - g|) + D(|g - h|) = 0 + 0 = 0$.

then $D(|f - h|) \le 0$ and since $|f - z| \ge 0$ then

$D(|f - h|) \ge D(0) = 0 \Longrightarrow D(|f - h|) \ge 0$, and hence

$D(|f - h|) = 0$ implies that $f \sim h$.

Theorem 1.9

the space of equivalent class $(\Omega, \mathcal{L}, D)$ is a subspace of (Ω, L, D).

Proof:

(1) Let $[f], [g] \in \mathcal{L}\ then\ [f] + [g] = \{h \in L: D(|f - h| = 0)\} + \{s \in L: D(|g - s| = 0)\} = \{h + s \in L: D(|f - h|) + D(|g - s|) = 0\} = \{h + s \in L: D(|f - h| + |g - s|) = 0\} = \{h + s \in L: D(|f + g| - |h + s|) = 0\} = [f + g]$.

Therefore $[f] + [g] \in \mathcal{L}$

(2) Let $[f] \in \mathcal{L}$ and $\lambda \in \mathbb{R}$, then $\lambda[f] = \{g \in L: D(|f - g| = 0)\}$

$= \{\lambda g \in L: \lambda D(|f - g| = 0)\} = \{h = \lambda g \in L: D(|\lambda f - h| = 0)\} = \{h \in L: D(|\lambda f - h| = 0)\} = [\lambda f]$. Therefore $\lambda[f] \in \mathcal{L}$.

Theorem 1.10

Let (Ω, L, D) be a Daniell space and let $f, g \in L$ then

(1) $f = g\ a.e$ if and only if $D(|f - g|) = 0$

(2) f is null sunction if and only if $f = 0\ a.e$.

(3) f and g are equivalent if and only if $f = g\ a.e$.

Proof:

(1) Let $A = \{x \in \Omega: f(x) \ne g(x)\}$

($\Longrightarrow$) suppose that $f = g\ a.e$. then $D(|I_A|) = D(I_A) = 0$

Then $|f - g| = I_A + I_A + \cdots$, implies that $D(|f - g|) = 0$

($\Leftarrow$) suppose that $D(|f-g|)=0$ then $I_A=|f-g|+|f-g|+\cdots$

There fore $D(I_A)=D(|I_A|)=0$ this implies that A is a null set,

and hence $f=g\ a.e.$

(2) Let $B=\{x\in\Omega: f(x)\neq 0\}$

($\Rightarrow$) suppose f is null function then $D(|f|)=0$, since $|f|\geq 0$

implies $f=0\ a.e.$, or in another proof, if $D(|f|)=0$

then $I_B=|f|+|f|+\cdots$, implies

$D(I_B)=D(|I_B|)=D(|f|)+D(|f|)+\cdots$ then $D(|I_B|)=0$

therefore B is a null set and then $f=0\ a.e.$

($\Leftarrow$) Let $f=0\ a.e.$ then $D(|I_B|)=D(I_B)=0$ Then

$|f|=I_B+I_B+\cdots$, implies that $D(|f|)=0$, then f is null function.

(3) ($\Rightarrow$) Suppose that $f\sim g$ then $D(|f-g|)=0$, then by (1), $f=g\ a.e.$

($\Leftarrow$) Let $f=g\ a.e.$, then by (1), $D(|f-g|)=0$ implies that $f\sim g$

2. Convergence Almost Everywhere

Definition 2.1

Let (Ω,L,D) be a Daniell space. A sequence $\{f_n\}$ in L is said to be

(1) Converges almost everywhere to the function f in L, denoted by $f_n\xrightarrow{a.e.}f$, if there is a null set $A\subseteq\Omega$ such that $f_n\to f$ on A^c.
(2) $\{f_n\}$ Cauchy almost everywhere, denoted by f_n Cauchy a.e. if there is a null set $A\subseteq\Omega$ such that f_n Cauchy on A^c.

Theorem 2.2

Let (Ω,L,D) be a Daniell space and let $f_n\in L$, $n\in\mathbb{N}$, if $f_n\xrightarrow{a.e.}f$, then $f\in L$.

Proof:

Let $A_n=\{x\in\Omega: \lim_{n\to\infty}f_n(x)\neq f(x)\}$, since $f_n\xrightarrow{a.e.}f$, then A is a null set

Define $h_n(x)=\begin{cases}f_n(x), & x\notin A\\ 0, & x\in A\end{cases}$, then if $x\notin A$ implies $f_n(x)=h_n(x)\Rightarrow$

$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}h_n(x)\Rightarrow f(x)=h(x)$ implies that $\{h_n\}$ convergenc pointwise to h on Ω. Also $h_n\in L$ for all n. Hence $h\in L$. Consequently $f\in L$.

Theorem 2.3

Let (Ω, L, D) be a Daniell space and let $f, f_n \in L, n \in \mathbb{N}$, if $f_n \xrightarrow{a.e.} f$ then

(1) f_n is a Cauchy a.e.
(2) If $g \in L$ and $f_n \xrightarrow{a.e.} g$ then $f = g$ a.e.
(3) If $g \in L$ and $f = g$ a.e. then $f_n \xrightarrow{a.e.} g$
(4) If $g_n \in L$ and $f_n = g_n \ a.e.$ then $g_n \xrightarrow{a.e.} f$

Proof: Let $A = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq f(x)\}, A \subseteq \Omega$.

(1) Since $f_n \xrightarrow{a.e.} f$ then $D(|I_A|) = D(I_A) = 0$ and $f_n(x) \to f(x)$ on A^c then $f_n(x)$ is a Cauchy sequence for all $x \in A^C$, there fore f_n is a Cauchy sequence.

(2) Since $f_n \xrightarrow{a.e.} f$ then $D(|I_A|) = D(I_A) = 0$ and $f_n(x) \to f(x)$ on A^c, since $f_n \xrightarrow{a.e.} g$ then there exist $B = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq g(x)\}, B \subseteq \Omega$ such that $D(I_B) = 0$ and $f_n(x) \to g(x)$ for all $x \in B^c$.

Let $C = A \cup B \Longrightarrow I_{C=A\cup B} = I_A + I_B - I_{A\cap B} = I_A + I_B - (I_A.I_B) \Longrightarrow D(I_C) = D(I_A) + D(I_B) - D(I_A.I_B) = 0$, and for any $x \in C^c f_n(x) \to f(x), f_n(x) \to g(x)$, then $f(x) = g(x)\ \forall x \notin C$ implies that $f = g\ a.e.$

(3) Since $f_n \xrightarrow{a.e.} f$ then $D(|I_A|) = D(I_A) = 0$ and $f_n(x) \to f(x)\ \forall \notin A^c$, $f = g$ a.e., then there exist $B = \{x \in \Omega: f(x) \neq g(x)\}, B \subseteq \Omega$ such that
$D(I_B) = 0$ and $f(x) = g(x)$ for all $x \notin B$. Let $C = A \cup B \Longrightarrow I_{C=A\cup B} = I_A + I_B - I_{A\cap B} = I_A + I_B - (I_A.I_B) \Longrightarrow D(I_C) =$

$D(I_A) + D(I_B) - D(I_A.I_B) = 0$, and $\forall x \notin D\ \lim_{n\to\infty} f_n(x) = f(x) = g(x)$,

So $\lim_{n\to\infty} f_n(x) = g(x) \forall x \notin D$. Therefore $f_n \xrightarrow{a.e.} g$.

(4) Since $f_n \xrightarrow{a.e.} f$ then $D(|I_A|) = D(I_A) = 0$ and $f_n(x) \to f(x)$ on A^c, and
$f_n = g_n\ a.e.$, let $B_n = \{x \in \Omega: f_n(x) \neq g_n(x)\}$ be a sequence in Ω such that $f_n(x) = g_n(x)$ $\forall x \notin B_n$ and $D(I_{B_n}) = 0$, let $C = A \cup (\bigcup_n B_n)$ then $I_{C=A\cup(\bigcup_{n=1}^{\infty} B_n)} = I_A + I_{\bigcup_{n=1}^{\infty} B_n} - I_{A\cap(\bigcup_{n=1}^{\infty} B_n)} = I_A + I_{\bigcup_{n=1}^{\infty} B_n} - (I_A.I_{\bigcup_{n=1}^{\infty} B_n})$, then

$D(I_C) = D(I_A) + D(I_{\bigcup_{n=1}^{\infty} B_n}) - D(I_A.I_{\bigcup_{n=1}^{\infty} B_n}) = 0$, and $\forall x \notin C, \lim_{n\to\infty} g_n(x) = lim_{n\to\infty} f_n(x) = f(x)$. So $g_n(x) \to f(x)\ \forall x \notin C$. Therefore $g_n \xrightarrow{a.e.} f$.

Theorem 2.4

Let (Ω, L, D) be a Daniell space and let $f, f_n \in L, n \in \mathbb{N}, and\ \lambda \in \mathbb{R}$,

if $f_n \xrightarrow{a.e.} f, and\ g_n \xrightarrow{a.e.} g$ then

(1) $\lambda f_n \xrightarrow{a.e.} \lambda f$
(2) $f_n + g_n \xrightarrow{a.e.} f + g$
(3) $|f_n| \xrightarrow{a.e.} |f|$

Proof:

(1) Since $f_n \xrightarrow{a.e.} f$ then there exist $A = \{x \in \Omega: f_n(x) \neq f(x)\}$ such that $D(|I_A|) = D(I_A) = 0 \; and \; f_n(x) \to f(x) \forall x \notin A$, then $\lambda f_n(x) \to \lambda f(x) \forall x \notin A$ there fore $\lambda f_n(x) \xrightarrow{a.e.} \lambda f(x) \forall x \notin A$.

(2) Since $f_n \xrightarrow{a.e.} f$ and $g_n \xrightarrow{a.e.} g$ then the sets $A = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq f(x)\}$, $A \subseteq \Omega$ and $B = \{x \in \Omega: lim_{n\to\infty} g_n(x) \neq g(x)\}$, $B \subseteq \Omega$ are null set, and $f_n(x) \to f(x) \forall x \notin A$ and $g_n(x) \to g(x) \forall x \notin B$.

Let $C = A \cup B \Longrightarrow I_{C=A\cup B} = I_A + I_B - I_{A\cap B} = I_A + I_B - (I_A . I_B) \Longrightarrow D(I_C) = D(I_A) + D(I_B) - D(I_A . I_B) = 0$ implies that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for all $x \in C^c$, so that $f_n(x) + g_n(x) \to f(x) + g(x)$ for all $x \in C^c$. Therefore $f_n + g_n \xrightarrow{a.e.} f + g$.

(3) Since $f_n \xrightarrow{a.e.} f$ then there exist $A = \{x \in \Omega: f_n(x) \neq f(x)\}$ such that $D(|I_A|) = D(I_A) = 0 \; and \; f_n(x) \to f(x) \forall x \notin A$, implies that

(4) $|f_n(x)| \to |f(x)| \; \forall x \notin A$. Therefore $|f_n| \xrightarrow{a.e.} |f|$.

Theorem 2.5

Let (Ω, L, D) be a Daniell space and let $f, f_n, g \in L$, $n \in \mathbb{N}$ such that $f_n \xrightarrow{a.e.} f$ then

(1) If $f_n \geq 0$ a.e. then $f \geq 0$ a.e.
(2) If $f_n \leq g$ a.e. for eah n then $f \leq g$ a.e.
(3) If $|f_n| \leq |g|$ a.e. then $|f| \leq |g|$a.e.
(4) If $f_n \leq f_{n+1}$ for each n, then $f_n \uparrow f$ a.e.

Proof:

Since $f_n \xrightarrow{a.e.} f$ then there is a set $A = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq f(x)\}$, $A \subseteq \Omega$.

such that $D(|I_A|) = D(I_A) = 0 \; and \; f_n(x) \to f(x) \forall x \notin A$.

(1) Since $f_n \geq 0$ a.e. then there exist $B_n = \{x \in \Omega: f_n(x) < 0\}, B_n \subset \Omega$, such that $D\left(I_{B_n}\right) = 0$ and $f_n(x) \geq 0$ for all $x \notin B_n$.

Let $C = A \cup (\bigcup_{n=1}^{\infty} B_n)$ and $I_{C=A\cup(\bigcup_n B_n)} = I_A + I_{\bigcup_n B_n} - I_{A\cap(\bigcup_n B_n)} = I_A + I_{\bigcup_{n=1}^{\infty} B_n} - \left(I_A . I_{\bigcup_{n=1}^{\infty} B_n}\right)$, then $D(I_C) = D(I_A) + D\left(I_{\bigcup_{n=1}^{\infty} B_n}\right) - D\left(I_A . I_{\bigcup_{n=1}^{\infty} B_n}\right) = 0$ implies hat $D(I_C) = 0$, then for any $x \notin C$

$f(x) = lim_{n\to\infty} g_n(x) \geq 0$ therefore $f \geq 0 \; a.e.$

(2) Since $f_n \leq g$ a.e. $\Longrightarrow g - f_n \geq 0 \; a.e.$ and since $f_n \xrightarrow{a.e.} f$ then $g - f_n \xrightarrow{a.e.} g - f$, by (1) $g - f \geq 0$ a.e. then $f \leq g$.

(3) Since $f_n \xrightarrow{a.e.} f$ then $|f_n| \xrightarrow{a.e.} |f|$ and since $|f_n| \leq |g|$ by (2) $|f| \leq |g|$a.e.

(4) Since $f_n \leq f_{n+1}$ a.e. for each n, then there exist $E_n = \{x \in \Omega: f_n(x) > f_{n+1}\}, E_n \subset \Omega$, such that $D\left(I_{E_n}\right) = 0$ and $f_n(x) \geq f_{n+1}(x)$ for all $x \notin E_n$.

Let $F = A \cup (\bigcup_{n=1}^{\infty} E_n)$, then $D(I_F) = 0$, and $f_n(x) \uparrow f(x)$ for all $x \notin F$ and $f_n(x) \to f(x)$ on A^c, therefore $f_n \uparrow f$ a.e.

Theorem 2.6

Let (Ω, L, D) be a Daniell space and let $f, f_n, g, g_n \in L, n \in \mathbb{N}$, then

(1) If $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$ and $f_n = g_n$ a.e. for all n, then $f = g$ a.e.
(2) If $f_n \xrightarrow{a.e.} f, f_n = g_n$ a.e. for all n, and $f = g\ a.e.$ then $g_n \xrightarrow{a.e.} g$.

Proof

(1) Since $f_n \xrightarrow{a.e.} f$ then there is a set $A = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq f(x)\}, A \subseteq \Omega$.

such that $D(|I_A|) = D(I_A) = 0\ and\ f_n(x) \to f(x) \forall x \notin A$, and $g_n \xrightarrow{a.e.} g$

then there is $B = \{x \in \Omega: lim_{n\to\infty} g_n(x) \neq g(x)\}$, $B \subseteq \Omega$ is a null set,

and $g_n(x) \to g(x) \forall x \notin B$, also $f_n = g_n$ a.e. $\Longrightarrow$ there exist

$C_n = \{x \in \Omega: f_n(x) \neq g_n(x)\}, C_n \subseteq \Omega$, which is a null set for all n and $f_n(x) = g_n(x)$ on ${C_n}^c$. Let $D = (A \cup B) \cup (\bigcup_{n=1}^{\infty} C_n)$ which is a null set, $f(x) = g(x) = lim_{n\to\infty} f_n(x) = lim_{n\to\infty} g_n(x) = g(x)$ for all $x \notin D$, so that $f(x) = g(x) for\ all\ x \notin D$. Therefore $f = g\ a.e.$

(2) Since $f_n \xrightarrow{a.e.} f$ then there is a set $A = \{x \in \Omega: lim_{n\to\infty} f_n(x) \neq f(x)\}, A \subseteq \Omega$. Such that $D(|I_A|) = D(I_A) = 0\ and\ f_n(x) \to f(x) \forall x \notin A$, and $f_n = g_n$ a.e. for all n $\Longrightarrow$ there exist $B_n = \{x \in \Omega: f_n(x) \neq g_n(x)\}, B_n \subseteq \Omega$,

which is a null set for all n and $f_n(x) = g_n(x)$ on ${B_n}^c$, also $f = g\ a.e \Longrightarrow$

there exist $C = \{x \in \Omega: f(x) \neq g(x)\}$ and $f(x) = g(x) on\ C^c$.

Let $D = A \cup C \cup (\bigcup_{n=1}^{\infty} B_n)$ which is a null set, $f(x) = g(x) = lim_{n\to\infty} f_n(x) = lim_{n\to\infty} g_n(x)$ for all $x \notin D$, there fore $g_n \to g$ on D^C. Therefore $g_n \xrightarrow{a.e.} g$ a.e.

Theorem 2.7

Let $\{f_n\}$ be sequence in L, if $(\lim_{n\to\infty} D(f_n)) < \infty$ then f_n converges a.e.

Proof: let $f(x) = \lim_{n\to\infty} f_n(x), f \in L \Longrightarrow D(f) = D(lim_{n\to\infty} f_n) = lim_{n\to\infty} D(f_n) < \infty$

Let $A = \{x \in \Omega: f(x) \neq lim_{n\to\infty} f_n(x)\}$ and since $f(x) = \lim_{n\to\infty} f_n(x)$ on A^c

There fore f_n converges.

3.Almost Uniformly Converence

Let (Ω, L, D) be a Daniell space. A sequence $\{f_n\}$ in L is said to be

(1) converges almost uniformly to the function $f \in L$, denoted by $f_n \xrightarrow{a.u.} f$, if there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c.

(2)$\{f_n\}$ Cauchy almost uniformly, denoted by f_n Cauchy a.u., if there is a null set $A \subseteq \Omega$ such that f_n Cauchy uniformly on A^c.

Theorem 3.1

Let (Ω, L, D) be a Daniell space and let $f, f_n, g, g_n \in L, n \in \mathbb{N}$, such that $f_n \xrightarrow{a.u.} f$, then

(1) Cauchy a.u.

(2) If $f_n \xrightarrow{a.u.} g$, then $f = g$ a.e.

(3) If $f = g$ a.e., then $f_n \xrightarrow{a.u.} g$

(4) If $f_n = g_n$ a.e. for all n, then $g_n \xrightarrow{a.u.} f$

(5) If $f_n = g_n$ $a.e.$ for all n and $f = g$ $a.e.$ then $g_n \xrightarrow{a.u.} g$

Proof:

(1) since $f_n \xrightarrow{a.u.} f$, then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c, thus f_n is uniformly Cauchy on A^c, therefore f_n Cauchy a.u.

(2) Since $f_n \xrightarrow{a.u.} f$, then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c, $f_n \xrightarrow{a.u.} g$ then there is a null set $B \subseteq \Omega$ such that $f_n \xrightarrow{u} g$ on B^c , $f_n(x) \to f(x)$ uniformly for any $x \notin A$ and $f_n(x) \to g(x)$

uniformly for any $x \notin B$. Let $C = A \cup B$ then C be a null set and

$f_n(x) \to f(x), f_n(x) \to g(x)$ uniformly for any $x \notin C$. Since C is a null set and $f(x) = g(x)$ for any $x \notin C$. Therefore $f = g$ a.e.

(3) Since $f_n \xrightarrow{a.u.} f$ then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c and since $f = g$ a.e. then there exist $B \subset \Omega$

which is a null set and $f(x) = g(x)$ for all $x \notin B$.

Let $C = A \cup B$ then C be a null set and $f_n(x) \to f(x) = g(x)$ for any $x \notin C$

and $f_n(x) \to g(x)$ uniformly. Therefore $f_n \xrightarrow{a.u.} g$.

(4) Since $f_n \xrightarrow{a.u.} f$ then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c and since $f_n = g_n$ a.e. for all n then there a sequence $B_n \subset \Omega$ and B_n be a null set for each n such that $f_n(x) = g_n(x)$ for all $x \in {B_n}^c$. Let $C = A \cup (\bigcup_{n=1}^{\infty} B_n)$ then C be a null set and since $g_n(x) = f_n(x) \to f(x)$ uniformly for any $x \notin C$. Therefore $g_n(x) \to f(x)$ uniformly for any $x \notin C$, thus $g_n \xrightarrow{a.u.} f$.

(5) Since $f_n \xrightarrow{a.u.} f$ then for any $\varepsilon > 0$ then there is a null set $A \subseteq \Omega$ such that

$f_n \xrightarrow{u} f$ on A^c and since $f_n = g_n$ $a.e.$ for all n then

there exist a sequence $B_n \subset \Omega$ and B_n be a null set for each n such that $f_n(x) = g_n(x)$ for all $x \in {B_n}^c$. And $f = g$ $a.e.$ then there exist $C \subset \Omega$

which is a null set and $f(x) = g(x)$ for all $x \notin C$, let $D = C \cup (\bigcup_{n=1}^{\infty} B_n)$

then D is a null set and $g_n(x) = f_n(x) \to f(x) = g(x)$ uniformly for any $x \notin D$. Therefore $g_n \xrightarrow{a.u.} g$.

Theorem 3.2

Let (Ω, L, D) be a Daniell space and let $f, f_n, g, g_n \in L, n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, such that $f_n \xrightarrow{a.u.} f$ and $g_n \xrightarrow{a.u.} g$, then

(1) $\lambda f_n \xrightarrow{a.u.} \lambda f$

(2) $f_n + g_n \xrightarrow{a.u.} f + g$

(3) $|f_n| \xrightarrow{a.u.} |f|$

Proof

(1) Since $f_n \xrightarrow{a.u.} f$ then there is a null set $A \subseteq \Omega$ such that and $f_n \xrightarrow{u} f$ on A^c this mean $f_n(x) \to f(x)$ uniformly for all $x \notin A$, then $\lambda f_n(x) \to \lambda f(x)$ uniformly for all $x \notin A$. Therefore $\lambda f_n \xrightarrow{a.u.} \lambda f$.

(2) Since $f_n \xrightarrow{a.u.} f$ then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c, $f_n(x) \to f(x)$ uniformly for all $x \notin A$ and since $g_n \xrightarrow{a.u.} g$ then there is a null set $B \subset \Omega$ such that $g_n \xrightarrow{u} g$ on B^c, $g_n(x) \to g(x)$ uniformly

for all $x \notin B$, let $C = A \cup B$ then C be a null set and

$f_n(x) + g_n(x) \to f(x) + g(x)$ uniformly for all $x \notin C$.

Therefore $f_n + g_n \xrightarrow{a.u.} f + g$

(3) Since $f_n \xrightarrow{a.u.} f$ then there is a null set $A \subseteq \Omega$ such that $f_n \xrightarrow{u} f$ on A^c, $f_n(x) \to f(x)$ uniformly for all $x \notin A$ then $|f_n(x)| \to |f(x)|$ uniformly for all $x \notin A$. Therefore $|f_n| \xrightarrow{a.u.} |f|$.

4.Convergence In Norm

Definition 4.1

Let (Ω, L, D) be a Daniell space. A norm on L is a function $\|\cdot\|: L \to \mathbb{R}$ which is defined by $\|f\| = D(|f|)$. The vector lattice L together with $\|\cdot\|$ is called a normed space in the Daniell space $(\Omega, L, D)$and is denoted by $(L, \|\cdot\|)$.

Remark

- $\|\cdot\|$ need not be norm since if $\|f\| = 0$ need not to be $f = 0$ only if $f = 0$ $a.e.$, that is $\|\cdot\|$ is a semi-norm but not a norm.
- the space of equivalent class in L is a normed space which is denoted by $(\mathcal{L}, \|\cdot\|)$ and $\|[f]\| = D(|f|)$.

Definition 4.2

Let (Ω, L, D) be a Daniell space and let $f, f_n \in L, n \in \mathbb{N}$, we say that

(1) f_n converges in norm to f, denoted by $f_n \overset{i.n.}{\rightarrow} f$, if $\|f_n - f\| \to 0$ as $n \to \infty$
(2) $\{f_n\}$ is a cauchy in norm , denoted by f_n Cauchy i.n., if
$\|f_n - f_m\| \to 0$ as $n, m \to \infty$.

Theorem 4.3

Let (Ω, L, D) be a Daniell space and let $f, f_n, g, g_n \in L, n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, such that $f_n \overset{i.n.}{\rightarrow} f$ and $g_n \overset{i.n.}{\rightarrow} g$, then

(1) f_n Cauchy a.u
(2) $\lambda f_n \overset{i.n.}{\rightarrow} \lambda f$
(3) $f_n + g_n \overset{i.n.}{\rightarrow} f + g$
(4) $|f_n| \overset{i.n.}{\rightarrow} |f|$
(5) $D(f_n) \overset{i.n.}{\rightarrow} D(f)$

Proof

(1) Since $f_n \overset{i.n.}{\rightarrow} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$ implies that f_n Cauchy sequence in norm.
(2) Since $f_n \overset{i.n.}{\rightarrow} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$,

since $\lambda\|f_n - f\| = \|\lambda f_n - \lambda f\| = D(|\lambda f_n - \lambda f|)\,) \to 0$ as $n \to \infty$.

therfore $\lambda f_n \overset{i.n.}{\rightarrow} \lambda f$

(3) Since $f_n \overset{i.n.}{\rightarrow} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$ and since $g_n \overset{i.n.}{\rightarrow} g$ then $\|g_n - g\| = D(|g_n - g|) \to 0$ as $n \to \infty$ therefore
$\|(f_n + g_n) - (f + g)\| = D(|(f_n + g_n) - (f + g)|)$
$= D(|(f_n - f) + (g_n - g)|) \leq D(|f_n - f|) + D(|g_n - g|) \to 0$ as $n \to \infty$,

then $\|(f_n + g_n) - (f + g)\| \to 0$ as $n \to \infty$. Therefore $f_n + g_n \overset{i.n.}{\rightarrow} f + g$

(4) Since $f_n \overset{i.n.}{\rightarrow} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$ then

$\||f_n| - |f|\| = D(\big||f_n| - |f|\big|) \leq D(|f_n - f|) \to 0$ as $n \to \infty$ then $\||f_n| - |f|\| \to 0$ as $n \to \infty$ Therefore $|f_n| \overset{i.n.}{\rightarrow} |f|$.

(5) Since $f_n \overset{i.n.}{\rightarrow} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$ then
$\|D(f_n) - D(f)\| = |D(f_n) - D(f)| = |D(f_n - f)| \leq D(|f_n - f|) \to 0$ as $n \to \infty$.
Therefore $D(f_n) \overset{i.n.}{\rightarrow} D(f)$.

Theorem 4.4

Let (Ω, L, D) be a Daniell space and let $f, f_n, g \in L$, if $f_n \overset{i.n.}{\rightarrow} f$ then $f_n \overset{i.n.}{\rightarrow} g$ if and only if $f = g$ a.e.

Proof

($\Longrightarrow$) Let $f_n \overset{i.n.}{\rightarrow} g$ and since $f_n \overset{i.n.}{\rightarrow} f$ then $f_n - f_n \overset{i.n.}{\rightarrow} f - g$ implies that $\|f - g\| = D(|f - g|) = D(|f_n - f_n - f + g|) \longrightarrow 0$ as $n \longrightarrow \infty \Longrightarrow D(|f - g|) = 0 \Longrightarrow f = g\ a.e.$

($\Longleftarrow$) Let $f = g$ a.e. then $\|f_n - g\| = D(|f_n - g|) = D(|f_n - g - f + f|) \leq D(|f_n - f|) + D(|f - g|) = D(|f_n - f|)) \longrightarrow 0\ as\ n \longrightarrow \infty$. Therefore $f_n \overset{i.n.}{\rightarrow} g$.

Theorem 4.5

Let (Ω, L, D) be a Daniell space and let $f \in L$ and $f = \lim_{n\to\infty} f_n$, then $f_n \overset{i.n.}{\rightarrow} f$

Proof:

Let $\varepsilon > 0$,since $f = \lim_{n\to\infty} f_n$ there is $k \in \mathbb{Z}^+$ such that $|f_n - f| < \varepsilon$ for all $n \geq k,$ then $D(|f_n - f|) < \varepsilon$ for all $n \geq k$. There fore $f_n \overset{i.n.}{\rightarrow} f$.

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