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On Quadruple Sequences Spaces $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ of fuzzy numbers Described by Double Young Functions Using Fuzzy Metric

Authors Names	**ABSTRACT**
Aqeel Mohammed Hussein ***Publication data***: 31 /8 /2023 ***Keywords:*** Metric space, completeness , quadruple sequences spaces , young function , double young functions , fuzzy metric.	In this paper, we introduce the quadruple sequences spaces $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ of fuzzy numbers defined by double young functions using fuzzy metric and discuss some properties like the space $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ is a metric space , the space $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ is complete metric space , and discuss other properties .

1. Introduction

In the year 1965, L.A. Zadeh established the concept of fuzzy set theory by Yu-ru [10], Tripathy and Baruah ([1], [2]), Tripathy and Borgohain ([3], Tripathy and Dutta ([4], [5]), Tripathy and Sarma ([7],[8],[9]), and many others. By generalizing the concept of the probabilistic metric space to the fuzzy situation, Kramosil and Michalek [6] established the fuzzy metric space . In this work, we offer and define the space $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ of fuzzy numbers determined by the double young functions using fuzzy metric.

2. Definitions and Preliminaries

If $\Omega : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing, and convex with $\Omega(0) = 0, \Omega(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\Omega(\mathfrak{A}) \to \infty$ as $\mathfrak{A} \to \infty$ then Ω is an Orlicz function.

If $\mathcal{H}: [0,\infty) \to [0,\infty) \ni \mathcal{H}(\mathfrak{A}) = \frac{\Omega(\mathfrak{A})}{\mathfrak{A}}$, $\mathfrak{A} \succ 0$ and $\mathcal{H}(0) = 0, \mathcal{H}(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\mathcal{H}(\mathfrak{A}) \to 0$ as $\mathfrak{A} \to \infty$ then $\mathcal{H}$ is a young function.

A double young function is a function $\mathbb{M} : [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty) \ni \mathbb{M}(\mathfrak{A},\mathfrak{S}) = (\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S}))$, where $\mathbb{M}_1 : [0,\infty) \to [0,\infty) \ni \mathbb{M}_1(\mathfrak{A}) = \frac{\Omega_1(\mathfrak{A})}{\mathfrak{A}}, \mathfrak{A} \succ 0$ and $\mathbb{M}_2 : [0,\infty) \to [0,\infty) \ni \mathbb{M}_2(\mathfrak{S}) = \frac{\Omega_2(\mathfrak{S})}{\mathfrak{S}}, \mathfrak{S} \succ 0$.These functions are non-decreasing , continuous , even , convex , and satisfy the following condition:

i) $\mathbb{M}_1(0) = 0, \mathbb{M}_2(0) = 0 \Longrightarrow \mathbb{M}(0,0) = (\mathbb{M}_1(0), \mathbb{M}_2(0)) = (0,0)$

ii) $\mathbb{M}_1(\mathfrak{A}) \succ 0, \mathbb{M}_2(\mathfrak{S}) \succ 0 \Longrightarrow \mathbb{M}(\mathfrak{A},\mathfrak{S}) = (\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S})) \succ (0,0)$, for $\mathfrak{A} \succ 0, \mathfrak{S} \succ 0$,
we mean by $(\mathfrak{A},\mathfrak{S}) \succ (0,0)$ implies that $\mathbb{M}_1(\mathfrak{A}) \succ 0, \mathbb{M}_2(\mathfrak{S}) \succ 0$.

iii) $\mathbb{M}_1(\mathfrak{A}) \to 0, \mathbb{M}_2(\mathfrak{S}) \to 0$ as $\mathfrak{A} \to \infty, \mathfrak{S} \to \infty$, then $\mathbb{M}(\mathfrak{A},\mathfrak{S}) = (\mathbb{M}_1(\mathfrak{A}), \mathbb{M}_2(\mathfrak{S})) \to (0,0)$ as $(\mathfrak{A},\mathfrak{S}) \to (\infty,\infty)$,we mean by $\mathbb{M}(\mathfrak{A},\mathfrak{S}) \to (0,0)$ as $\mathbb{M}_1(\mathfrak{N}) \to 0, \mathbb{M}_2(\mathfrak{M}) \to 0$

Let $\mathbb{T},\mathbb{S}: [0,1] \times [0,1] \to [0,1]$ be non-decreasing , symmetric, and satisfies $\mathbb{T}[0,0] = 0$ and $\mathbb{S}[1,1] = 1$, where $\mathbb{T} = \min\{\mathfrak{p},\mathfrak{q}\}$ and $\mathbb{S} = \max\{\mathfrak{p},\mathfrak{q}\}$, where $\mathfrak{p},\mathfrak{q} \in [0,1]$.

Assume $\mathcal{T}: \mathbb{R}(\mathbb{I}) \times \mathbb{R}(\mathbb{I}) \to \mathbb{R} \ni \mathcal{T}(\mathbb{X},\mathbb{Y}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes)$,and $\mathcal{T}_\ltimes: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ni \mathcal{T}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes) = \min\{|\mathbb{X}_1^\ltimes - \mathbb{Y}_1^\ltimes|, |\mathbb{X}_2^\ltimes - \mathbb{Y}_2^\ltimes|\}$.

Suppose $\mathcal{S}: \mathbb{R}(\mathbb{I}) \times \mathbb{R}(\mathbb{I}) \to \mathbb{R} \ni \mathcal{S}(\mathbb{X},\mathbb{Y}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{S}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes)$ and $\mathcal{S}_\ltimes: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ni \mathcal{S}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes) = \max\{|\mathbb{X}_1^\ltimes - \mathbb{Y}_1^\ltimes|, |\mathbb{X}_2^\ltimes - \mathbb{Y}_2^\ltimes|\}$.

The distance between the fuzzy numbers $\mathbb{X},\mathbb{Y}$ which are symbolized by $d_{\mathbb{F}}$ has the following definition $d_{\mathbb{F}}(\mathbb{X},\mathbb{Y}) = [\mathcal{T}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes), \mathcal{S}_\ltimes(\mathbb{X}^\ltimes,\mathbb{Y}^\ltimes)], 0 \leqslant\ltimes\leqslant 1$. The quadruple $(\mathbb{R}(\mathbb{I}), d_{\mathbb{F}}, \mathbb{T}, \mathbb{S})$ be a referred to as fuzzy metric space and $d_{\mathbb{F}}$ is a fuzzy metric if :

i) $d_{\mathbb{F}}(\mathbb{X},\mathbb{Y}) = 0$ iff $\mathbb{X} = \mathbb{Y}$, for everybody $\mathbb{X},\mathbb{Y} \in \mathbb{R}(\mathbb{I})$.

University of Al-Qadisiyah , Faculty of Education, Department of Mathematics, Iraq. E-mail: aqeel.Hussein@qu.edu.iq

ii) $d_{\mathbb{F}}(\mathbb{X},\mathbb{Y}) = d_{\mathbb{F}}(\mathbb{Y},\mathbb{X})$, for everybody $\mathbb{X},\mathbb{Y} \in \mathbb{R}(\mathbb{I})$.

iii) $\forall\ \mathbb{X},\mathbb{Y},\mathbb{Z} \in \mathbb{R}(\mathbb{I})$,

a) $d_{\mathbb{F}}(\mathbb{X},\mathbb{Y})(\mathbb{s}+\mathbb{r}) \succcurlyeq \mathbb{T}\big(d_{\mathbb{F}}(\mathbb{X},\mathbb{Z})(\mathbb{s}), d_{\mathbb{F}}(\mathbb{Z},\mathbb{Y})(\mathbb{r})\big)$, whenever $\mathbb{s} \preccurlyeq \mathcal{T}_1(\mathbb{X},\mathbb{Z})$, $\mathbb{r} \preccurlyeq \mathcal{T}_1(\mathbb{Z},\mathbb{Y})$and $\mathbb{s}+\mathbb{r} \preccurlyeq \mathcal{T}_1(\mathbb{X},\mathbb{Y})$.

b) $d_{\mathbb{F}}(\mathbb{X},\mathbb{Y})(\mathbb{s}+\mathbb{r}) \preccurlyeq \mathbb{T}\big(d_{\mathbb{F}}(\mathbb{X},\mathbb{Z})(\mathbb{s}), d_{\mathbb{F}}(\mathbb{Z},\mathbb{Y})(\mathbb{r})\big)$, whenever $\mathbb{s} \succcurlyeq \mathcal{T}_1(\mathbb{X},\mathbb{Z})$, $\mathbb{r} \succcurlyeq \mathcal{T}_1(\mathbb{Z},\mathbb{Y})$ and $\mathbb{s}+\mathbb{r} \succcurlyeq \mathcal{T}_1(\mathbb{X},\mathbb{Y})$.

If it is satisfies the following conditions:

1. $\mathbb{F}$ is a convex if for each $\mathbb{F}(\mathbb{r}_2) \succcurlyeq \mathbb{F}(\mathbb{r}_1) \wedge \mathbb{F}(\mathbb{r}_3) = \min\{\mathbb{F}(\mathbb{r}_1), \mathbb{F}(\mathbb{r}_3)\}$, $\forall \mathbb{r}_1 \prec \mathbb{r}_2 \prec \mathbb{r}_3, \forall\ \mathbb{r}_1,\mathbb{r}_2,\mathbb{r}_3 \in \mathbb{R}$.
2. $\mathbb{F}$ is normal if there is a $\mathbb{r}_0 \in \mathbb{R}$ and $\mathbb{F}(\mathbb{r}_0) = 1$.
3. $\mathbb{F}$ is upper-semi-continuous $\forall\ \mathbb{a} \in \mathbb{I}, \forall\ \varepsilon \succ 0$ and $\mathbb{F}^{-1}([0, \mathbb{a}+\varepsilon))$ is open in the usual topology of $\mathbb{R}$
4. $\mathbb{F}$ is a non-negative fuzzy number $\forall\ \mathbb{r} \prec 0$ implies $\mathbb{F}(\mathbb{r}) = 0$ then $\mathbb{F} : \mathbb{R} \to [0,1]$ is a fuzzy real number .

In this study, we introduce and define these space as follows:

$$m(\mathbb{M},\varphi)_{\mathbb{F}}^4 = \Big\{(\mathfrak{X}_{abce}) = ((\mathfrak{X}_1)_{abce}, (\mathfrak{X}_2)_{abce}), (\mathfrak{X}_{abce}) \in \mathbb{W}_{\mathbb{F}}^4 : \sup_{\mathfrak{s},r,i,j\succcurlyeq 1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}((\mathfrak{X}_1)_{abce},\bar{0})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{T}((\mathfrak{X}_2)_{abce},\bar{0})}{\rho}\right)\right] \prec (\infty,\infty) \text{ and } \sup_{\mathfrak{s},r,i,j\succcurlyeq 1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{S}((\mathfrak{X}_1)_{abce},\bar{0})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{S}((\mathfrak{X}_2)_{abce},\bar{0})}{\rho}\right)\right] \prec (\infty,\infty)\Big\}$$

, for some $\rho \succ 0$, where $\mathbb{M} = (\mathbb{M}_1, \mathbb{M}_2)$.

3. Main Results

Theorem 3.1: $m(\mathbb{M},\varphi)_{\mathbb{F}}^4$ is a metric space by the metric :

$$\bar{d}(\mathfrak{A},\mathfrak{U}) = \inf\Big\{(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{s},r,i,j\succcurlyeq 1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})}{\rho}\right)\right] \preccurlyeq (1,1) \text{ and } \sup_{\mathfrak{s},r,i,j\succcurlyeq 1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})}{\rho}\right)\right] \preccurlyeq (1,1)\Big\}, \forall\ \mathfrak{A},\mathfrak{U} \in m(\mathbb{M},\varphi)_{\mathbb{F}}^4 .$$

Proof:

Let $\mathfrak{A}, \mathfrak{U}, \mathcal{U} \in m(\mathbb{M},\varphi)_{\mathbb{F}}^4$

i) If $\bar{d}(\mathfrak{A},\mathfrak{U})_{\mathbb{M}} = 0$ implies that $\mathcal{T}\big(((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})\big) = 0, \mathcal{T}\big(((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})\big) = 0$, and $\mathcal{S}\big(((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})\big) = 0, \mathcal{S}\big(((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})\big) = 0, (\mathbb{M}_1(0) = 0, \mathbb{M}_2(0) = 0)$

This leads that , $\forall \ltimes \in (0,1]$,

$$\sup_{0\preccurlyeq\ltimes\preccurlyeq 1} \mathcal{T}_{\ltimes}((\mathfrak{A}_1)_{abce}^{\ltimes},(\mathfrak{U}_1)_{abce}^{\ltimes}) = 0 \Rightarrow \mathcal{T}_{\ltimes}((\mathfrak{A}_1)_{abce}^{\ltimes},(\mathfrak{U}_1)_{abce}^{\ltimes}) = 0, \forall \ltimes \in (0,1]$$

$$\Rightarrow \min\{|(\mathfrak{A}_1)_{abce1}^{\ltimes} - (\mathfrak{U}_1)_{abce1}^{\ltimes}|, |(\mathfrak{A}_1)_{abce2}^{\ltimes} - (\mathfrak{U}_1)_{abce2}^{\ltimes}|\} = 0, \forall \ltimes \in (0,1] \ldots (1$$

$$\sup_{0\preccurlyeq\ltimes\preccurlyeq 1} \mathcal{T}_{\ltimes}((\mathfrak{A}_2)_{abce}^{\ltimes},(\mathfrak{U}_2)_{abce}^{\ltimes}) = 0 \Rightarrow \mathcal{T}_{\ltimes}((\mathfrak{A}_2)_{abce}^{\ltimes},(\mathfrak{U}_2)_{abce}^{\ltimes}) = 0, \forall \ltimes \in (0,1]$$

$$\Rightarrow \min\{|(\mathfrak{A}_2)_{abce1}^{\ltimes} - (\mathfrak{U}_2)_{abce1}^{\ltimes}|, |(\mathfrak{A}_2)_{abce2}^{\ltimes} - (\mathfrak{U}_2)_{abce2}^{\ltimes}|\} = 0, \forall \ltimes \in (0,1] \ldots (2$$

Similarly , $\forall \ltimes \in (0,1]$,

$\sup_{0\leqslant\ltimes\leqslant1} \mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0 \Longrightarrow \mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0, \forall \ltimes \in (0,1]$

$\Longrightarrow \max\{|(\mathfrak{A}_1)^{\ltimes}_{abce1} - (\mathfrak{U}_1)^{\ltimes}_{abce1}|, |(\mathfrak{A}_1)^{\ltimes}_{abce2} - (\mathfrak{U}_1)^{\ltimes}_{abce2}|\} = 0, \forall \ltimes \in (0,1] \ldots$ (3

$\sup_{0\leqslant\ltimes\leqslant1} \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0 \Longrightarrow \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0, \forall \ltimes \in (0,1]$

$\Longrightarrow \max\{|(\mathfrak{A}_2)^{\ltimes}_{abce1} - (\mathfrak{U}_2)^{\ltimes}_{abce1}|, |(\mathfrak{A}_2)^{\ltimes}_{abce2} - (\mathfrak{U}_2)^{\ltimes}_{abce2}|\} = 0, \forall \ltimes \in (0,1] \ldots$ (4

From (1) and (2) and (3) and (4), it follows that, $\mathfrak{A}_{abce} = \mathfrak{U}_{abce} \Longrightarrow \mathfrak{A} = \mathfrak{U}, \forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{e} \in \mathbb{N}$, where ($\mathbb{N}$ is the natural numbers) .

Conversely, assume that, $\mathfrak{A} = \mathfrak{U}$. Then, using the definition of $\mathcal{T}$ and $\mathcal{S}$, we obtain ,

$\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0, \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0$ and

$\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0, \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0, \forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{e} \in \mathbb{N}, \forall \ltimes \in (0,1]$.

This means that,

$\sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0$, $\sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0$ and $\sup_{0\leqslant\ltimes\leqslant1} \mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) = 0$, $\sup_{0\leqslant\ltimes\leqslant1} \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) = 0$. $\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{e} \in \mathbb{N}$. Consequently,

$\mathcal{T}((\mathfrak{A}_1)_{abce}, (\mathfrak{U}_1)_{abce}) = 0, \mathcal{T}((\mathfrak{A}_2)_{abce}, (\mathfrak{U}_2)_{abce}) = 0$ and

$\mathcal{S}((\mathfrak{A}_1)_{abce}, (\mathfrak{U}_1)_{abce}) = 0, \mathcal{S}((\mathfrak{A}_2)_{abce}, (\mathfrak{U}_2)_{abce}) = 0$.

By the continuity of $\mathbb{M}$, we obtain , $\bar{d}(\mathfrak{A}, \mathfrak{U})_{\mathbb{M}} = 0$. Therefore $\bar{d}(\mathfrak{A}, \mathfrak{U})_{\mathbb{M}} = 0 \Leftrightarrow \mathfrak{A} = \mathfrak{U}$.

ii) $\bar{d}(\mathfrak{A}, \mathfrak{U})_{\mathbb{M}} =$

$\inf\Big\{(\rho,\rho) \succ (0,0) :$

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})}{\rho}\right)\right] \preccurlyeq$ (1,1) and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})}{\rho}\right)\right] \preccurlyeq (1,1) \Big\}$.

From the definition of $\mathcal{T}$, we get,

$\mathcal{T}((\mathfrak{A}_1)_{abce}, (\mathfrak{U}_1)_{abce}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abce}, (\mathfrak{U}_1)^{\ltimes}_{abce}) =$
$\sup_{0\leqslant\ltimes\leqslant1} (\min\{|(\mathfrak{A}_1)^{\ltimes}_{abce1} - (\mathfrak{U}_1)^{\ltimes}_{abce1}|, |(\mathfrak{A}_1)^{\ltimes}_{abce2} - (\mathfrak{U}_1)^{\ltimes}_{abce2}|\}) =$
$\sup_{0\leqslant\ltimes\leqslant1} (\min\{|(\mathfrak{U}_1)^{\ltimes}_{abce1} - (\mathfrak{A}_1)^{\ltimes}_{abce1}|, |(\mathfrak{U}_1)^{\ltimes}_{abce2} - (\mathfrak{A}_1)^{\ltimes}_{abce2}|\}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{U}_1)^{\ltimes}_{abce}, (\mathfrak{A}_1)^{\ltimes}_{abce}) =$
$\mathcal{T}((\mathfrak{U}_1)_{abce}, (\mathfrak{A}_1)_{abce})$.

$\mathcal{T}((\mathfrak{A}_2)_{abce}, (\mathfrak{U}_2)_{abce}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abce}, (\mathfrak{U}_2)^{\ltimes}_{abce}) =$
$\sup_{0\leqslant\ltimes\leqslant1} (\min\{|(\mathfrak{A}_2)^{\ltimes}_{abce1} - (\mathfrak{U}_2)^{\ltimes}_{abce1}|, |(\mathfrak{A}_2)^{\ltimes}_{abce2} - (\mathfrak{U}_2)^{\ltimes}_{abce2}|\}) =$
$\sup_{0\leqslant\ltimes\leqslant1} (\min\{|(\mathfrak{U}_2)^{\ltimes}_{abce1} - (\mathfrak{A}_2)^{\ltimes}_{abce1}|, |(\mathfrak{U}_2)^{\ltimes}_{abce2} - (\mathfrak{A}_2)^{\ltimes}_{abce2}|\}) = \sup_{0\leqslant\ltimes\leqslant1} \mathcal{T}_{\ltimes}((\mathfrak{U}_2)^{\ltimes}_{abce}, (\mathfrak{A}_2)^{\ltimes}_{abce}) =$
$\mathcal{T}((\mathfrak{U}_2)_{abce}, (\mathfrak{A}_2)_{abce})$.

Continuing in the same way, we have,

$\mathcal{S}((\mathfrak{A}_1)_{abce}, (\mathfrak{U}_1)_{abce}) = \mathcal{S}((\mathfrak{U}_1)_{abce}, (\mathfrak{A}_1)_{abce})$, $\mathcal{S}((\mathfrak{A}_2)_{abce}, (\mathfrak{U}_2)_{abce}) = \mathcal{S}((\mathfrak{U}_2)_{abce}, (\mathfrak{A}_2)_{abce})$.

Then we arrive that,

$\bar{d}(\mathfrak{A}, \mathfrak{U})_{\mathbb{M}} = \inf\Big\{(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant1,\sigma\in\mathfrak{Y}_{srij}} \frac{1}{\varphi_{srij}} \sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abce},(\mathfrak{U}_1)_{abce})}{\rho}\right) \curlyvee \mathbb{M}_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abce},(\mathfrak{U}_2)_{abce})}{\rho}\right)\right] \preccurlyeq$

$(1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big)\curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big]\preccurlyeq(1,1)\Big\}$

$=\inf\Big\{(\rho,\rho)\succ(0,0):\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{U}_1)_{\mathfrak{abce}},(\mathfrak{A}_1)_{\mathfrak{abce}})}{\rho}\Big)\curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{U}_2)_{\mathfrak{abce}},(\mathfrak{A}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big]\preccurlyeq$
$(1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{U}_1)_{\mathfrak{abce}},(\mathfrak{A}_1)_{\mathfrak{abce}})}{\rho}\Big)\curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{U}_2)_{\mathfrak{abce}},(\mathfrak{A}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big]\preccurlyeq(1,1)\Big\}=\bar{d}(\mathfrak{U},\mathfrak{A})_{\mathbb{M}}$. Consequently $\bar{d}(\mathfrak{A},\mathfrak{U})_{\mathbb{M}}=\bar{d}(\mathfrak{U},\mathfrak{A})_{\mathbb{M}}$.

iii) Let's assume $\rho_1,\rho_2\succ 0$, then

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big]\preccurlyeq(1,1),$

and

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big]\preccurlyeq(1,1).$

Assume $\rho=\rho_1+\rho_2$, then we have

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big]\preccurlyeq$

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\Big(\frac{\mathcal{T}(((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}}),((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}}))}{\rho_1+\rho_2}\Big)+$
$\Big(\frac{\mathcal{T}(((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}}),((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}}))}{\rho_1+\rho_2}\Big)\Big)\curlyvee\mathbb{M}_2\Big(\Big(\frac{\mathcal{T}(((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}}),((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}}))}{\rho_1+\rho_2}\Big)+$
$\Big(\frac{\mathcal{T}(((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}}),((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}}))}{\rho_1+\rho_2}\Big)\Big)\Big]\preccurlyeq$

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}(((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}}),((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}}))}{\rho_1}\Big)+$
$\Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}(((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}}),((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}}))}{\rho_2}\Big)\Big)\curlyvee\mathbb{M}_2\Big(\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}(((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}}),((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}}))}{\rho_1}\Big)+$
$\Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}(((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}}),((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}}))}{\rho_2}\Big)\Big)\Big]\preccurlyeq$

$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}}\frac{\rho_1}{\rho_1+\rho_2} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big]+$
$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}}\frac{\rho_2}{\rho_1+\rho_2} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big]\preccurlyeq$
$(1,1).$

Since the ρ's are non-negative , so taking the infimum of such ρ's, we get

$\inf\Big\{(\rho,\rho)\succ(0,0):$
$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big)\curlyvee\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big]\preccurlyeq$
$(1,1)\Big\}\preccurlyeq$

$\inf\Big\{(\rho_1,\rho_1)\succ(0,0):\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big)\curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big]\preccurlyeq(1,1)\Big\}+$

$\inf\Big\{(\rho_2,\rho_2) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big] \preccurlyeq (1,1)\Big\}$.

Continuing in the same way, we obtain that,

$\inf\Big\{(\rho,\rho) \succ (0,0) :$
$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big] \preccurlyeq$
$(1,1)\Big\} \preccurlyeq$
$\inf\Big\{(\rho_1,\rho_1) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big] \preccurlyeq (1,1)\Big\} +$
$\inf\Big\{(\rho_2,\rho_2) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big] \preccurlyeq (1,1)\Big\}$.

Moreover, we have

$\inf\Big\{(\rho,\rho) \succ (0,0) :$
$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big] \preccurlyeq$
$(1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho}\Big)\Big] \preccurlyeq (1,1)\Big\} \preccurlyeq$
$\inf\Big\{(\rho_1,\rho_1) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big] \preccurlyeq$
$(1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abce}},(\mathcal{U}_1)_{\mathfrak{abce}})}{\rho_1}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abce}},(\mathcal{U}_2)_{\mathfrak{abce}})}{\rho_1}\Big)\Big] \preccurlyeq (1,1)\Big\} +$
$\inf\Big\{(\rho_2,\rho_2) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{T}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big] \preccurlyeq$
$(1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathcal{U}_1)_{\mathfrak{abce}},(\mathfrak{U}_1)_{\mathfrak{abce}})}{\rho_2}\Big) \curlyvee$
$\mathbb{M}_2\Big(\frac{\mathcal{S}((\mathcal{U}_2)_{\mathfrak{abce}},(\mathfrak{U}_2)_{\mathfrak{abce}})}{\rho_2}\Big)\Big] \preccurlyeq (1,1)\Big\} \Rightarrow \bar{d}(\mathfrak{A},\mathfrak{U})_{\mathbb{M}} \preccurlyeq \bar{d}(\mathfrak{A},\mathcal{U})_{\mathbb{M}} + \bar{d}(\mathcal{U},\mathfrak{U})_{\mathbb{M}}$.

Thus, $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ is a metric space.

Theorem 3.2: $m(\mathbb{M},\varphi)^4_{\mathbb{F}}$ is complete metric space by the metric :

$\bar{d}(\mathbb{X},\mathbb{Y})_{\mathbb{M}} = \inf\Big\{(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{T}((\mathbb{X}_1)_{\mathfrak{abce}},\bar{0})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathcal{T}((\mathbb{X}_2)_{\mathfrak{abce}},\bar{0})}{\rho}\Big)\Big] \preccurlyeq (1,1)$ and $\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Big[\mathbb{M}_1\Big(\frac{\mathcal{S}((\mathbb{X}_1)_{\mathfrak{abce}},\bar{0})}{\rho}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathcal{S}((\mathbb{X}_2)_{\mathfrak{abce}},\bar{0})}{\rho}\Big)\Big] \preccurlyeq (1,1)\Big\}, \forall\, \mathbb{X},\mathbb{Y} \in m(\mathbb{M},\varphi)^4_{\mathbb{F}}$.

<u>Proof:</u>

Assume that $(\mathbb{X}^{(ij\ell k)})$ is a Cauchy quadruple sequence in $m(\mathbb{M},\varphi)^4_{\mathbb{F}} \ni \mathbb{X}^{(ij\ell k)} = \big(\mathbb{X}^{(ij\ell k)}_{wvut}\big)^{\infty}_{w,v,u,t=1}$.

Let $\varepsilon \succ 0$. For a fixed exist $x_0 \succ 0$, choose $\mathfrak{p} \succ 0 \ni \Big[\mathbb{M}_1\Big(\frac{\mathfrak{p}x_0}{2}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathfrak{p}x_0}{2}\Big)\Big] \succcurlyeq (1,1)$. Then a positive integer exists $n_0 = n_0(\varepsilon) \ni \bar{d}\big(\mathbb{X}^{(ij\ell k)},\mathbb{X}^{(gfed)}\big)_{\mathbb{M}} \prec \Big(\frac{\varepsilon}{\mathfrak{p}x_0},\frac{\varepsilon}{\mathfrak{p}x_0}\Big), \forall\, i,j,\ell,k,g,f,e,d \geqslant n_0$.

By the definition of $\bar{d}(\mathbb{X},\mathbb{Y})_{\mathbb{M}}$, we obtain :

$$\inf\Big\{(\rho,\rho) \succ (0,0) : \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq (1,1) \text{ and } \sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{S}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{S}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq (1,1)\Big\} \prec \varepsilon, \forall\, i,j,\ell,k,g,f,e,d \geqslant n_0 \qquad (1)$$

It follows that ,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq (1,1) \qquad (2)$$

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{S}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{S}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq (1,1) \qquad (3)$$

From (2) , we have ,

$$\sup_{\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}} \frac{1}{\varphi_{\mathfrak{srij}}} \sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq (1,1)\ldots \qquad (4)$$

On taking $\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j} = 1$ and varying σ over $\mathfrak{Y}_{\mathfrak{srij}}$, we arrive that ,

$$\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\rho}\Bigg)\Bigg] \preccurlyeq \varphi_{1111}, \forall\, i,j,\ell,k,g,f,e,d \geqslant n_0,$$

$$\Bigg[\mathbb{M}_1\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_1)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_1)^{(gfed)}_{\mathfrak{abce}}\big)}{\bar{d}\big(\mathbb{X}^{(ij\ell k)}_1,\mathbb{X}^{(gfed)}_1\big)}\Bigg) \curlyvee \mathbb{M}_2\Bigg(\frac{\mathcal{T}\big((\mathbb{X}_2)^{(ij\ell k)}_{\mathfrak{abce}},\ (\mathbb{X}_2)^{(gfed)}_{\mathfrak{abce}}\big)}{\bar{d}\big(\mathbb{X}^{(ij\ell k)}_2,\mathbb{X}^{(gfed)}_2\big)}\Bigg)\Bigg] \preccurlyeq \varphi_{1111} \preccurlyeq \Big[\mathbb{M}_1\Big(\frac{\mathfrak{p}x_0}{2}\Big) \curlyvee \mathbb{M}_2\Big(\frac{\mathfrak{p}x_0}{2}\Big)\Big].$$

By the continuity of $\mathbb{M}$, we obtain ,

$\mathcal{T}_{\ltimes}\left(\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)},\ (\mathbb{X}_1)_{\mathfrak{abce}}^{(gfed)}\right),\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)},\ (\mathbb{X}_2)_{\mathfrak{abce}}^{(gfed)}\right)\right)_{\mathbb{M}} \prec \left(\frac{\mathfrak{p}x_0}{2},\frac{\mathfrak{p}x_0}{2}\right).\left(\frac{\varepsilon}{\mathfrak{p}x_0},\frac{\varepsilon}{\mathfrak{p}x_0}\right)=\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)$.

Therefore $\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)}\right),\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)}\right)$ are Cauchy quadruple sequence in $\mathbb{R}(\mathbb{I})$,so is convergent in $\mathbb{R}(\mathbb{I})$ by the completeness property of $\mathbb{R}(\mathbb{I})$.

Assume $\lim_{ij\ell k}(\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)}=(\mathbb{X}_1)_{\mathfrak{abce}}$ and $\lim_{ij\ell k}(\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)}=(\mathbb{X}_2)_{\mathfrak{abce}}$, $\forall\ \mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{e}\in\mathbb{N}$.

We must to prove that,

$\lim_{ij\ell k}(\mathbb{X}_1)^{(ij\ell k)}=\mathbb{X}_1$ and $\lim_{ij\ell k}(\mathbb{X}_2)^{(ij\ell k)}=\mathbb{X}_2$, $\forall\ \mathbb{X}_1,\mathbb{X}_1\in m(\mathbb{M},\varphi)_{\mathbb{F}}^4$.

Since $\mathbb{M}$ is a continuous , so on taking $g,f,e,d\longrightarrow\infty$ and fixing i,j,ℓ,k .

From (2) ,we obtain,

$\sup_{\mathfrak{s,r,i,j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}}\frac{1}{\varphi_{\mathfrak{srij}}}\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_1)_{\mathfrak{abce}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_2)_{\mathfrak{abce}}\right)}{\rho}\right)\right]\preccurlyeq$ (1,1) for some $\rho\succ 0$ and $\forall\ i,j,\ell,k\succcurlyeq n_0$.

Continuing in the same way, from (2) we obtain,

$\sup_{\mathfrak{s,r,i,j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}}\frac{1}{\varphi_{\mathfrak{srij}}}\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{S}\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_1)_{\mathfrak{abce}}\right)}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{S}\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_2)_{\mathfrak{abce}}\right)}{\rho}\right)\right]\preccurlyeq$ (1,1) for some $\rho\succ 0$ and $\forall\ i,j,\ell,k\succcurlyeq n_0$.

Now on taking the infimum of such ρ's, we obtain ,

$\inf\left\{(\rho,\rho)\succ(0,0):\ \sup_{\mathfrak{s,r,i,j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}}\frac{1}{\varphi_{\mathfrak{srij}}}\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_1)_{\mathfrak{abce}}\right)}{\rho}\right)\curlyvee\right.\right.$

$\left.\mathbb{M}_2\left(\frac{\mathcal{T}\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_2)_{\mathfrak{abce}}\right)}{\rho}\right)\right]\preccurlyeq$

(1,1) and $\sup_{\mathfrak{s,r,i,j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}}\frac{1}{\varphi_{\mathfrak{srij}}}\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{S}\left((\mathbb{X}_1)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_1)_{\mathfrak{abce}}\right)}{\rho}\right)\curlyvee\right.$

$\left.\left.\mathbb{M}_2\left(\frac{\mathcal{S}\left((\mathbb{X}_2)_{\mathfrak{abce}}^{(ij\ell k)},(\mathbb{X}_2)_{\mathfrak{abce}}\right)}{\rho}\right)\right]\preccurlyeq(1,1)\right\}\prec(\varepsilon,\varepsilon)$, $\forall\ i,j,\ell,k\succcurlyeq n_0$.

Which demonstrates that,

$\bar{d}\left(\left((\mathbb{X}_1)^{(ij\ell k)},\mathbb{X}_1\right),\left((\mathbb{X}_2)^{(ij\ell k)},\mathbb{X}_2\right)\right)_{\mathbb{M}}\prec(\varepsilon,\varepsilon),\forall\ i,j,\ell,k\succcurlyeq n_0$.

That is $\lim_{ij\ell k}(\mathbb{X}_1)^{(ij\ell k)}=\mathbb{X}_1$ and $\lim_{ij\ell k}(\mathbb{X}_2)^{(ij\ell k)}=\mathbb{X}_2$.

Now, to prove that $\mathbb{X}_1,\mathbb{X}_2\in m(\mathbb{M},\varphi)_{\mathbb{F}}^4$. We have,

$\bar{d}\left((\mathbb{X}_1,\bar{\theta}),(\mathbb{X}_2,\bar{\theta})\right)_{\mathbb{M}}\preccurlyeq$

$\bar{d}\left(\left(\mathbb{X}_1,(\mathbb{X}_1)^{(ij\ell k)}\right),\left(\mathbb{X}_2,(\mathbb{X}_2)^{(ij\ell k)}\right)\right)_{\mathbb{M}}+\bar{d}\left(\left((\mathbb{X}_1)^{(ij\ell k)},\bar{\theta}\right),\left((\mathbb{X}_1)^{(ij\ell k)},\bar{\theta}\right)\right)_{\mathbb{M}}\prec(\varepsilon,\varepsilon)+(\mathbb{M}_1,\mathbb{M}_1),\forall\ i,j,\ell,k\succcurlyeq n_0(\varepsilon)$.

That is $\bar{d}\left((\mathbb{X}_1,\bar{\theta}),(\mathbb{X}_2,\bar{\theta})\right)_{\mathbb{M}}$ is a finite.

Consequently $\mathbb{X}_1,\mathbb{X}_2\in m(\mathbb{M},\varphi)_{\mathbb{F}}^4$. Thus, $m(\mathbb{M},\varphi)_{\mathbb{F}}^4$ is complete metric space.

Proposition 3.3: $m(\mathbb{M},\varphi)_{\mathbb{F}}^4\subseteq m(\mathbb{M},\varphi,p)_{\mathbb{F}}^4$, $\forall\ 1\preccurlyeq p\prec\infty$.

Proof: Let $\mathbb{X}=(\mathbb{X}_{\mathfrak{abce}})=((\mathbb{X}_1)_{\mathfrak{abce}},(\mathbb{X}_2)_{\mathfrak{abce}})\in m(\mathbb{M},\varphi)_{\mathbb{F}}^4$, then we have, for some $\rho\succ 0$,

$\sup_{\mathfrak{s,r,i,j}\geqslant 1,\sigma\in\mathfrak{Y}_{\mathfrak{srij}}}\frac{1}{\varphi_{\mathfrak{srij}}}\sum_{\mathfrak{a}\in\sigma}\sum_{\mathfrak{b}\in\sigma}\sum_{\mathfrak{c}\in\sigma}\sum_{\mathfrak{e}\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}((\mathbb{X}_1)_{\mathfrak{abce}}\ ,\ \bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}((\mathbb{X}_2)_{\mathfrak{abce}}\ ,\ \bar{0})}{\rho}\right)\right]=\mathbb{K}\prec\infty.$

Hence, for fixed $\mathfrak{s},\mathfrak{r},\mathfrak{i},\mathfrak{j}$, we have

$$\sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left[\mathbb{M}_1\left(\frac{\mathcal{T}(\,(\mathbb{X}_1)_{abce}\;,\;\bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}(\,(\mathbb{X}_2)_{abce}\;,\;\bar{0})}{\rho}\right)\right]\preccurlyeq\mathbb{K}\,\varphi_{srij}\,,\forall\;\sigma\in\mathfrak{Y}_{srij}\,.$$

$$\left[\sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left(\mathbb{M}_1\left(\frac{\mathcal{T}(\,(\mathbb{X}_1)_{abce}\;,\;\bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}(\,(\mathbb{X}_2)_{abce}\;,\;\bar{0})}{\rho}\right)\right)^{\wp}\right]^{\frac{1}{\wp}}\preccurlyeq\mathbb{K}\varphi_{srij}\,,\forall\,\sigma\in\mathfrak{Y}_{srij}$$

$$\sup_{s,r,i,j\geqslant1,\sigma\in\mathfrak{Y}_{srij}}\frac{1}{\varphi_{srij}}\left[\sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left(\mathbb{M}_1\left(\frac{\mathcal{T}(\,(\mathbb{X}_1)_{abce}\;,\;\bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}(\,(\mathbb{X}_2)_{abce}\;,\;\bar{0})}{\rho}\right)\right)^{\wp}\right]^{\frac{1}{\wp}}\preccurlyeq\mathbb{K}\,.$$

$$\sup_{s,r,i,j\geqslant1,\sigma\in\mathfrak{Y}_{srij}}\frac{1}{\varphi_{srij}}\left[\sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left(\mathbb{M}_1\left(\frac{\mathcal{T}(\,(\mathbb{X}_1)_{abce}\;,\;\bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{T}(\,(\mathbb{X}_2)_{abce}\;,\;\bar{0})}{\rho}\right)\right)^{\wp}\right]^{\frac{1}{\wp}}\prec\infty\,.$$

Continuity in the same way , we get ,

$$\sup_{s,r,i,j\geqslant1,\sigma\in\mathfrak{Y}_{srij}}\frac{1}{\varphi_{srij}}\left[\sum_{a\in\sigma}\sum_{b\in\sigma}\sum_{c\in\sigma}\sum_{e\in\sigma}\left(\mathbb{M}_1\left(\frac{\mathcal{S}(\,(\mathbb{X}_1)_{abce}\;,\;\bar{0})}{\rho}\right)\curlyvee\mathbb{M}_2\left(\frac{\mathcal{S}(\,(\mathbb{X}_2)_{abce}\;,\;\bar{0})}{\rho}\right)\right)^{\wp}\right]^{\frac{1}{\wp}}\prec\infty\,.$$

Therefore $\mathbb{X}=(\mathbb{X}_{abce})=((\mathbb{X}_1)_{abce}\,,(\mathbb{X}_2)_{abce})\in m(\mathbb{M},\varphi,\wp)^4_{\mathbb{F}}\,,\forall\;1\preccurlyeq\wp\prec\infty$

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