Weak Equality Co-Neighbourhood Domination in Graphs

Authors Names

Sahib Sh. Kahat[1]
Hayder H. Mohamme[2]

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ABSTRACT

When $G = (V, E)$ be undirected simple and connected graph. And $D \subset V$ is seed to be weak equality co-neighborhood dominating set of G dented by symbol (WENDS), if the following condition that (every vertex in the set D it is associated with the same number of vertices in the remaining set $(V - D)$ such that degree of vertex (v) $\forall v \in D$ is lessthan the degree of vertex (u) $\forall u \in D$ is achieved. In this paper, we defined WENDS in graphs, and determined new properties of WENDS for some types of graphs.

1.Introduction

Due to the importance of graph theory in various aspects of science and life, its study has become important and widely [6,8,20]. And it has branched into several branches, the most important of which is dominance. Research in this branch has varied, including in the domination polynomial as in [5,10,11,12]. including new definitions in dominance after adding some conditions to it with some properties of it as in [1,2,3,4,7] and [16,17,18,19], applying it, and linking it with other branches of mathematics such as fuzzy [15] and topology as in [9]. And others as weak and strong research presented carries the condition of equal dominance, which is the weakest, as the degree of each head in the dominant group D is less than the degree of each head in the remaining group $V - D$ [4,7]. Also from other branches such as Chromatic polynomials as in [17], and others. Berge is the first person who introduced the domination in graphs [6]. In this research, we presented a study on weak dominance in the new definition of dominance over equal neighbours [21] Structure

Definition.1.1[21].

In $G(V, E)$ be a undirected simple graph, a subset $D \subset V$ is called equally co-neighborhood dominating set of G dented by symbol (ENDS), if every vertex $v \in D$ is adjacent to equally number of vertices in $V - D$. The set D is called minimal $ENDS(\ MENDS)$ if it has no proper subset ENDS. The equally domination number denoted by simplicity $\gamma_{enw\ (ally\ co-neighborhood\ dominating\ s\ twaphs}(G)$ is the minimum cardinality of a $MENDS$. The $MENDS$ of cardinality γ_{en} is called γ_{en}- set.

2. Results for some certain graphs in the WENDS

Definition 2.1.

Consider D ⊂ V is an equality co-neighborhood dominating set of the graph G, it is said to be weak equality co-neighborhood dominating set of G dented by symbol (WENDS), if the following

Sahib Sh. Kahat[1], Department of Professional Supervision, The General Director of Education Al-Najaf Al-Ashraf, Najaf, Iraq, sgehet@gmail.com
Hayder H. Mohammed[2] , Department of Professional Supervision, The General Director of Education Al-Najaf Al-Ashraf, Najaf, Iraq, heyderhassoon95@gmail.com

condition that, $degree(v) < degree(u)$ $\forall v \in D$ and $\forall u \in V - D$. $\gamma_{en}^{w}(G)$ it is mean the weak co-equally domination number. (see Fig .1(b))

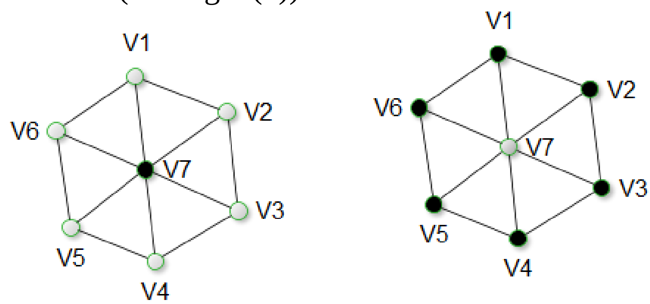

Figure 1: (a) $\gamma_{en}(G)$ (b) $\gamma_{en}^{w}(G)$

Theorem 2.2. Let G_n be a connected graph with order n and let D= γ_{en}^{w}-set, then

1. $v \in D$, for every pendant vertex $v \in G$ such that $n \geq 3$.
2. $v \notin D$, if degree(v)=n-1 $\forall$ $n \geq 2$
3. $\gamma_{en}^{w}(G) \neq 1$
4. $2 \leq \gamma_{en}^{w}(G) \leq n - 1$.
5. $\gamma_{en}^{w}(G) \geq \gamma_{en}(G)$.
6. G is not necessary has WENDS, if G it has ENDS.

Proof.

1. Let D be a $\gamma_{en}^{w}(G) -$ set, and v be pendant vertex in G ($degree(v) = 1$). in connected graph when $n \geq 3$, then exist vertex $u \in G$ such that $degree(u) \geq 2$, then $degree(u) > degree(v)$ therefore v in D, by definition of ENDS and definition of WENDS.
2. let $v \in V(G)$ and $deg(v) = n - 1$, then $deg(u) \leq deg(v)$ $\forall u \in V(G)$, then $v \notin D$, according to definition of weak.
3. If $\gamma_{en}^{w}(G) = 1$, then exist vertex $v \in V(G)$ such that $\{v\} = D$, and it is adjacent to all vertices in V(G), hence degree(v)=n-1, this is contradiction 2. $v \notin D$, therefore $\gamma_{en}^{w}(G) \neq 1$.
4. The lower bound occurs by (3) and if $G = P_3$ and upper bound occurs when $G = S_n$.
5. If $G_{4,5}$ be bipartite graph, then $\gamma_{en}(G_{4,5}) = 3$ but $\gamma_{en}^{w}(G_{4,5}) = 5$, then $\gamma_{en}^{w}(G) > \gamma_{en}(G)$ and by (2) therefore $\gamma_{en}^{w}(G) \geq \gamma_{en}(G)$. (For example see Fig .2)
6. It is straightforward if $G = K_n$, we have $\gamma_{en}^{w}(G) = 1$ and since degree(v)=n-1 $\forall v \in K_n$, then G has no WENDS according to 2 and 3

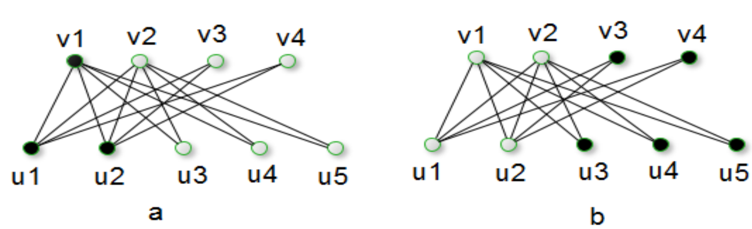

Figure 2: (a) $\gamma_{en}(G) = 3$ (b) $\gamma_{en}^{w}(G) = 5$

Proposition 2.3. For wheel W_n and star S_n graphs, then

1. $\gamma_{en}^{w}(S_n) = n - 1, \forall n \geq 3$
2. $\gamma_{en}^{w}(W_n) = n - 1, \forall n \geq 3$.

Proof.

1. It is straightforward from Proposition **2.2.**(1,2).
2. Let v be centre of wheel W_n such that $n \geq 3$, then $\text{degree}(v) > \text{degree}(u)$ and $\text{degree}(u) = 3$, $\forall u \in V - \{v\}$, in this case only the set $V - \{v\}$ is WENDS, therefore $\gamma_{en}^{w}(W_n) = n - 1$.

Theorem 2.4. The r-regular graph G_n of order n, has no WENDS, $\forall n \geq 2$.

Proof.

Since $\deg(v) = r \ \forall v \in G$, since $D \subset v(G)$ and must be $\text{degree}(v) < \text{degree}(u)$ $\forall v \in D$ and $\forall u \in V - D$ according to Definition 1.1 and Definition 2.1. Therefore G has no WENDS.

Proposition 2.5. For a complete bipartite graph $K_{n,m}$

$$\gamma_{en}^{w}(K_{n,m}) = \begin{cases} \max\{n, m\}, & \text{if } n \neq m \\ \text{has no WENDS} & \text{if } n = m \end{cases}.$$

Proof.

Let $X, \text{and } Y$ be partite subsets of $K_{n,m}$ such that X has m of vertices and Y has n of vertices. By Proposition 1.2, $X, \text{and } Y$ are ENDS, by definition of complete bipartite graph $\text{degree}(v) > \text{degree}(u)\ \forall\ v \in X$ and $u \in Y$ if $m > n$, then only the vertices in X is WENDS, therefore $\gamma_{en}^{w}(K_{n,m}) = \max\{n, m\}$.
All the vertices has equal degree in $K_{n,m}$ If n=m, then $K_{n,m}$ is n-regular graph therefore it has no WENDS according to Theorem 2.4

Proposition 2.6.

1. For a cycle graph C_n with order n, C_n has no WENDS, $\forall n \geq 3$
2. For a complete graph K_n with order n, K_n has no WENDS, $\forall n \geq 2$

Proof.

Since $\deg(v) = 2 \ \forall v \in C_n$, then C_n =2-regular graph and since $\deg(v)$ = n-1 $\forall v \in K_n$, then K_n =n-1-regular graph, then C_n and K_n have no WENDS according to Theorem 2.5.

Proposition 2.7. For a path graph P_n

1. $\gamma_{en}^{w}(P_n) = 2, \forall 3 \leq n \leq 4$.
2. P_n has no WENDS, $\forall n \neq 3,4$

Proof.

1. $\text{degree}(v) = 2\ \forall v \in P_n$ except for the pendant vertices will be $\deg(v) = 1$, then the pendant vertices are in D according to Definition 2.1 and Proposition 2.2, then $\gamma_{en}^{w}(P_n) = \gamma_{en}(P_n) = \left\lceil \frac{n}{3} \right\rceil = 2$ if n=3,4.
2. If $n \geq 5$, then $\gamma_{en}(P_n) = \left\lceil \frac{n}{3} \right\rceil > 2$, and the pendant vertices are not in D, therefore P_n has no WENDS, $\forall n \geq 5$ according to Definition 1.1. And if $n = 2$, then P_n is 1-regular graph, then P_n have no WENDS according to Theorem 2.5
3.

Proposition 2.8.

1. $\gamma_{en}^{w}(F_n) = \begin{cases} n & \forall\, n \geq 5 \\ 2 & \forall\, 3 \geq n \geq 4 \\ \text{has no WENDS} & \forall\, n = 2 \end{cases}$

   for $F_n = P_n + K_1$ be a fan graph with order $n + 1$ such that $n \geq 2$ (See Fig .3 (a,b,c))

2. $\gamma_{en}^{w}(C_{n,m}) = \begin{cases} m, & \text{if } m > n - 2 \\ n, & \text{if } m < n - 2 \\ \text{has no WENDS}, & \text{if } m = n - 2 \end{cases}$

   $C_{n,m} = C_n + \overline{K_m}$ be a cone graph with order n + m, $\overline{K_m} \equiv N_m$ (See Fig .4 (a,b,c))

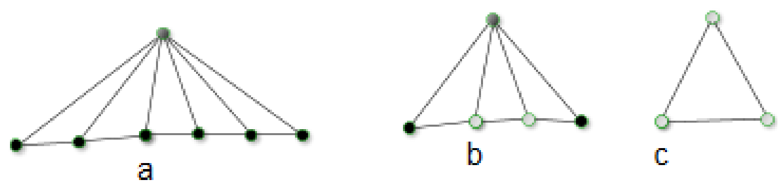

Figure 3: (a) WENDS in fan F_6 (b) WENDS in fan F_4 (c) WENDS in fan F_2

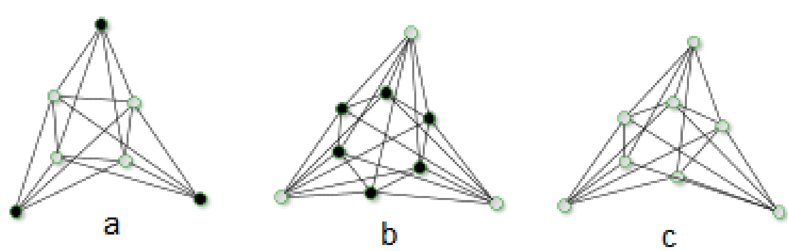

Figure 4: (a) WENDS in cone $C_{4,3}$ (b) WENDS in cone $C_{6,3}$ (c) WENDS in cone $C_{5,3}$

Proof.

1) i) By definition of fan graph $F_n = P_n + K_1$, for $n \geq 5$ then, a vertex (v) in the K_1 with $\text{degree}(v) \geq 5$ and $\text{degree}(u) \leq 3\ \forall u \in (P_n)$ since P_n has no WENDS, $\forall n \neq 3,4$, then $\gamma_{en}^{w}(F_n) = |P_n| = n$ according to Definition 2.1 and Proposition 2.8. (see Fig .3(a))

   ii) According to Proposition 2.8. in (1) $\gamma_{en}^{w}(P_n) = \gamma_{en}(P_n) = 2, \forall 3 \leq n \leq 4$ and definition of fan graph $F_n = P_n + K_1$ then $\gamma_{en}^{w}(F_n) = 2$. (see Fig .3(b))

iii) Since $F_2 = K_3$ for $n = 2$, then it has no WENDS according to Proposition 2.7. (see Fig .3(c))

2) Since $\text{degree}(v) = n\ \forall v \in \overline{K_m}$ and $\text{degree}(u) = m + 2\ \forall u \in C_n$ in cone graph $C_{n,m}$ According to definition of a cone graph $C_{n,m} = C_n + \overline{K_m}$, then they prove is done according to Definition 2.1. (see Fig .4(a,b,c))

3 . Results of complement of some types of graphs in the WENDS

Proposition 3.1.

1. $\overline{P_n}$ has no WENDS $\forall n \leq 3$
2. $\gamma_{en}^{w}(\overline{P_n}) = n - 2$, for $n \geq 4$.
3. $(\overline{C_n})$, has no WENDS.

Proof.

1. Since $\overline{P_n}$ has isolated vertex $\forall n \leq 3$, then has no ENDS and has no WENDS.
2. Since v_1, and v_n are two pendants vertices in P_n then, degree(v_1)=degree(v_n)=1 in P_n, then degree(v_1)=degree(v_n)=n-2 in $\overline{P_n}$, and in P_n $\text{degree}(v_i) = 2$ $\forall i > 1$ and $\forall i < n$, then $\text{degree}(v_i) = n - 3$ in $\overline{P_n}$, therefore $\{ v_i : \forall 1 < i < n\}$ is WENDS in $\overline{P_n}$ and $\gamma_{en}^{w}(\overline{P_n}) = n - 2$
3. Since C_n is 2-regular graph, then $\overline{C_n}$ is n-3-regular graph and have no WENDS according to Theorem 2.4.

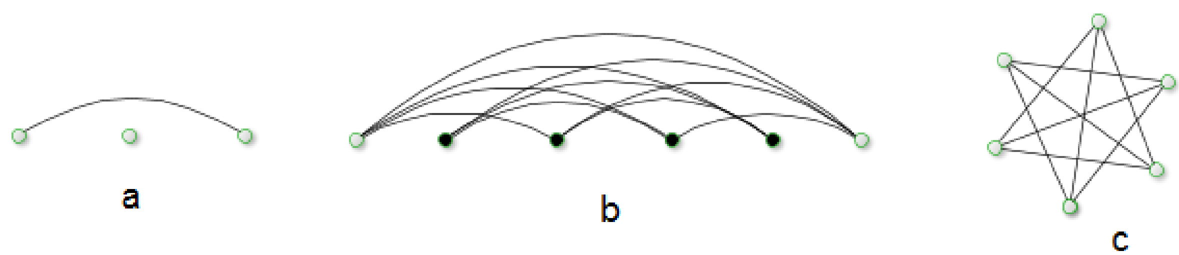

Figure 5: (a) $\overline{P_3}$ has no WENDS (b) $\gamma_{en}^{w}(\overline{P_6}) = 4$ (c) $\overline{C_6}$ has no WENDS

Proposition 3.2.

1. $(\overline{S_n})$, $(\overline{W_n})$ and $(\overline{K_n})$ have no WENDS
2. $(\overline{K_{n,m}})$ has no WENDS

Proof.

1. $(\overline{S_n})$, $(\overline{W_n})$ and $(\overline{K_n})$ have isolated vertex $\forall n \leq 3$, then have no WENDS
2. Since $\overline{K_{n,m}} \equiv K_n \cup K_m$, and since K_n and K_m are n-regular, and m-regular, then $(\overline{K_{n,m}})$= has no WENDS according to Theorem 2.5.

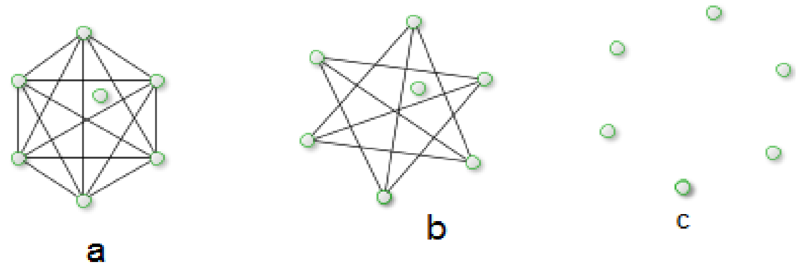

Figure 6: (a)) $\overline{S_6}$ has no WENDS (b)) $\overline{W_6}$ has no WENDS (c)) $\overline{K_6}$ has no WENDS

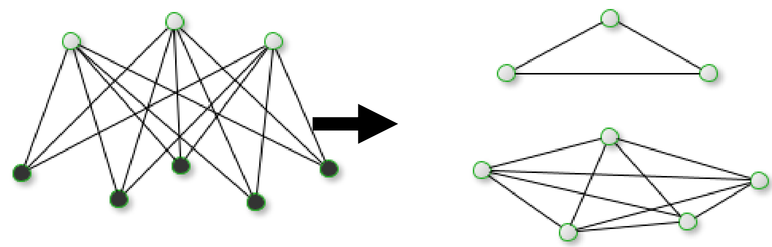

Figure 7: (a)$K_{3,5}$ (b) $\overline{K_{3,5}} \equiv K_3 \cup K_5$ has no WENDS

4. Some results of a corona graphs in the WENDS

Theorem 4.1. Let G_1 and G_2 are two graphs, then

1. $\gamma_{en}^{w}(G_1 \odot G_2) = |G_1|(\gamma_{en}^{w}(G_2))$ if G_2 has WENDS
2. $\gamma_{en}^{w}(G_1 \odot G_2) = |G_1||G_2|$ if G_2 has no WENDS
3. $(G_1 \odot G_2)$ has no WENDS if G_1 is null graph and G_2 is complete graph

Proof.

If G_1 with order $n \geq 2$ and G_2 with order $m \geq 1$, then $\text{degree}(v) \geq m$ and $\text{degree}(u) \leq m$ $\forall v \in G_1$ and $\forall u \in G_2$ in $(G_1 \odot G_2)$ by definition of corona, then $\text{degree}(u) \leq \text{degree}(v)$ hence:

1. If $\text{degree}(u) < \text{degree}(v)$, and G_2 has WENDS, then $\gamma_{en}^{w}(G_1 \odot G_2) = |G_1|(\gamma_{en}^{w}(G_2))$.
2. Now if G_2 has no WENDS and $\text{degree}(u) < \text{degree}(v)$, then $\gamma_{en}^{w}(G_1 \odot G_2) = |G_1||G_2|$
3. If G_1 is null graph and G_2 is complete graph, then $\text{degree}(u) = \text{degree}(v)$ and $(G_1 \odot G_2)$ is r-regular graph, therefore $(G_1 \odot G_2)$ has no WENDS according to Theorem 2.5.

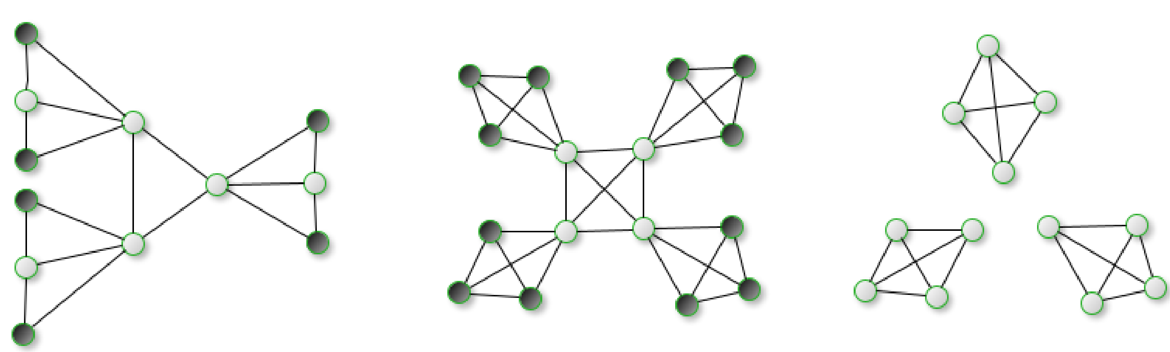

Figure 8: (a) $\gamma_{en}^{w}(K_3 \odot P_3)$=3x2=6 (b) γ_{en}^{w} $(K_4 \odot C_3)$=4x3=12 (c) $(N_3 \odot K_3)$ has no WENDS

5. Conclusion

In this research, the weak dominance in equality co-neighborhood dominating set was defined and studied on some special graphs and its complements, as well as the study of composite graphs such as the corona graph, fan graph and cone graph.

REFERENCES

[1]. M. N. Al-Harere and M. A. Abdlhusein, Pitchfork domination in graphs, Discrete Mathematics, Algorthem and Applications, 12 (2) (2020) 2050025.

[2]. M. N. Al-Harere and P. A. KhudaBakhash, Tadpole domination in graphs, Baghdad Sci. J. 15 (2018) 466–471.

[3]. M. A. Abdlhusein, Doubly connected bi-domination in graphs, Discrete Mathematics, Algorthem and Applications,13(2(2021))2150009**.**

[4]. M. A. Abdlhusein and M. N. Al-Harere, Weak pitchfork domination and its inverse in graphs, Discrete Mathematics, Algorthem and Applications, (2021)2150038.

[5]. I. A. Alwan and A. A. Omran, Domination polynomial of the composition of complete graph and star graph, J. Phys.: Conf. Ser. 1591 012048, (2020).

[6]. C. Berge, The theory of graphs and its applications, Methuen and Co, London,1962.

[7]. E. J. Cockayne, R. M. Dawes S. T. Hedetniemi, Weak domination in graphs, Networks:Int.J. 10(3) (1980) 211.219.

[8]. F. Harary, Graph Theory (Addison-Wesley, Reading Mass, 1969).

[9]. A. A. Jabour and A. A. Omran, Domination in discrete topology graph, AIP 11 Conf.Proc. 2183 (2019) 030006.

[10]. S. Sh. Kahat, A. M. Khalaf, R. Hasni, Dominating sets and domination polynomials of Stars, Australian Journal of Basic and Applied Sciences, 8(6)(2014) 383-386.

[11]. S. Sh. Kahat, A. M. Khalaf, Dominating sets and domination polynomial of complete graphs with missing edges, Journal of Kufa for Mathematics and Computer,2(1)(2014) 64-68.

[12]. S. Sh. Kahat, A. M. Khalaf, R. Hasni, Dominating sets and domination polynomial of wheels, Asian Journal of Applied Sciences, 2 (3)(2014)287-290.

[13]. Sh. Kahat, A. M. Khalaf Chromatic polynomials and chromaticity of zero-divisor graphs II ,Far East Journal of Mathematical Sciences, 2016, 100(12), pp. 2143–2151.

[14]. S. Sh. Kahat, M. N. Al-Harere, Inverse equality co-neighborhood domination in graphs, Journal of Physics: Conference Series this link is disabled, 2021, 1879(3), 032036.

[15]. S. Sh. Kahat, A. A. Omran and M. N. Al-Harere, Fuzzy equality co-neighborhood domination in graphs, Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 537-545 ISSN: 2008-6822 (electronic) http://dx.doi.org/10.22075/ijnaa.2021.5101.

[16]. S. Sh. Kahat, M. N. Al-Harere, Total equality Co-neighborhood domination in a graph, AIP Conference Proceedings 2414 (1) (2023).

[17]. S. Sh. Kahat, M. N. Al-Harere, Dominating sets and polynomial of equality co-neighborhood domination of graphs, AIP Conference Proceedings 2398 (1), 060063 (2022).
[18]. S. Sh. Kahat, Strong Equality Co-Neighborhood Domination in Graphs, Eurasian Journal of Physics, Chemistry and Mathematics 4, 39-45, 2022.
[19]. S. Sh. Kahat, DOMINATING SETS AND DOMINATION POLYNOMIAL OF CORONA OF GRAPHS, European Scholar Journal (ESJ)Available Online at: https://www.scholarzest.comVol. 3No.4,April 20225562-2660ISSN:
[20]. O. Ore, Theory of Graphs (American Mathematical Society, Providence, R.I. (1962).
[21]. A. A. Omran, M. N. Al-Harere, and Sahib Sh. Kahat,Equality co-neighborhood domination in graphs, Discrete Mathematics, Algorithms and Applications, 2021. https://doi.org/10.1142/S1793830921500981