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Daniell Integral With Some Of Their Properties

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ABSTRACT

In this work, we presented both the concept of Daniell space and the extension Daniell space and some basic results related to them, and we proved that the extension Daniel space is a complete space, and then we introduced the concept of Danielly integrable functions by introducing the concept of the upper and lower Daniel integration. Finally, some properties of this integration have been proven

1.Introduction

Definite integral has an interesting history as the idea of definite integral arose from problems of calculating the lengths, areas, and volumes of curved geometric shapes. These problems were solved for the first time by Greek mathematicians and passed through multiple stages until in 1868 Riemann presented the concept of integration over a period and Libeck introduced the integral based on the concept of measurement in 1902.

In 1918, Daniell published his paper entitled "A general form of integral" where he defined the integral as a positive linear function defined on a vector lattice (Riesz space) whose elements are real valued function defined on a set that has no conditions only being a non-empty set. Daniell integral was more general than the Riemann integral and Lebesgue integral.

This research includes four other items in addition to the introduction. In the first item, we touched on some basic definitions and concepts related to vector space and lattice space, and we demonstrated some theorems related to this concept. The second item included the definition of Daniell space and the Daniell function as well, and we proved some theorems related to this space, either in item Third, we have expanded Daniell space and proved that the extension space is a complete space. As for the fourth item, it included the definition of the integration of the upper and lower Daniell, and we have mentioned some definitions and proofs related to this concept.

2.Fundamental Concepts

The letters $\mathbb{R}$ and $\mathbb{C}$ will always denote the field of real numbers and the field of complex numbers, respectively. For the moment, let F stand for either $\mathbb{R}$ or $\mathbb{C}$. A scalar is a member of the scalar field F.

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Definition 2.1[7]

A linear space over F is a set X, whose elements are called ***vector***, and in which two operations, addition $(+: X \times X \to X)$ and scalar multiplication $(\cdot: F \times X \to X)$ such that

(1) $x + y \in X$ for all $x, y \in x$.

(2) $x + y = y + x$ for all $x, y \in X$.

(3) $x + (y + z) = (x + y) + z$ for all $x, y, z \in X$.

(4) There exists $0 \in X$ such that $x + 0 = 0 + x$ for all $x \in X$ and 0 is the zero vector or the origin.

(5) For all $x \in X$,there exist $-x \in X$ such that $x + (-x) = (-x) + x = 0$

(6) $\lambda \cdot x \in X$ for all $\lambda \in F$ and for all $x \in X$.

(7) $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ for all $\lambda \in F$ and for all $x, y \in X$.

(8) $(\alpha + \beta) \cdot x = \alpha x + \beta x$ for all $\alpha, \beta \in F$ and for all $x \in X$.

(9) $(\alpha \cdot \beta) = \alpha \cdot (\beta \cdot x)$ for all $\alpha, \beta \in F$ and for all $x \in X$.

$I \cdot x = x$ for all $x \in X$ and I is the unity element of the field F. [8]

Remark:

A real linear space is one for which $F = \mathbb{R}$, a complex is linear space is one for which $F = \mathbb{R}$.

Definition 2.2[1]

Let Ω be an arbitrary set. Suppose f and g are real valued functions on Ω $(f, g: \Omega \to \mathbb{R})$, . we define

$f \vee g = \max\{f, g\} = \{f, g\} = max\{f + g, 0\} + g$ and

$f \wedge g = \min\{f, g\} = (f + g) - \max\{f, g\}$, where 0 is the zero function.

Definition 2.3[1]

Let L be a set of real valued function defined on Ω. we say that L is a lattice if $\max\{f, g\}$, $\min\{f, g\} \in L$ for all $f, g \in L$. Further, a linear L of real valued functions on a set Ω over the real field is called a vector lattice (or Riesz space) provided L is also a lattice.

Notice that if f is in some Riesz space, then $|f|$ is also in that Riesz space.

Theorem 2.4[1]

Suppose L is a linear space of real valued functions on a set Ω. Then L is a vector lattice if $\max\{f, 0\} \in L$ for all $f \in L$.

Proof:

Let $f, g \in L$, Since L is a linear space, then $f - g \in L$, thus $max\{f - g, 0\} \in L$ and $max\{f - g, 0\} + g \in L$.

Hence $max\{f, g\} \in L$. Similarly, $f + g \in L$ and $max\{f, g\} \in L$, so $min\{f, g\} \in L$.

Definition 2.5

Let Ω be any set and $f : \Omega \to \mathbb{R}$ a function, we define the positive and negative parts $f^+$and f^- by $f^+ = max\{f, 0\} +$ and $f^- = -min\{f, 0\} = max\{-f, 0\}$

i.e.

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) \leq 0 \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

It follows that

(1) f^+ and f^- are non-negative

(2) $f = f^+ - f^-$and $|f| = f^+ + f^- = f^+ + (-f)^-$

(3) $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$

(4) $(-f)^+ = f^-$ and $(-f)^- = f^+$

(5) If $\lambda > 0$, then $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- = \lambda f^-$

Now if we define, for $f, g: \Omega \to \mathbb{R}$, $x \in \Omega$

$(f \vee g) = max\{f(x), g(x)\} =$ and $(f \wedge g) = min\{f(x), g(x)\}$.

It follows that

(1) $f^+ = f \vee 0$ and $f^- = -(f \wedge 0) = (-f) \vee 0$

(2) $f \vee g = (f - g) \vee 0 + g$ and $f \wedge g = f + g - (f \vee g)$

Theorem 2.6

Let L is a linear space of real valued functions on Ω. Then L is a Riesz space if and only if for every $f \in L$, then $|f| \in L$.

Proof:

Suppose that L is a Riesz space

Let $f \in L$, since L is a linear space, then $-f \in L$

by theorem 2.4, we have max{ f ,0} $\in L$ and max{ $-f$,0} $\in L$

since $f^+ = max\{f, 0\} +$ and $f^- = max\{-f, 0\}$, then $f^+, f^- \in L$, so $f^+ + f^- \in L$, since $|f| = f^+ + f^-$, then $|f| \in L$.

Conversely, suppose $|f| \in L$ for all $f \in L$. To prove L is a vector lattice.

Suppose $f \in L$, then $|f| \in L$, so $\frac{1}{2}(|f| + f) \in L$

Since $max\{f, 0\} = f^+ = \frac{1}{2}(|f| + f)$, then $max\{f, 0\}$, by theorem 2.4, L is a vector lattice.

Definition 2.7

Let $\mathbb{R}$ be the set of real numbers. The extended real numbers system consists of the real numbers system to be the real number with two symbols, $+\infty$ and $-\infty$. and it is denoted by $\overline{\mathbb{R}}$, i.e

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\} = \{-\infty, \infty\}$$

The following algebraic relation among them and real numbers x: $-\infty < x < \infty$

(1) $x + \infty = \infty + x = \infty$, $x + (-\infty) = -\infty + x = -\infty$

(2) If $x = 0$, then $x(\infty) = 0$ and $x(-\infty) = 0$

(3) If $x > 0$, then $x(\infty) = \infty$ and $x(-\infty) = \infty$

(4) If $x < 0$, then $x(\infty) = -\infty$ and $x(-\infty) = -\infty$

(5) $\infty + \infty = \infty, -\infty + (-\infty) = -\infty, \infty - (-\infty) = \infty, -\infty - \infty = -\infty$

3. Daniell Space

We will introduce the concept of Daniell integration as presented by Daniel in his research paper (A General Form of Integral).

Recall that a functional I on a linear space L over the field F is called a linear functional if $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for all $f, g \in L$ and $\alpha, \beta \in F$.

Definition 3.1

Let L be a Riesz space of real valued functions defined on a set Ω. A linear functional $D: \Omega \to \mathbb{R}$ is called

(1) Positive if $D(f) \geq 0$ whenever $f \in L$ and $f \geq 0$.

(2) Continuous under monotone limits if, for every increasing sequence $\{f_n\}$ of functions in L and $f \in L$ such that $f(x) \leq \lim_{n\to\infty} f_n(x)$ for all $x \in \Omega$, then $D(f)\ lim_{n\to\infty} D(f_n)$.

Note that if I is positive, then $D(f) \leq D(g)$ for each $f \in L$ and $f \leq g$.

(3) Daniell functional (or Daniell integral) if D is positive and continuous under monotone limit.

Remark

D is continuous under monotone limit iff $D(f_n) \downarrow 0$ whenever $f_n \downarrow 0$ and each $f_n \in L$.

Definition 3.2

A triple (Ω, L, D) is called a Daniell space if Ω is a nonempty set, L is a Riesz space of real valued functions on Ω, and $D: L \to \mathbb{R}$ is a Daniell functional.

Theorem 3.3[1]

Let L be a vector lattice of real valued function on a set Ω. Suppose that D is a Daniell integral on L. Then $D(f) \leq \sum_{n=1}^{\infty} D(f_n)$ whenever $\{f_n\}$ is a sequence of nonnegative functions in L and $f \in L$ such that $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \Omega$.

Proof:

Let $\{f_n\}$ is a sequence of nonnegative functions in L and $f \in L$ such that $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \Omega$.

Define $g_n(x) = \sum_{i=1}^{n} f_i(x)$ for all $x \in \Omega$. Then $g_n \in L$ for all n and $\lim_{n\to\infty} g_n(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \Omega$.

Since $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$ for all $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$, then $f(x) \leq lim_{n\to\infty} g_n(x)$ for all $x \in \Omega$. Thus $D(f) \leq lim_{n\to\infty} D(g_n)$.

But $D(g_n) = \sum_{i=1}^{n} D(f_i)$. Hence $lim_{n\to\infty} D(g_n) = \sum_{n=1}^{\infty} D(f_n)$, so $D(f) \leq \sum_{n=1}^{\infty} D(f_n)$. [1]

4. Complete Daniell Space

Definition 4.1[1]

Let L^* be the class of all those extended real -valued functions on Ω each of which is the limit of a monotone increasing sequence of functions in the vector lattice L.

In other words: Let L be a vector lattice, then $f \in L^*$ iff $f: \Omega \to \overline{\mathbb{R}}$ is a function and there exists a sequence $\{f_n\}$ of monotone increasing sequences of functions in L *such that* $f = \lim_{n\to\infty} f_n$. [1]

Definition4.2[2]

Let f be a real function on Ω. if there exist function $f_n \in L, n \in \mathbb{N}$, such that

(1) $\sum_{n=1}^{\infty} D(|f_n|) < \infty$

(2) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in \Omega$ and $\sum_{n=1}^{\infty} |f_n(x)| < \infty$

then we write $f = \sum_{n=1}^{\infty} f_n$ or $f = f_1 + f_2 + f_3 + \cdots$

Definition 4.3[2]

A Daniell space (Ω, L, D) is called complete if $f = \sum_{n=1}^{\infty} f_n$ for some $f_1, f_2, , \ldots \in L$, implies that $f \in L$.

Theorem4.4 [1]

Let $\{f_n\}$ and $\{g_m\}$ are monotone increasing sequences such that f_n and g_m are in L for all n and m. Let $lim f_n \le lim g_m$. Then $\lim D(f_n) \le \lim D(g_m)$. Further if f is in L^*, $f_n \uparrow f$ and $g_m \uparrow f$, then $\lim D(f_n) = \lim D(g_m)$.

Proof: Let n arbitrary and fixed. Then $f_n \le lim f_n \le lim g_n, so\ D(f_n) \le \lim D(g_m)$ for each n. Hence $lim\, D(f_n) \le lim\, D(g_m)$.

Let f is in L^*, $f_n \uparrow f$ and $g_m \uparrow f$, then $\lim f_n = f \le \lim g_m$, so $\lim D(f_n) \le \lim D(g_m)$.

But $\lim g_m = f \le \lim f_n$, so $\lim D(g_m) \le \lim D(f_n)$. Therefore $lim\, D(f_n) = lim\, D(g_m)$.

Remark[1]

If f is in L^*, then there exist an increasing sequence $\{f_n\}$ such that f_n is in L for all n and $f = lim f_n$. Then $D(f) = lim\, D(f_n)$.

Theorem 4.5 [1]

Let f in L^*, then f is in L^* if and only if there exist a sequences $\{f_n\}$ of non-negative functions in L with $f = \sum_{n=1}^{\infty} f_n$. Further, $D(f) = \sum_{n=1}^{\infty} D(f_n)$.

Proof:($\Longrightarrow$) Let f is in L^* then there exist asequences $\{g_n\}$ of functions in L such that $g_n \uparrow f$. Suppose that g_n are non-negative.Let $f_1 = g_1$, and $f_n = g_n - g_{n-1}$ for $n \ge 2$.

Then $\sum_{n=1}^{k} f_n = \sum_{n=2}^{k} (g_n - g_{n-1}) + g_1 = g_k$. Thus $\sum_n f_n = \lim_k \sum_{n=1}^{k} f_n = \lim_k g_k = f$.

Therefore, $D(f) = \lim D(g_n) = \lim D(\sum_{i=1}^{n} f_i) = \lim \sum_{i=1}^{n} D(f_i) = \sum_{i=1}^{\infty} D(f_i)$.

($\Longleftarrow$) Let $\{f_n\}$ be a sequences of non-negative functions in L with $f = \sum_{n=1}^{\infty} f_n$ and $D(f) = \sum_{n=1}^{\infty} D(f_n)$, then f is in L_u, since $L \subset L^*$.

Theorem 4.6

(Ω, L^*, D) is a complete Daniell space.

Proof

i. First we must prove that L^* is a Riesz space

Let $f, g \in L^*, \lambda, \beta \in \mathbb{R}$, then $f = lim_{n\to\infty} f_n\ and\ g = lim_{m\to\infty} g_n$ *, where* f_n *and* g_n are increasing sequences of function in L, then

$\lambda f(x) + \beta g(x) = \lambda \lim_{n\to\infty} f_n(x) + \beta \lim_{m\to\infty} g_n(x) =$

$lim_{n\to\infty} \lambda f(x) + lim_{m\to\infty} \beta g_n(x) = \lim_{n,m\to\infty} (\lambda f_n(x) + \beta g_m(x))$.

there fore $\lambda f + \beta g$ is in L^*.

Since L is Riesz then $f_n \vee g_n$ is in L for all n then $\lim_{n\to\infty} f_n \vee g_n)$ is in L^* to show that $f_n \vee g_n$ is monotone increasing, let $x \in \Omega$ then $f_n(x) \leq f_{n+1}(x) \leq (f_{n+1} \vee g_{n+1})(x)$
and $g_n(x) \leq g_{n+1}(x) \leq (f_{n+1} \vee g_{n+1})(x)$, so that $(f_n \vee g_n)(x) \leq (f_{n+1} \vee g_{n+1})(x)$.
Therefore $f_n \vee g_n$ is monotone increasing. If $\lim f_n(x) = \infty$ or
$lim g_n(x) = \infty, \lim(f_n \vee g_n)(x) = \infty$ and $(f \vee g)(x) = \infty$ then $(f \vee g)(x) = \lim(f_n \vee g_n)(x)$.
Let $\lim f_n(x) \neq \infty$ and $lim g_n(x) \neq \infty$, then $\lim(f_n \vee g_n)(x) \neq \infty$ and $(f \vee g)(x) \neq \infty$.
Now define $h_n = f_n \vee g_n$ for all n. Then h_n is in L, and $\lim h_n(x) = \lim(f_n \vee g_n)(x)$.
Let $h_n(x) > \max\big(lim f_n(x), lim g_n(x)\big) = \max\big(f(x), g(x)\big)$.
Suppose that there exist $N \in \mathbb{Z}^+$ such that $n \geq N$ implies that $h_n(x) > max\big(f(x), g(x)\big)$, since h_n is monotone. But $\{f_n\}, \{g_n\}$ monotone increasing implies $lim g_n(x) \geq g_n(x)$ and $lim f_n(x) \geq f_n(x)$ for all n.
Thus $h_n(x) > max\big(f_i(x), g_i(x)\big)$ for all i and for all $n \geq N$ which is a contradication.
Therefore, $lim\, h_n(x) \leq max\big(f(x), g(x)\big)$. But $h_n(x) \geq f_n(x)$ for all n implies $h_n(x) \geq f(x)$, and $h_n(x) \geq g_n(x)$ for all n implies $h_n(x) \geq g(x)$, thus $lim\, h_n(x) \geq max\big(f(x), g(x)\big)$, so $lim\, h_n(x) = max\big(f(x), g(x)\big)$.
Therefore, $f \vee g$ is in L^*, by the same way we can prove that $f \wedge g$ is in L^*. Since $L \subset L^*$, then L^* is a lattice contained L.

ii. to prove that $D: L^* \to \mathbb{R}$ is a positive linear function on L^*

Let $f \in L^*, f \geq 0$, there is an increasing sequence $\{f_n\}$ such that f_n is in L for all n=1,2,3,…. and $0 \leq f = lim f_n \Longrightarrow \lim D(f_n) \geq D(0) = 0$.
Let $f, g \in L^*$ then $f = lim_{n\to\infty} f_n\ and\ g = lim_{m\to\infty} g_m\ , where\ f_n\ and\ g_n$ are are monotone increasing sequences of function in L.
Suppose that $f \leq g$ implies that $lim_{n\to\infty} f_n \leq lim_{m\to\infty} g_m$ then
$D(f) = D(\underset{n\to\infty}{lim} f_n) \leq D(lim_{m\to\infty} g_m) = D(g)$ implies
$D(f) = lim_{n\to\infty} D(f_n) \leq lim_{m\to\infty} D(g_m) = D(g)$ there fore $D(f) \leq D(g)$.
Let $f, g \in L^*\ and\ \alpha, \beta \in \mathbb{R}$ then $D(\alpha f + \beta g) = D(\alpha(\underset{n\to\infty}{lim} f_n) + \beta\,(\underset{m\to\infty}{lim} g_m))$

$$= \alpha D(\lim_{n\to\infty} f_n) + \beta D(\lim_{m\to\infty} g_m) = \alpha D(f) + \beta D(g).$$

iii. To prove that D is a Daniell integral on L^*.

Let $\{f_n\}$ be an increasing sequence in L^* and f in L^* with $f \leq \underset{n\to\infty}{lim} f_n$
Let $g_n = f_n - f_1, g_n \geq f$ then $D(f_n) = D(g_n) + D(f_1) \Rightarrow$
$\lim D(f_n) = \lim D(g_n) + D(f_1)$.
Let $g = \lim g_n + f_1 = \lim f_n$ then g in L^*. Then $f \leq \lim f_n = g$.
Then $D(f) \leq D(g) = lim\, D(g_n) + D(f_1) = lim\, D(f_n)$
There fore $D(f) \leq \underset{n\to\infty}{lim} D(f_n)$.
Hence (Ω, L^*, D) is a Daniell space.

iv. To prove that (Ω, L^*, D) is complete.

Let $f_n \in L^*, f_n \geq 0\,for\ each\, n = 1,2, \ldots$ and $f = \sum_{n=1}^{\infty} f_n$, we must prove that $f \in L^*$.

Then there is a sequence of non-negative functions $\{g_{m,n}\}$ In L for each n such that $f_n = \sum_{m=1}^{\infty} g_{n,m}$

Then $f = \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} g_{n,m} = \sum_{n,m=1}^{\infty} g_{n,m}$ then by (Thm.4.5), $f \in L^*$.

Therefore (Ω, L^*, D) is a complete Daniell space.

5. Upper and Lower Daniell Integral

Defintion 5.1

Let $f \in L^*$. Define the upper Daniell integral of f by $\overline{\mathrm{D}}(\mathrm{f}) = \inf\{\mathrm{D(g)}: \mathrm{g} \in L^*, \mathrm{g} \geq \mathrm{f}\}$ (i.e $\inf(\emptyset) = \infty$).

Theorem 5.2:

Let $f_1, f_2, g\ L^*$, then

(1) If $a > 0, \overline{\mathrm{D}}(af) = a\overline{\mathrm{D}}(f)$.

(2) $\overline{\mathrm{D}}(f_1 + f_2) \leq \overline{\mathrm{D}}(f_1) + \overline{\mathrm{D}}(f_2)$.

(3) If $f \leq g$ then $\overline{\mathrm{D}}(f) \leq \overline{\mathrm{D}}(g)$.

Proof:

(1) Let $c > 0. Then$

$\overline{\mathrm{D}}(af) = inf\{D(a\ g): a\ g \in\ L^*, a\ g \geq af\} =$

$\overline{\mathrm{D}}(af) = inf\{a\ D(g): g \in\ L^*, g \geq f\}$

$\overline{\mathrm{D}}(f) = a\ \overline{\mathrm{D}}(f)$.

(2) Let $g_1, g_2 \in L_u$ such that $f_1 \leq g_1$ and $f_2 \leq g_2$.

Then $f_1 + f_2 \leq g_1 + g_2$ therefore $D(f_1 + f_2) \leq D(g_1 + g_2)$= $D(g_1) + D(g_2)$.

For fixed g_1 and for all $g_2 \geq f_2$, the inequality holds. Thus $\overline{\mathrm{D}}(f_1 + f_2) \leq \overline{D}(f_1) + \overline{\mathrm{I}}(f_2)$. Since this inequality holds for all $g_1 \geq f_1$ it follows that $\overline{\mathrm{D}}(f_1 + f_2) \leq \overline{\mathrm{D}}(f_1) + \overline{\mathrm{D}}(f_2)$.

(3) Let $e \in L^*$ and $e \geq g$. Then $e \geq f$ so $\{e: e \in L^*,\ e \geq f\} \supset \{e: e \in L^*, e \geq g\}$.

Definition 5.3

Let $f \in L^*$. the lower Daniell integral of f is defined by $\underline{D}(f) = sup\{I(l): l \in L^*, l \leq f\}$.

i.e. $\underline{\mathrm{D}}(f) = -\overline{\mathrm{D}}(-f)$

Theorem 5.4

Let f_1, f_2 and g are extended real-valued function on Ω. then

(1) $\underline{\mathrm{D}}(f) \leq \overline{\mathrm{D}}(f)$ for all $f \in L^*$.

(2) $\overline{\mathrm{D}}(f \vee g) + \overline{\mathrm{D}}(f \wedge g) \leq \overline{\mathrm{D}}(f) + \overline{\mathrm{D}}(g)$.

(3) $\overline{\mathrm{D}}(|f|) - \underline{\mathrm{D}}(|f|) \leq \overline{\mathrm{D}}(f) - \underline{\mathrm{D}}(f)$.

Proof:

(1) $0 = \overline{\mathrm{D}}(0) = \overline{\mathrm{D}}(f - f) \leq \overline{\mathrm{D}}(f) + \overline{\mathrm{D}}(-f)$ but $\overline{\mathrm{D}}(-f) = -\underline{\mathrm{D}}(f)$ (by definition). Thus $\underline{\mathrm{D}}(f) \leq \overline{\mathrm{D}}(f)$.

(2) Let $h_1, h_2 \in L^*$, $h_1 \geq f, h_2 \geq g$. Then $h_1 \wedge h_2 \geq f \wedge g$ $and\ h_1 \vee h_2 \geq f \vee g$. Therefore $\overline{\mathrm{D}}(f \vee g) + \overline{\mathrm{D}}(f \wedge g) \leq D(h_1 \vee h_2) + D(h_1 \wedge h_2) = D(h_1) + D(h_2)\ since\ (h_1 \vee h_2) + (h_1 \wedge h_2) = D(h_1) + D(h_2)$.

(3) $|f| = f \vee (-f)$ and $-|f| = f \wedge (-f)$.

Thus $\overline{D}(|f|)+\overline{D}(-|f|)\le\overline{D}(f)+\overline{D}(-f)$. That is, $\overline{D}(|f|)-\underline{D}(|f|)\le\overline{D}(f)-\underline{D}(f)$.

Theorem 5.5[1]

Let $\{f_n\}$ be a sequence of non-negative extended real-valued functions on Ω, and that $f=\sum_n f_n$. Then $D(f)\le\sum_n\overline{D}(f_n)$.

Proof:

Suppose that $\overline{D}(f_n)=\infty$ for some n. Then $\sum_n\overline{D}(f_n)=\infty$. Let $\overline{D}(f_n)\ne\infty$ for all n, given $\in>0$, there exists a g_n in L^* such that $f_n\le g_n$ and with $D(g_n)\le\overline{D}(f_n)+\frac{\in}{2^n}$. Then

$g_n\ge f_n\ge 0$ for all n and hence, $g=\sum_n g_n$ is in L_u . and $D(g)=\sum_n D(g_n)$. But $D(g)=\sum_n D(g_n)\le\sum_n(\overline{D}(f_n)+\frac{\in}{2^n})=\sum_n D(f_n)+\in$.

$g_n\ge f_n$ implies that $g=\sum_n g_n\ge\sum_n f_n=f$. Therefore, $D(g)\ge\overline{D}(f)$.

Hence, $\overline{D}(f)\le D(g)\le\sum_n\overline{D}(f_n)+\in$. But $\in>0$ was arbitrary. Hence, $\overline{D}(f)\le\sum_n\overline{D}(f_n)$. [1]

Definition 5.6[1]

We say that a function $f\in L^*$is D-integrable if $\overline{D}(f)=\underline{D}(f)$ and is finite.we will denote to the class of all D-integrable function by L^1.

If $f\in L^1$ we write $D(f)$ for $\overline{D}(f)$.

In this case define $D(f)=\overline{D}(f)=\underline{D}(f)$. $D(f)$ is called the Daniell integral.. [1]

Theorem 5.7[1]

the set L^1 is a Riesz space of functions containing L, D is a positive linear functional on L^1, $L\subset L^1$

Proof:

We want to show that if f in L^1 implies that cf is in $L^1$for all real c.

Let $f\in L^1, c\ge 0$ then $\overline{D}(cf)=c\overline{D}(f)=c\underline{D}(f)=\underline{D}(cf)$.

Suppose that $c\le 0$. Then $\overline{D}(cf)=\overline{D}(-|c|f)=|c|\,\overline{D}(-f)=-|c|\underline{D}(f)=-|c|\overline{D}(f)=-\overline{D}(|c|f)=\underline{D}(-|c|f)=\underline{D}(cf)$.Therefore, cf is in L^1.

we want to show that if $f\ and\ g$ in L^1 then $f+g$ is in L^1.

Let $f,g\in L^1, then\ \overline{D}(f+g)\le D(f)+D(g)$

Also, $-f\ and-g$ are in L^1, hence $\overline{D}(-f-g)\le D(-f)+D(-g)=-D(f)-D(g)$.

Thus $\underline{D}(f+g)=-\overline{D}(-f-g)\ge D(f)+D(g)\ge\overline{D}(f+g)$. But $\underline{D}(f+g)\le\overline{D}(f+g)$.

Therefore $\underline{D}(f+g)=\overline{D}(f+g)$, this implies that $f+g$ is in L^1.

Third we must proof that if $f\ and\ g$ in $L_1$and a,b are real numbers, then $af+bg$ is in L^1.

$$D(af+bg)=D(af)+D(bg)=aD(f)+bD(g).$$

Hence L^1 is a vector space and D is a linear functional.

If $f\le g$ then if h is in L^* and $h\ge g$ then $h\ge f$. Hence, $\overline{D}(f)\le\overline{D}(g)$.

If $f\le g$ then $-f\ge -g$ and $-\overline{D}(-f)\le -\overline{D}(-g)$. Thus $\underline{D}(f)\le\underline{D}(g)$.

By above, we see that D is positive.

if f is in L^*, then there is a sequence $\{g_n\}$, $g_n\subset L$, such that $g_n\uparrow f$ and $f\ge g_n$ for all n.

Hence $\underline{D}(f)\ge\underline{D}(g_n)=I(g_n)$ for all n. Therefore $\underline{D}(f)\ge\lim\underline{D}(g_n)=D(f)$.

But $\underline{D}(f) \leq \overline{D}(f) = D(f)$. Hence, But $\underline{D}(f) = D(f) = \overline{D}(f)$.
Let $f \in L$ then $D(f) < \infty$. Hence $f \in L^1$, implies that L^1 is an extension of L.
To prove that L^1 is a vector lattice we must show that if $f \in L^1$ then $f \vee 0$ is in L^1.
Let $f \in L^1$ and $g \in L^*$with $g \geq f$. Then $g \vee 0 \geq f \vee 0$ and $g \wedge 0 \geq f \wedge 0$
Then $\overline{D}(f \vee 0) + \overline{D}(f \wedge 0) \leq \overline{D}(g \vee 0) + \overline{D}(g \wedge 0) = D(g)$ and since $g \geq f$
then $\overline{D}(f \vee 0) + \overline{D}(f \wedge 0) \leq D(f)$.
If we replace f by $-f$ in the above inquality, we must replace $f \vee 0$ by $-(f \wedge 0)$ and $f \wedge 0$ by $-(f \vee 0)$. Then $\overline{D}(-(f \wedge 0) + \overline{D}(-(f \vee 0)) \leq D(-f)$.
Thus $-\underline{D}(f \wedge 0) - \underline{D}(f \vee 0) \leq D(-f) = -D(f)$.
Therefore, $\overline{D}(f \vee 0) + \overline{D}(f \wedge 0) \leq D(f) \leq \underline{D}(f \wedge 0) + \underline{D}(f \vee 0)$ which implies that $\overline{D}(f \vee 0) - \underline{D}(f \vee 0) + \overline{D}(f \wedge 0) - \underline{D}(f \wedge 0) \leq 0$. But $\underline{D}(h) \leq \overline{D}(h)$ for all h in L^*, then $\overline{D}(h) - \underline{D}(h) \geq 0$ for all h in L^*.
Hence $\overline{D}(f \vee 0) - \underline{D}(f \vee 0) = 0$.
Therefore, $f \vee 0$ is in L^1 and $L^1$is a vector lattice. [1]

Theorem 5.8

(1) If f is D-integrable, and $f \geq 0$ then $D(f) \geq 0$.
(2) If c is a real number and f is I integrable, then cf is I integrable and $D(cf) = cD(f)$.
(3) $If\ f_1 and\ f_2\ are$ I integrable, then $f_1 + f_2$ are I integrable and $D(f_1 + f_2) = D(f_1) + D(f_2)$.
(4) If $f_1\ and\ f_2\ are$ I integrable, then $f_1 \vee f_2\ and\ f_1 \wedge f_2$ are D-integrable.
(5) If f is D integrable, $|f|$ is D integrable and $(|D(f)|) \leq D(|f|)$.

Proof:

(1) Since $0 \leq f, 0 = \overline{D}(0) \leq \overline{D}(f)$. Therefore $D(f) = \overline{D}(f) \geq 0$.
(2) Let $c \geq 0. By\ theorem\ \overline{D}(cf) = c\overline{D}(f) = cD(f)$.
There fore $\underline{D}(cf) = cD(f)$, and so for $c \geq 0, D(cf) = \overline{D}(cf) = \underline{D}(cf) = cD(f). Let\ c < 0.$ Then $\overline{D}(cf) = \overline{D}\big((-c)(-f)\big) = -c\overline{D}(-f) = c\underline{D}(f) = cD(f)$ and $-\underline{D}(cf) = \overline{D}(-cf) = -c\overline{D}(f) = -cD(f) >$ That is, $\underline{D}(cf) = cD(f)$.

Thus for all real c, $D(cf) = \underline{D}(cf) = \overline{D}(cf) = cD(f)$.

(3) since $\overline{D}(f_1 + f_2) \leq \overline{D}(f_1) + \overline{D}(f_2) = D(f_1) + D(f_2)$.
$$-\underline{D}(f_1 + f_2) = \overline{D}(-f_1 - f_2) \leq \overline{D}(-f_1) + \overline{D}(-f_2) = -D(f_1) - D(f_2).$$

Therefore $\underline{D}(f_1 + f_2) \geq D(f_1) + I(f_2)$.
Since $\overline{D}(f_1 + f_2) \geq \underline{D}(f_1 + f_2) \Rightarrow \overline{D}(f_1 + f_2) = D(f_1) + D(f_2)$, and also $\underline{D}(f_1 + f_2) = D(f_1) + D(f_2)$,therefore $f_1 + f_2$ is D-integrable and $D(f_1 + f_2) = D(f_1) + D(f_2)$.

(4) Suppose $f_1\ and\ f_2\ are$ I integrable. Then $\overline{D}(f_1 \vee f_2) + \overline{D}(f_1 \wedge f_2) \leq \overline{D}(f_1) + \overline{D}(f_2) = D(f_1) + D(f_2)$,

and $\overline{D}(-f_1 \vee -f_2) = \overline{D}(-f_1 \wedge -f_2) \leq \overline{D}(-f_1) + \overline{I}(-f_2)$.
Now $-f_1 \vee -f_2 = -(f_1 \wedge f_2)$, and $-f_1 \wedge -f_2 = -(f_1 \vee f_2)$ therefore

$-\underline{D}(f_1 \wedge f_2) - \underline{D}(f_1 \vee f_2) \leq -\underline{D}(f_1) - \underline{D}(f_2) = -I(f_1) - I(f_2)$.

Hence $\underline{D}(f_1 \wedge f_2) + \underline{D}(f_1 \vee f_2) \geq D(f_1) + D(f_2) \geq \overline{D}(f_1 \vee f_2) + \overline{D}(f_1 \wedge f_2)$.

That is $\left[\overline{D}(f_1 \vee f_2) - \underline{D}(f_1 \vee f_2)\right] + \left[\overline{D}(f_1 \wedge f_2) - \underline{D}(f_1 \wedge f_2)\right] \leq 0$.

Since each of these differences is non-negative, each must be zero, and so the theorem is proved.

(5) $\overline{D}(|f|) - \underline{D}(|f|) \leq \overline{D}(f) - \underline{D}(f) = D(f) - D(f) = 0$, but $\overline{D}(|f|) - \underline{D}(|f|) \geq 0$, thus $\overline{D}(|f|) = \underline{D}(|f|)$ and $|f|$ is D -integrable .Since $-|f| \leq f \leq |f| \Longrightarrow -D(|f|) \leq D(f) \leq D(|f|)$.That is, $|D(f)| \leq D(|f|)$.

Theorem 5.9[1]

If $\{f_n\}$ is a sequence of functions in L^1 such that $f_n \uparrow f$. Then f is in L^1 if and only if $\lim D(f_n)$ is finite. If f in L^1, then $D(f) = \lim D(f_n)$.

Proof:

($\Longrightarrow$) suppose that $\lim D(f_n) = \infty$ and since $f_n \uparrow f$, $f \geq f_n$ for all n then $D(f) \geq \lim D(f_n) = \infty$. this implies that f is not in L^1.

($\Longleftarrow$) Suppose that $\lim D(f_n)$ is finite, since $f \geq f_n$ then $-f \leq -f_n$ for all n.

Thus $\underline{D}(f) \leq \overline{D}(f) = D(f_n)$ for all n which implies that $\underline{D}(f) \geq \lim D(f_n)$.

Let $\varepsilon > 0$ and let $\{g_n\}$ be a sequence of functions in L^* such that $g_1 \geq f_1$, $g_n \geq f_n - f_{n-1}$ for $n \geq 2$ and with $D(g_1) < D(f_1) + {}^{\epsilon}/_{2}$ and $D(g_n) < D(f_n - f_{n-1}) + {}^{\epsilon}/_{2}$ for $n \geq 2$, $g_n \geq f_n - f_{n-1} \geq 0$. Now define $h_n = \sum_{i=1}^{n} g_i$ for all n. then h_n is in L^* for all n and the sequence $\{h_n\}$ is monotone increasing, which implies that $\lim h_n$ is in L^* and $I(\lim h_n) = \lim I(h_n)$.

Then $h_n = \sum_{i=1}^{n} g_i \geq \sum_{i=2}^{n}(f_i - f_{i-1}) + f_1 = f_n$ which implies that $\lim h_n \geq \lim f_n = f$, so that $\lim I(h_n) \geq \bar{I}(f)$.

But $D(h_n) = \sum_{i=1}^{n} D(g_i) < \sum_{i=2}^{n} D(f_i - f_{i-1}) + D(f) + \sum_{i=1}^{n} {}^{\epsilon}/_{2^i}$

$$= D(f_n) + \sum_{i=1}^{n} {}^{\epsilon}/_{2^i} .$$

Thus $\lim D(h_n) < \lim D(f_n) + \epsilon$.

Hence $\overline{D}(f) \leq \lim D(h_n) \leq \lim D(f_n) + \epsilon \leq \underline{D}(f) + \epsilon$.

But ϵ was arbitrary, so $\overline{D}(f) \leq \underline{D}(f)$.

Thus $\overline{D}(f) = lim\, D(f_n) = D(f) = \underline{D}(f)$ which is finite.

Therefore f is in L^1.

Theorem 5.10 [1]

let $\boldsymbol{f_m}$ be a sequence of non-negative functions in L^1. Then $inf\, f_m$ *is in* L^1. Further, if $\underline{lim}\, I(f_m)$ is finite then is in L^1 and I($\underline{lim}\ f_m) \leq \underline{lim}\, D(f_m)$.

Proof:

Let $g_n = f_1 \wedge f_2 \wedge \dots \dots \wedge f_n$. Then the sequence $\{g_n\}$ is a decreasing sequence of non-negative function in L^1. At any point x of Ω, $g_n(x) = g1b_{1 \leq i \leq n}\, f_i(x)$

Hence $g = \lim g_n = \inf f_m$ then $-g_n \uparrow -g$. Since $g_n \geq 0$, $-g_n \leq 0$ then $D(-g_n) \leq 0$ for all n. Thus $\lim D(-g_n) \leq 0$ and is finite, so, $-g$ is in L^1, this implies that $g = \inf f_m$ is in L^1.

Define the sequence $\{h_n\}$ by $h_n = inf_{m \geq n} f_m$, h_n is in L^1 for all n.
Since $f_m \geq 0$, $h_n \geq 0$, $\lim h_n = \underline{\lim}\, f_m$, and $h_n \leq f_m$ if $n \leq s$, thus
$\lim D(h_n) \leq \underline{\lim}\, D(f_m)$ and is finite.
Hence $\lim h_n = \underline{lim}\, f_m$ is in L^1, and $D\left(\underline{\lim}\, f_m\right) = D((\lim h_n) = \lim\ D(h_n) \leq \underline{\lim}\, D(f_m)$.

Theorem 5.11

Let $\{f_n\}$ be a sequence of functions in L^1, g in L^1 such that $|f_n| \leq g$ for all n. then if
$f = lim_{n\to\infty} f_n$, $D(f) = lim_{n\to\infty} D(f_n)$.

Proof:

Since $-f_n \leq |f_n| \leq g$ implies that $f_n + g \geq 0$ for all n, $f_n + g$ is in L^1 for all n, $f_n \leq |f_n| \leq g$ for all n, and $D(f_n) \leq D(g)$ for all n.
Therefore, $D(f_n + g) = D(f_n) + D(g) \leq 2D(g)$ for all n.
Thus $\underline{\lim}\, D(f_n + g) \leq 2D(g) < \infty$.
Then $\underline{lim}\ (f_n + g) = \lim f_n + g = f + g$ is in L^1.
Thus $(f + g) - g = f$ is in L^1.
Therefore, $D(f) + D(g) = D(f + g) = D\left(\underline{lim}\ (f_n + g)\right) \leq\ \underline{lim}\, D(f_n + g) = \underline{lim}\, D(f_n) + D(g)$.
Therefore, $D(f) \leq \underline{lim}\ D(f_n)$.
But $g - f_n \geq 0$ implies that

$$D(g) - D(f) = D(g - f) = D(\underline{\lim}(g - f_n)) \leq \underline{lim}\ D(g - f_n) = D(g) + \underline{lim}\ D(-f_n)$$
$$= D(g) + \underline{lim}(-D(f_n)) = D(g) - \overline{lim}\, D(f_n).$$

Therefore, $\overline{lim}\, D(f_n) \leq D(f) \leq \underline{lim}\, D(f_n)$ which implies that $\lim D(f_n)$ exist and $D(f) = \lim D(f_n)$.

Theorem 5.12[1]

let f is a real-valued function on Ω. then f is in L^1 if and only if there exists a sequence $\{f_n\}$ of functions in L such that $\overline{D}(|f - f_n|) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof:

($\Longrightarrow$) Suppose that $\overline{D}(|f - f_n|) \longrightarrow 0$ then $\overline{D}(f) = \lim D(f_n) < \infty$.
But $\overline{D}(f) = \lim D(f_n)$ then $-\underline{D}(f) = \overline{D}(-f) = \lim D(-f_n) = \lim\ (-D(f_n)) = -\lim\ D(f_n) = -\overline{D}(f)$.
Hence $\underline{D}(f) = \overline{D}(f)$ and is finite then f is in L^1.
($\Longleftarrow$) suppose that f is in L^1. Then there exists a sequence $\{g_n\}$ of functions in L^* with $g_n \uparrow f$ and such that $D(f) \leq D(g_n) < D(f) + {}^1\!/_n$.
But g_n in L^* then there exist a sequence $\{h_{n,m}\}$ of functions in L for each n, such that $h_{n,m} \uparrow g_n$ and $D(h_{n,m}) \uparrow D(g_n)$.
For each n, we choose f_n in L by $f_n = h_{n,m}$ for some m with

$$D(g_n) \geq D(f_n) = D(h_{n,m}) > D(g_n) - {}^1\!/_n.$$

Then f_n is in L for all n and for each n, $D(f - f_n) = D(f) - D(f_n) > D(g_n) - {}^1/_n - D(f_n) \geq D(g_n) - {}^1/_n - D(g_n) = - {}^1/_n$.

$D(f - f_n) = D(f) - D(f_n) \leq D(g_n) - D(f_n) < D(g_n) - D(g_n) + {}^1/_n = {}^1/_n$.

Therefore, $- {}^1/_n < D(f - f_n) < {}^1/_n$ so $D(f - f_n) \to 0$.

Consider $(f_n \vee f) - f$ in L^1 and $f_n \leq g_n$ implies that $f \leq f_n \vee f \leq g_n \vee f = g_n$.

Hence $0 \leq D\big((f_n \vee f) - f\big) \leq D(g_n) - D(f)$.

But $D(g_n) \downarrow D(f)$ and hence $D(g_n) - D(f) \downarrow 0$.

Thus $D\Big(\big((f_n \vee f) - f\big) - f\Big) \downarrow 0$.

Suppose x is in Ω. then

$\big((f_n \vee f) - f\big)(x) > 0 \; iff \; (f_n)(x) > f(x)$, and $f_n(x) \leq f(x)$

implies that $\big((f_n \vee f) - f\big) - f)(x) = f(x)$

and hence $\big((f_n \vee f) - f\big) - f)(x) = 0$.

Thus $(f_n \vee f) - f = (f_n - f)^+$.

Then $(f_n - f)^- = (f_n - f)^+ - (f_n - f) = (f_n - f)^+ + (f_n - f)$.

Hence $D((f_n - f)^-) \to 0$.

Thus, since $|f - f_n| = |f_n - f| = (f_n - f)^+ + (f_n - f)^-$,

And since $D(|f - f_n|) \to 0$. [1]

Theorem 5.13[1]

Let h is a real-valued function defined on Ω. and let $g \wedge h$ is in L for all functions g in L. Then $f \wedge h$ is in L^1 for all functions f in L^1.

Proof:

Suppose that f is in L^1. Then there exists a sequence $\{f_n\}$ of functions in L with $D(|f - f_n|) \to 0$.

And since f_n is in L then $f_n \wedge h$ is in L, and by the inequality

$0 \leq |(f \wedge h) - (f_n \wedge h)| \leq |f - f_n|$

and since $\overline{D}(|(f \wedge h) - (f_n \wedge h)| \leq |f - f_n|) \longrightarrow 0$.

Hence $f \wedge h$ is in in L^1. [1]

Theorem 5.14 [1]

The functional D is a Dniell integral on the vector lattice L^1.

Proof:

Since D is a positive linear functional and L^1 is a vector lattice

Let $\{f_n\}$ is an increasing sequence of functions in L^1, and f in L^1 with $f \leq \lim f_n$.

Let $g_n = f_n - f_1$ for all n. Then $D(f_n) = D(f_1) + D(g_n)$

then $\lim D(f_n) = D(f_1) + lim \; D(g_n)$.

Let $g = lim\, I(f_n) = f_1 + lim\, g_n = \lim f_n$.

Since $g_n \geq f$, and $lim g_n$ in L^1 and hence h in L^1. Then $f \leq \lim f_n = h$, so $D(f) \leq D(g) = D(f_1) + \lim D(g_n) = \lim D(f_n)$.

Therefore D is a Daniell integral. [1]

References

[1] E. M.Wadsworth, Daniell integral, University of Montana,1965.

[2] E. Blackstone and P. Mikusinski, The Daniell integral, arXiv:1401.0310v1[math CA] 1 Jan, 2014.

[3] R. M.Ingle, The Daniell approach to integration and measure, The faculty of the graduate division,1969.

[4] S.J. Taylor, Introduction to Measure and integration, Cambridge Press, 1972.

[5] R.B. Ash, Real Analysis and Probability, Academic Press, INC, University of Illinois, 1972.

[6] R.E. Shermoen, An Introduction to General Integrals, the faculty of the Oklahoma state university, 1965.

[7] N.F. AlMayahi, Measure Theory And Applications, Deposit number in the National Library and Archiver in Baghdad (1780). First edition, Iraq, 2018.