Some Properties of S-closed Fuzzy Submodules

Authors Names

[a]*Hassan K. Marhon*
[b]*Maysoun A. Hamel*

Publication data: 31 /8 /2023
Keywords: F-Module, Closed F-submodule, S-closed F-submodule, essential F-submodule.

ABSTRACT

In this paper we introduce and study S-closed F-submodules of a F-module μ as a generalization of notion closed F-submodules. We prove many basic properties about this concept.

1. Introduction

The notion of a F-subset of a nonempty set S as a function from S into [0,1], was first developed by Zadeh [1]. The concept of F-modules was introduced by Zahedi in [2]. The concept of F-submodules was introduce by Martines in [3]. Goodeal in [4], introduced and studied the concept of closed submodule, where a submodule A of an R-module M is said to be closed submodule of M ($A\leq_c M$), if has no proper essential extension. Hassan in [5] fuzzify this concept to closed F-submodule of a F-module μ.

In this paper, we introduce the concept of S-closed F-submodules which is stronger than closed F-submodule and we investigate the main properties of S-closed F-submodules such as the transitive property. Moreover we study the relationships between S-closed F-submodules and other submodules.

Throughout this paper R commutative ring with unity. M is an R-module and μ is a F-module of an R-module M. Finally,(shortly fuzzy set, fuzzy submodule and fuzzy module is F-set, F-submodule and F- module).

2. Preliminaries

In this section, the concepts of F-sets and operations on F-sets are given, with some properties, which are useful in the next work.

Definition 1.1 [1]: Let M be a non-empty set and let I be the closed interval [0,1] of the real line (real numbers). An F-set μ in M (a F-subset μ of M) is a function from M into I ."

Definition 1.2 [2]: Let $x_t: M \longrightarrow I$ be a F-set in M, where $x \in M$, $t \in [0,1]$, defined by:

$$x_t = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \quad \text{for all } y \in M$$

Then x_t is called a F-singleton.

If x = 0 and t = 1 then $0_1(y) = \begin{cases} 1 & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}$

We shall call such F-singleton the F-zero singleton.

Definition 1.3[2]: Let ν and ρ be two F-sets in M, then :

1.$\nu = \rho$ if and only if $\nu(x) = \rho(x)$,for all $x \in M$.

2. $\nu \subseteq \rho$ if and only if $\nu(x) \leq \rho(x)$, for all $x \in M$.

If $\nu \subseteq \rho$ and there exists $x \in M$ such that $\nu(x) < \rho(x)$, then we write $\nu \subset \rho$ and ν is called a proper F-subset of ρ .

3. $x_t \subseteq \nu$ if and only if $x_t(y) \leq \nu(y)$, for all $y \in M$ and if $t > 0$, then $\nu(x) \geq t$. Thus $x_t \subseteq \nu$ ($x \in \nu_t$), (that is $x \in \nu_t$ iff $x_t \subseteq \nu$)."

Next, we give some operation on F-sets:

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[a]*Ministry of Education, Rusafa1, Baghdad,Iraq, E-mail: hassanmath316@gmail.com*
[b]*University of Baghdad, College of Education for pure Sciences, Ibn-Al-Haitham, Baghdad,Ira , E-mail: maysoon.a.h@ihcodu.uobaghdad.edu.iq*

"Definition 1.4 [2]: Let ν and ρ be two F-sets in M, then:
1.$(\nu \cup \rho)(x) = \max\{\nu(x), \rho(x)\}$, for all x ∈ M.
2.$(\nu \cap \rho)(x) = \min\{\nu(x), \rho(x)\}$, for all x ∈ M.
ν ∪ ρ and ν ∩ ρ are F-sets in M
In general if $\{\nu_\alpha, \alpha \in \Lambda\}$, is ν family of F-sets in M, then:

$$\left(\bigcap_{\alpha\in\Lambda} \nu_\alpha\right)(x) = \inf\{\nu_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in S.$$

$(\bigcup_{\alpha\in\Lambda} \nu_\alpha)(x) = \sup\{\nu_\alpha(x), \alpha \in \Lambda\}$, for all x ∈ S."

Now, we give the definition of the level subset, which is a set between a F-set and ordinary set.
"Definition 1.5 [3]: Let ν be a F-set in M, for all t ∈ [0,1], the set $\nu_t = \{x \in M, \nu(x) \geq t\}$ is called a level subset of ν ".

Note that, ν_t is a subset of M in the ordinary sense. The following are some properties of level subset:
"Remark 1.6 [1]: Let ν and ρ be two F-subsets of a set M, then:
1. $(\nu \cap \rho)_t = \nu_t \cap \rho_t$ for any t ∈ [0,1] .
2.$(\nu \cup \rho)_t = \nu_t \cup \rho_t$ for any t ∈ [0,1] .
3. $\nu = \rho$ if and only if $\nu_t = \rho_t$, for all t ∈ [0,1]."
"Definition 1.7 [2]: Let M be an R-module. A F-set μ of M is called F-module of an R-module if :
1. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, for all x, y ∈ M.
2. $\mu(rx) \geq \mu(x)$, for all x ∈ M, r ∈ R.
3. μ (0) = 1, (0 is the zero element of M)."
"Definition 1.8 [3]:
Let μ, ν be two F-module of an R-module M. ν is called a F-submodule of μ if ν ⊆ μ."
"Definition 1.9 [2]: If ν is F-submodule of an R-module M, then the submodule ν_t of M is called level submodule of M, where t ∈ [0,1]."
"Definition1.10 [2]: Let ν, ρ be two F-subsets of an R-module M. Then $(\nu + \rho)(x) = \sup\{\min\{\nu(a), \rho(b), x = a + b\}\}$ a, b ∈ M, for all x ∈ M. ν + ρ is a F-subset of M ."
"Proposition 1.11 [2]: Let ν and ρ be two F-submodule of a F-module μ, then ν + +ρ is a F-submodule of μ ."
"Definition 1.12 [7]: Let μ and γ be two F-modules of an R-module M_1 and M_2 respectively. Define $\mu \oplus \gamma : M_1 \oplus M_2 \longrightarrow [0,1]$ by:
$(\mu\oplus\gamma)(a, b) = \min\{\mu(a), \gamma(b)$, for all (a, b) ∈ $M_1 \oplus M_2\}$.
$\mu\oplus\gamma$ is called a F-external direct sum of μ and γ .
If ν and ρ are F-submodules of μ, γ respectively. Define $\nu \oplus \rho : M_1 \oplus M_2 \longrightarrow [0,1]$ by :

$$(\nu\oplus\rho)(a, b) = \min\{\nu(a), \rho(b), \text{for all } (a, b) \in M_1\oplus M_2\}.$$

Note that, if μ = ν + ρ and ν ∩ ρ = 0, then μ is called the internal direct sum of ν and ρ which is denoted by ν ⊕ ρ. Moreover, ν and ρ are called direct summand of μ ."
"Definition 1.13 [7]: Let ν be a F-submodule of F-module μ of an R-module M, then ν is called an essential F-submodule (briefly $\nu \leq_e \mu$), if $\nu \cap \rho \neq 0_1$, for any non-trivial F-submodule ρ of μ. Equivalent a F-submodule ν of μ is called essential if $\nu \cap \eta = 0_1$, implies to $\eta = 0_1$, for all F-submodule η of μ."
"Definition 1.14 [2]: Suppose that ν and ρ be two F-submodules of a F-module of μ an R-module M. The residual quotient of ν and ρ. We define (ν: ρ) by: $(\nu: \rho) = \{r_t : r_t$ is a F-singleton of R such that $r_t\rho \subseteq \nu\}$."
"Definition 1.15 [7]: Let μ be non-empty F-module of an R-module M. The F-annihilator of ν denoted by (F-$ann_R\nu$) is defined by $\{x_t : x \in R, x_t\nu \subseteq 0_1\}$ t ∈ [0,1], where ν is a proper F-submodule of μ."
"Definition 1.16 [7]:

A F-module μ of an R-module M is said to be prime F-module if F-ann ν = F-annμ for any non-trivial F-submodule ν of μ."

"Definition 1.15: [8] Let ν be a F-module of an R-module M, if ρ a F-submodule of ν, then ρ is called a semi-essential F-submodule of ν if for all prime F-submodule η of ν and $\rho \cap \eta = 0_1$, then $\eta = 0_1$."

"Definition 1.19:[7] Let μ be a F-module of an R-module M is called fully prime F-module, if every proper F-submodule of μ is prime F-submodule."

3. S-Closed F-Submodules

In this section, we introduce the notion of S-closed F-submodule of a F-module as a generalization of (ordinary) notion S-closed submodule, where a submodule N of M is called S-closed submodule in M (briefly $N \leq_{Sc} M$), if has no proper semi-essential in M[9]. We shall give some properties of this concept.

Definition (2.1): Let ν be a F-submodule of a fuzzy module of μ. ν is called S-closed F-submodule (briefly $\nu \leq_{Sc} \mu$) , if has no proper semi-essential F-submodule of μ.

Proposition (2.2): Let ν be a F-submodule of a F-module μ, if ν_t is S-closed submodule of μ_t, ∀ t ∈ [0,1], then ν is S-closed submodule of μ.

Proof: Suppose there exists $\rho \leq \mu$, such that ν is a semi-essential of ρ, then ν_t semi-essential of ρ_t, ∀ t ∈ [0,1]. But ν_t is S-closed submodule of μ_t ∀ t, so that $\nu_t = \rho_t$, hence by Remark (1.6)(3), $\nu = \rho$. Thus ν is S-closed submodule of μ .

Remarks and Examples (2.3):

1. Every F-module μ is an S-closed F-submodule it self.
2. 0_1 may be not S-closed F-submodule of μ , for example:

Example:

Let M = Z_2 as Z-module. Let μ: M ⟶ [0,1], define by:
μ (a) = 1, ∀ a ∈ Z_2.

Note that $(0_1)_t$ = (0), for all t ∈ [0,1]. But (0) is a semi-essential in $Z_2 = \mu_t$, thus 0_1 is semi-essential F-submodule of μ [8, Proposition(3.4)] and so 0_1 is not S-closed F-submodule.

3. A direct summand of S-closed F-submodule is not necessary S-closed F-submodule of μ, for example:

Let M = Z_{12} as Z-module and let μ : M ⟶[0,1] such that:
μ (a) = 1, for all a ∈ [0,1]

And $\nu(x) = \begin{cases} 1 & if \quad x \in \langle\bar{3}\rangle \\ 0 & otherwise \end{cases}$

It is clear that ν is a F-submodule of μ and $\nu_t = \langle\bar{3}\rangle$. ν_t is a direct summand of $\mu_t = Z_{12}$, ∀ t ∈ [0,1]. Since ν_t is semi-essential [9, Remarks and Examples (3.1.2.)(6). A is not S-closed [See Proposition (2.2)]

4. Let μ be a F-module of an R-module M and ν is an S-closed F-submodule in μ and $\rho \leq \mu$ such that $\nu \cong \rho$, then it is not necessary that ρ is an S-closed F-submodule of μ , for example:

Example: Let M be Z as Z-module. Let μ : M ⟶[0,1], such that: μ (a) = 1, ∀ a ∈ Z.

And ν : M ⟶[0,1], define by : $\nu(x) = \begin{cases} 1 & if \quad x \in 3Z \\ 0 & otherwise \end{cases}$

It is clear that ν is a F-submodule of μ and $\nu \cong \mu$. But ν is not S-closed F-submodule since $\nu \leq_e \mu$ (and $\nu \leq_{sem} \mu$). Moreover μ is S-closed in μ.

Remarks (2.4):

1. Let ν be a F-submodule of a F-module μ of an R-module M. If ν is an essential F-submodule in μ, then ν is a semi-essential F-submodule in μ.

Proof: It is easy, so it omitted.

"Recall that a F-submodule ν of μ of an R-module M ,then ν is called closed F-submodule of μ (shortly $\nu \leq_c \mu$), if ν has no proper essential extension; that is $\nu \leq_e \rho \leq \mu$, then $\nu = \rho$" [5].

2. Every S-closed F-submodule of a F-module μ is a closed F-submodule of μ.

Proof: Let ν be S-closed F-submodule of μ and let $\rho \leq \mu$, with $\nu \leq_e \rho \leq \mu$, the by part (1), ν is a semi-essential in ρ. But ν S-closed F-submodule in μ imply $\nu = \rho$. That is ν is a closed F-submodule.

Lemma (2.5): Let μ be a F-module of an R-module M with a F-submodules ν and ρ where $\nu \leq \rho \leq \mu$. If ν is a prime F-submodule in μ and ρ is a prime F-submodule in μ, then ν is a prime F-submodule in ρ.

Proof: Let $x_t \subseteq \rho$, $r_t \subseteq R$ such that $(rx)_t \subseteq \nu$. To show that ν is a prime F-submodule of ρ, we must prove that either Let $x_t \subseteq \nu$ or $r_t \subseteq (\nu:\rho)$. Suppose that $r_t \subseteq (\nu:\mu)$, (i.e $r_t\mu \subseteq \nu$). But $\rho \subseteq \mu$, so $r_t\rho \subseteq \nu$. Hence $r_t \subseteq (\nu:\rho)$. Thus ν is a prime F-submodule in ρ .

If $x_t \subseteq \nu$, then nothing to prove.

Proposition (2.6): Let μ be fully prime F-module of an R-module M and let $0_1 \neq \nu \leq \mu$. Then ν is a semi-essential F-submodule of ρ if and only if ν is an essential F-submodule of ρ, for every F-submodule ρ of μ .

Proof: ($\Rightarrow$) Assume that ν is a semi-essential F-submodule of ρ and let β be a F-submodule of ρ such that $\nu \cap \rho = 0_1$. We have two cases if $\rho = \beta$, then $\nu = 0_1$ which is a contradiction. Otherwise since μ is a fully prime F-module, then β is a prime F-submodule of μ and by Lemma (2.5) ν is a prime F-submodule of ρ. But ν is a semi-essential F-submodule of ρ, therefore $\beta = 0_1$. That is ν is an essential F-submodule of ρ.

($\Leftarrow$) It is clear.

Proposition (2.7): Let ν, ρ be a F-submodules of μ of an R-module M and $0_1 \neq \nu \leq \rho \leq \mu$. If ν is a semi-essential ρ and ρ is a semi-essential in μ, then ν is a semi-essential in μ.

Proof: Let η be a prime F-submodule of μ such that $\nu \cap \eta = 0_1$. Note that $0_1 = \nu \cap \eta = (\nu \cap \eta) \cap \rho = \nu \cap (\eta \cap \rho)$. But η is a prime F-submodule of μ, so we have two cases. If $\rho \leq \eta$, then $0_1 = \nu \cap (\eta \cap \rho) = \nu \cap \rho$, hence $\nu \cap \rho = 0_1$, but $\nu \subseteq \rho$, so $\nu \cap \rho = \nu$, implies that $\nu = 0_1$ which is a contradiction with our assumptions. Thus $\rho \nsubseteq \eta$ and by [10, Lemma (2.3.11)], $\eta \cap \rho$ is a prime F-submodule of ρ. But ν is a semi-essential F-submodule of ρ. Therefore $\eta \cap \rho = 0_1$ and since ρ is a semi-essential F-submodule of μ. That is ν is a semi-essential F-submodule of μ.

Proposition (2.8): Let μ be a F-module of an R-module M and let $0_1 \neq \nu \leq \mu$, then there exists an S-closed F-submodule ρ in μ such that ν is a semi-essential F-submodule of ρ.

Proof: Consider the set S = $\{\eta | \eta$ is a F-submodule of μ such that ν is a semi-essential F-submodule of $\eta\}$. It is clear that S is a non-empty By [10, Proposition (2.2.13)], S has maximal element say ρ. In order to prove that ρ is an S-closed F-submodule in μ assume that there exists a F-submodule γ of μ such that ρ is a semi-essential F-submodule in γ, where γ a F-submodule of μ. Since ν is a semi-essential F-submodule in ρ and ρ is a semi-essential F-submodule in γ, so by Proposition (2.7), ν is a semi-essential F-submodule in γ. But this is a contradiction with the maximality. Thus $\rho = \gamma$, that is ρ is an S-closed F-submodule of μ with ν is a semi-essential F-submodule of ρ.

The following results show that the transitive property for S-closed F-submodule certain.

Proposition (2.9): Let ν, ρ be a F-submodules of μ of an R-module M. If $\nu \leq_{Sc} \rho$ and $\rho \leq_{Sc} \mu$, then $\nu \leq_{Sc} \mu$, provided that ρ contained in any semi-essential F-submodule of ν.

Proof: Let $\eta \leq \mu$ such that ν is a semi-essential F-submodule of η, where η F-submodule in μ. By assumption, we have two cases. If $\eta \leq \rho$. Since $\nu \leq_{Sc} \rho$, then $\nu = \eta$ and since ν semi-essential F-submodule of η, so by Proposition (2.7), ρ is a semi-essential F-submodule in η. But $\rho \leq_{Sc}, \mu$ then $\rho = \eta$. That is ν is a semi-essential F-submodule in ρ other hand $\nu \leq_{Sc} \rho$, so $\nu = \rho$, hence $\nu \leq_{Sc} \mu$.

"Recall that a F-module μ is a chained, if for each F-submodule ν *and* ρ of μ either $\nu \leq \rho$ or $\rho \leq \nu$ [11]".

Corollary (2.10): Let μ be a chained F-module and ν, ρ be a F-submodules of μ such that $\nu \leq \rho \leq \mu$. If $\nu \leq_{Sc} \rho$ and $\rho \leq_{Sc} \mu$, then $\nu \leq_{Sc} \mu$.

Proof: Let $\eta \leq \mu$ such that ν is a semi-essential F-submodule of η, where η F-submodule in μ . Since μ is a chained F-module, then either $\eta \leq \rho$ or $\rho \leq \eta$ and the result follows.

As the same argument which used in the proof of Proposition (2.9).

"Recall that a non-zero R-module M is called fully essential, if every non-zero semi-essential submodule of M is an essential submodule of M"[12].

We fuzzify the following:

Definition (2.11): A F-module μ is called fully essential F-module, if every a non-trivial semi-essential F-submodule of μ is an essential F-submodule of μ.

Proposition (2.12): Let ν be a non-trivial closed F-submodule of a F-module μ. If every semi-essential of ν is fully essential F-submodule of μ, then ν is an S-closed F-submodule of μ.

Proof: Let ν be a non-trivial and closed F-submodule of μ and let ν is a semi-essential F-submodule of ρ, where ρ is F-submodule of μ. By assumption ν is a fully essential F-submodule. Therefore ν is an essential F-submodule of ρ. But ν is a closed F-submodule in μ, that is $\nu = \rho$. Thus ν is an S-closed F-submodule of μ.

Remark (2.13): If μ is a fully prime F-module, then every non-trivial closed F-submodule in μ is an S-closed F-submodule in μ.

Proof: Let ν be a non-trivial closed F-submodule in μ and let ν is a semi-essential F-submodule in ρ, where ρ a F-submodule in μ, then by Proposition (2.6), ν is an essential F-submodule of ρ. But ν is closed F-submodule of μ, then $\nu = \rho$ and we are done.

We can replace the condition by other conditions to get the transitive property of S-closed F-submodule.

Proposition (2.14): Let μ be a F-module and $0_1 \neq \nu \leq \rho \leq \mu$. Assume that every semi-essential F-submodule of ν is fully essential F-submodule of μ. If $\nu \leq_{sc} \rho$ and $\rho \leq_{stc}, \mu$ then $\nu \leq_{sc}. \mu$

Proof: Since $\nu \leq_{sc} \rho$ and $\rho \leq_{sc} \mu$, then by Remarks and Examples (2.3)(5), ν is a closed F-submodule in ρ and ρ is a closed F-submodule in μ, imply ν is a closed F-submodule in μ (see [10, Proposition (2.1.20)]). Therefore $\nu \leq_{sc} \mu$ by Proposition (2.12)

The following Remark shows that the property of S-closed F-submodule is not hereditary.

Remark (2.15): Let ν, ρ be a F-submodules of μ such that$\nu \leq \rho \leq \mu$. If $\rho \leq_{sc} \mu$, then ν need not S-closed F-submodule in μ, for example:

Let M = Z as Z-module. Let $\mu : \longrightarrow[0,1]$, defined by:

μ (a) = 1, for all a ∈ Z. Let ν : M $\longrightarrow[0,1]$, such that: $\nu(a) = \begin{cases} 1 & if \quad a \in 2Z \\ 0 & otherwise \end{cases}$

It is clear that ν F-submodule of μ and μ is S-closed. But ν is an essential F-submodule of μ (so ν is a semi-essential F-submodule in μ), hence ν is not S-closed F-submodule in μ.

Proposition (2.16):

If every F-submodule of a F-module μ is an S-closed F-submodule, then every F-submodule of μ is a direct summand of μ.

Proof: Since every F-submodule of μ is an S-closed F-submodule, then by Remarks (2.4)(2), every S-closed F-submodule is a closed F-submodule, so every F-submodule is a direct summand. Hence the result follows from by [5, Remarks and Examples (3.3)(2)].

Proposition (2.17): Let ν, ρ be a F-submodules of μ such that $\nu \leq \rho \leq \mu$. If ν is an S-closed F-submodule in μ, then ν is an S-closed F-submodule in ρ.

Proof: Suppose that ν is a semi-essential F-submodule in η and η F-submodule of ρ (where $\rho \leq \mu$). But ν is an S-closed F-submodule in μ, then $\nu = \eta$.

Corollary (2.18): If ν, ρ be S-closed F-submodules of μ, then ν *and* ρ are S-closed F-submodules in $\nu + \rho$.

Proof: Since $\nu \leq \nu + \rho \leq \mu$, so by Proposition (2.17), we are done.

In the following Proposition we see that the converse of Proposition (2.2) is true when the condition (*) holds, where condition (*):Let μ be a F-module of an R-module M and ν, ρ are a non-trivial F-submodules of μ , if $\nu_* \subseteq \rho_*$, then $\nu \subseteq \rho$.

Proposition (2.19): Let μ be a F-module of an R-module M and ν be a F-submodule of μ. If ν is an S-closed F-submodule of μ if and only if ν_* is an S-closed submodule in μ_*, provided that μ satisfies condition (*).

Proof: (⇒) Suppose that $\nu_* \leq_{sem} N \leq \mu_*$. We have to show that $\nu_* = N$. Let

$$\rho(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$

It is clear that ρ is a F-submodule of μ and $\rho_* = N$, hence $\nu_* \subseteq \rho_* = N$ and so by condition $(*)$, $\nu \subseteq \rho$. But $\nu_* \leq_{sem} N = \rho_*$, then by [8, Proposition(3.4)], $\nu \leq_{sem} \rho$, since ν is an S-closed F-submodule in μ, therefore $\nu = \rho$, so $\nu_* = \rho_* = N$, thus $\nu_* = N$.

(⇐) It clear by Proposition (2.2).

Proposition (2.20): Let μ, γ be F-modules of an R-module M_1, M_2 respectively and let annν_* +ann$\gamma_* = R$ and every submodules of $\nu_* \oplus \gamma_*$ fully essential module, if ν, ρ be S-closed F-submodules of μ, γ respectively, then $\nu \oplus \rho$ is an S-closed F-submodule in $\mu \oplus \gamma$.

Proof: Since ν and ρ are an S-closed F-submodules, then μ_* and ρ_* are an S-closed submodules by Proposition (2.19) we have $\nu_* \oplus \rho_*$ is an S-closed submodule in $\mu_* \oplus \gamma_*$ see [9, Proposition (3.1.17)], so $(\nu \oplus \rho)_*$ is an S-closed submodule in $(\mu \oplus \gamma)_*$ therefore $\nu \oplus \rho$ is an S-closed F-submodule in $\mu \oplus \gamma$ see Proposition (2.19).

"Recall that a F-submodule ν of μ is called y-closed F-submodule, if μ/ν is non-singular F-module, where a F-module μ of an R-module M such that : Z(μ) = {$x_t \subseteq \mu$: F-ann(x_t) is an essential F-ideal of R} is called F-singular submodule of μ. If Z(μ) = μ, then μ is called singular F-module. If Z(μ) = 0_1, then μ is called non-singular F-module.

The following proposition show that under certain condition S-closed F-submodules implies y-closed F-submodules.

Proposition (2.21): Let μ be a non-singular F-module of an R-module M, if a F-submodule ν of μ is an $\mathrm{S-closed}$, then ν is a $\mathrm{y-closed\ F-submodule}$.

Proof: Let ν be an S-closed F-submodule in μ, by Remarks (2.4)(2), ν is a closed F-submodule in ν. But μ is a non-singular F-module, so by [10, Proposition (2.5.9)], ν is a y-closed F-submodule.

Proposition (2.22): Let μ be a fully prime F-module of an R-module M and let ν be a non-trivial F-submodule of μ. Consider the following statements.

1. ν is a y-closed F-sub module.
2. ν is a closed F-submodule.
3. ν is an S-closed F-submodule.

Then (1)$\Rightarrow$ (2) $\Leftrightarrow$ (3), and if μ is a non-singular F-module, then (3) $\Rightarrow$ (1).

Proof: $\Rightarrow$ (2) It follows by [10, Remarks and Examples (2.5.8)(3).

(1) $\Leftrightarrow$ (3) Since μ is a fully prime F-module, then by Remark (2.13), ν is an S-closed F-submodule. The converse is clear.

(2) $\Rightarrow$ (1) Since μ is a non-singular F-module, then by Proposition (2.21), ν is a y-closed F-submodule.

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