New largest bounds of (d, m)-arcs in projective plane of order seventy-nine

Authors Names	ABSTRACT
Salam Abdulqader Falih Alabdullah ***Article History*** *published on: 25/8 / 2023* ***Keywords:*** *Finite projective plane, Arcs in projective plane of order ω.*	A projective plane is defined as a structural geometry comprising points and lines related through a specific mathematical relationship. As a case study, a projective plane of order (ω) is considered in this research, including a number of points signified by a quadratic equation $(\omega^2 + \omega + 1)$ and a number of lines signified by the same equation and represented by $\mathrm{PG}(2, \omega)$. Each line can convey $(\omega + 1)$ points, and each point can pass through $(\omega + 1)$ lines. The blocking set S is defined by a group of points, where every single line can have at least ℓ points of S, and other lines can have exactly ℓ number of points of S. It is worth noting that the blocking set S is a complementary part of a (d, m)-arc D taking into account that $\ell = \omega + 1 - \mathrm{m}$. In short, this study aims to prove that (d, m)-arcs are not existing at ω equal to seventy nine.

1- Introduction

A projective plane is regarded in this study, including $\omega^2 + \omega + 1$ points and $\omega^2 + \omega + 1$ lines. For each line, an exact number of points is identified by the equation $\omega + 1$. The ℓ-fold blocking set S is recognized through a group of points in $PG(2, \omega)$. The associated lines can involve ℓ points as a minimum number in S, and some lines can have exactly ℓ number of points in S. Significantly, there is no line in the blocking set $PG(2, \omega)$. Despite that, it still represents the complementary set of a (d, m)-arc D in $PG(2, \omega)$, where $\ell = \omega + 1 - m$. It is also worth mentioning that the most miniature blocking set, called trivial, comprises only lines.

The blocking set was first explored by Di Paola in 1969 [14]. The author worked on a non-trivial blocking set in $PG(3, \omega)$ to determine the smallest size. Bruen [10], [11] in 1970 and 1971, respectively, demonstrated that $|S| \geq \left(\omega + \sqrt{\omega} + 3\right)$ corresponding to any non-trivial blocking set. Bruen and Thas [12] in 1977 proved that it is essential for the blocking set of $(\omega + \sqrt{\omega} + 2)$ points to have a Baer sub plane, where $|S| \geq \left(\omega + \sqrt{\omega} + 3\right)$ will detain. It was deduced in 1994 [7] that in $PG(2,11)$, the largest complete arcs are (78,8) and (90,9), whereas the arcs of the size (106,9), (110,10), (134,11) do not exist for $PG(2,13)$. Ball [8] discovered a new lower bound for $\omega < 11$ in 1996. Diskalov [13] started by studying $PG(2,17)$ in 2004. Alabdullah [1] classified (k,3)-arcs in projective plane of order eight and found the maximal arcs which is 15. Then Alabdullah [2] and Alabdullah and Hirschfeld [3] came up with new upper and lower bounds for $PG(2, \omega)$ in 2018 and 2019, respectively. In 2021, Alabdullah and Hirschfeld [4] reached new upper bounds $\left(\beta_m(2, \omega) \leq \frac{1}{2}(\omega + 1)(2m - 3)\right)$ in $PG(2, \omega)$ corresponding to some values of ω. After that, Alabdullah [5], [6] found new largest bounded of (k, n)-arcs in $PG(2,73)$ and classified $(k, 4)$-arcs in $PG(2,8)$. In this article, new upper bounds have been concluded for (d, m)-arcs in $PG(2,79)$.

2. Background

Definition 2.1. [15]

A set D including d points is a (d, m)-arc in projective plane of order ω. In this regard, no $m + 1$ of this set is collinear, but one set or more of m points are collinear. A $(d, 2)$-arc is named as a d-arc. A (d, m)-arc is considered complete if it is not involved within a $(d + 1, m)$-arc.

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Theorem 2.2. (Ball [8]) Assume that D represents (d, m)-arc in projective plane of order ω. Here, ω is prime.

1. If m $< \frac{\omega+1}{2}$, then $d \leq (\ell - 1)\omega + 1$.
2. Ifm $> \frac{\omega}{2}$, then $d \leq (\ell - 1)\omega + \ell - (\omega + 1)/2$.

Theorem 2.3. (Ball [8]) Assume that S represents ℓ -fold blocking set in projective plane of order ω. Here, ω prime and greater than three.

1. If $\ell < \omega/2$, then $|S| \geq (\ell + \frac{1}{2})(\omega + 1)$.
2. If $\ell > \omega/2$, then $|S| \geq (\ell + 1)\omega$.

Theorem 2.4. (Ball [9]) Assume that S represents a ℓ -fold blocking set in projective plane of order ω, in which the line is included.

1. If $(\ell - 1, \omega) = 1$, then $|S| \geq \omega(\ell + 1)$.
2. If $(\ell - 1, \omega) > 1$ and $\ell \leq \frac{\omega}{2} + 1$, then $|S| \geq \ell\omega + \omega - \ell + 2$.
3. If $(\ell - 1, \omega) > 1$ and $\ell \geq \frac{\omega}{2} + 1$, then $|S| \geq \ell(\omega + 1)$.

Theorem 2.5. (Daskalov [13]). Let S an $\{h, \ell\}$-blocking set in projective plane of order ω, where ω is prime.

1. If $\ell < \frac{\omega}{2}$, and $\omega > 3$, then $h \geq \ell(\omega + 1) + (\omega + 1)/2$.
2. If $h = \ell(\omega + 1) + (\omega + 1)/2$, then
   a. An exact number of lines (ω +3)/2, which are not ℓ-secants, exists at every point of S.
   b. An exact number of lines (ω -3)/2, which are ℓ-secants, exists at every point of S.
   c. $\gamma = h(\omega - 1)/2\ell$ refers to the total number of ℓ-secants.

Lemma 2.6. ([15]). For any set of d points in projective plane of order ω, the following holds:

$$\sum_{j=0}^{r+1} \delta_j = \omega^2 + \omega + 1 \qquad \text{... ... ... (1).}$$
$$\sum_{j=1}^{r+1} j\delta_j = |S|(\omega + 1) \qquad \text{... ... ... (2).}$$
$$\sum_{j=2}^{r+1} j(j-1)\delta_j = |S|(|S| - 1) \qquad \text{... ... ... (3).}$$

Theorem 2.7. (Alabdullah [4]). For $(\omega + 3/2) < m < \omega$, with ω prime, $\beta_m(2, \omega) \leq \frac{(\omega+1)(2m-3)}{2}$.

Notation 2.8. For a (d, m)-arc D in $PG(2, \omega)$, assume

- δ_j = the total number of j-secants of D,
- ε_j = the number of j-secants through a point P of D,
- $\beta_m(2, \omega)$ = the maximum size of a (d, m)-arc in $PG(2, \omega)$.

3. New Largest Bound

In this section, there are some (d, m)-arcs proved not exist in projective plane of order seventy nine.

Theorem 3.1. No (d, m)-arc is available for the values of d in projective plane of order seventy nine, where the associated upper bounds are given for $\beta_m(2,79)$.

3.1.1. Case I: Bounds for the complete (d, m)-arcs corresponding to a non-integer value of γ:

d	3241	3321	3401	3481	3561	3641	3721	3801
m	42	43	44	45	46	47	48	49
$\beta_m(2,79) \leq$	3240	3320	3400	3480	3560	3640	3720	3800
d	3961	4041	4121	4281	4441	4521	4601	4761
m	51	52	53	55	57	58	59	61
$\beta_m(2,79) \leq$	3960	4040	4120	4280	4440	4520	4600	4760
d	4841	4921	5001	5161	5401	5561	5721	
m	62	63	64	66	69	71	73	
$\beta_m(2,79) \leq$	4840	4920	5000	5160	5400	5560	5720	

Table (1)

Proof.

The above problems can be proved based on either Theorem 2.3, Theorem 2.4 and Theorem 2.5 or Theorem 2.7.

The first arc will be proved based on Theorem 2.3, Theorem 2.4 and Theorem 2.5. While the second arc will be calculated by using Theorem 2.7.

For $d = 3241$ and $m = 42$.

The largest size of (3241,42)-arc in $PG(2,79)$ is equivalent to estimating the largest 38-fold blocking set. Theorem 2.3 provides that $\boldsymbol{S}$ must be larger than or equal to 3080 points. Theorem 2.5 provides $\gamma = (3080 * 78)/76$. It is clearly γ is not integer. Thus, a (3241,42)-arc does not exist and $\beta_{42}(2,79) \leq 3240$.

For $d = 3321$ and $m = 43$, then we have

$h = 3000,\ \ell = 37;$

$|S| = \ell(\omega + 1) + \frac{\omega+1}{2}$

$|S| = 37 * 80 + 40$

$|S| = 3000$

That mean $|\boldsymbol{S}| = h$. So, $\gamma = \frac{|S|(\omega-1)}{2\ell} = 3000 * \frac{78}{74}$.

It is clearly γ is not integer, therefore (3321,42)-arc does not exist. So, $\beta_{43}(2,79) \leq 3320$.

The other cases are demonstrated based on the same way.

3.1.2. Case II: Bounds for the complete (d, m)-arcs corresponding to an integer value of γ:

d	3881	4201	4361	4681	5081	5241	5321
m	50	54	56	60	65	67	68
$\beta_m(2,79) \leq$	3880	4200	4360	4680	5080	5240	5320
d	5481	5641	5801	5881	6961	6041	6121
m	70	72	74	75	76	77	78
$\beta_m(2,79) \leq$	5480	5640	5800	5880	5960	6040	6120

Table (2)

Proof.

Estimating the largest size of (3881,50)-arc is analogous to calculating the largest $\{2440,30\}$-blocking set $\boldsymbol{S}$. Since , $\ell = 30$Theorem 2.5 provides γ is 3172. Assume that α is the length of the longest secant. If $\alpha = 80$, then S has a line. However, Theorem 2.4 gives $|\boldsymbol{S}| \geq 2449$. If $71 \leq \alpha \leq 79$, then taking into account there are lines passing through a point on the longest secant, but not in $\boldsymbol{S}$. Therefore, $\boldsymbol{S}$ must be larger or equal to $30 * 79 + \alpha$ points. This is impossible with $|\boldsymbol{S}| = 2440$.

In the case of taking the intersection of the 30-secants through $P \notin \boldsymbol{S}$ with the longest secant, so,

$\varepsilon_{30} \geq 39\ \alpha + (80 - \alpha)(79 - j)$ ... ... 4.

The values of ε_{30} are calculated from Equation 4 for $j \leq 38$ and give the below table.

$\boldsymbol{\alpha}$	70	69	68	67	66	65	64	63	62	61	60	59	58
$\boldsymbol{j}$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\boldsymbol{\varepsilon_{30}}$	3520	3549	3576	3601	3624	3645	3664	3681	3696	3709	3720	3729	3736
$\boldsymbol{\alpha}$	57	56	55	54	53	52	51	50	49	48	47	46	45
$\boldsymbol{j}$	13	14	15	16	17	18	19	20	21	22	23	24	25
$\boldsymbol{\varepsilon_{30}}$	3741	3744	3745	3744	3741	3736	3729	3720	3709	3696	3681	3664	3645
$\boldsymbol{\alpha}$	44	43	42	41	40	39	38	37	36	35	34	33	32
$\boldsymbol{j}$	26	27	28	29	30	31	32	33	34	35	36	37	38
$\boldsymbol{\varepsilon_{30}}$	3624	3601	3576	3549	3520	3489	3456	3421	3384	3345	3304	3261	3216

Table (3)

Table (3) illustrates that all values of α for $\alpha = 32, 21, \ldots, 70$ show a contrast. This is because the total number of 30-seants is 80. However, for $\alpha = 30$ and 31, now by using Equations 1, 2 and 3 of Lemma 2.1 get the following:

$\varepsilon_{30} + \varepsilon_{31} = 6321;$

$30\,\varepsilon_{30} + 31\varepsilon_{31} = 195200;$

$870\varepsilon_{30} + 930\varepsilon_{31} = 5951160.$

There is no solution for this system. Therefore, no (3881,50)-arc exists.

The other cases can be proved in the same method.

4. Conclusion

In this paper, new some $\beta_m(2,\omega)$ are proved in the largest (d,m)-arcs in $PG(2,79)$.

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