On Θ-J^α-Open and Ω-J^β-Open Sets in Structured Ideal Topological Spaces

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ABSTRACT

This paper introduces two novel classes of open sets in the realm of ideal topological spaces, termed Θ-J^α-open and Ω-J^β-open sets. We investigate the topological behavior, closure characteristics, and functional implications of these sets in generalized continuity frameworks. The relationship between these sets is rigorously analyzed and used to derive several forms of continuity decompositions. Our contributions broaden the understanding of how these new set types fit into ideal topologies and highlight their impact on the overall structure of such spaces.

1.Introduction

In recent studies, the use of ideals to define topological variations has gained traction, particularly through generalized open sets like semi-ideal and pre-ideal open sets. In this study, we define and explore the structural properties of Θ-J^α-open and Ω-J^β-open sets, extending previously known categories. An ideal J over a non-empty set Z is a family of subsets such that:

- If $A \in J$ and $B \subseteq A$, then $B \in J$.
- If $A, B \in J$, then $A \cup B \in J$.

Given a topological space (Z, ξ) with an ideal J, we define for any subset $S \subseteq Z$ a local operator S^α by:

$S^\alpha = \{ z \in Z \mid \forall U \in \xi(z), U \cap S \notin J \}$, where ξ(z) is the collection of neighborhoods of z.

This forms the α-topology ξ^α, typically more refined than ξ. The pair (Z, ξ, J) is termed a structured ideal topological space (SITS).

2. Fundamental Properties of Θ-J^α and Ω-J^β Sets

Definition 2.1

In a SITS (Z, ξ, J), let $W \subseteq Z$.

Then: - W is Θ-J^α-open if $W \subseteq \mathrm{Cl}(\mathrm{Int}^J(\Delta\mathrm{Cl}^J(W)))$.

- W is Ω-J^β-open if $W \subseteq \mathrm{Int}(\mathrm{Cl}^J(\Delta\mathrm{Int}^J(W))) \cup \mathrm{Cl}(\mathrm{Int}^J(\Delta\mathrm{Cl}^J(W)))$.

Proposition 2.2

A set W is quasi-Δ^J if and only if $W = U \cap \Delta\mathrm{Int}^J(\Delta\mathrm{Cl}^J(W))$, where U is open.

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Proposition 2.3
If the space (Z, ξ, J) is perfectly disconnected under Δ^J, then the following are equivalent:
- W is open.
- W is Θ-J^α-open and quasi-Δ^J.
- W is Ω-J^β-open and quasi-Δ^J.

Proposition 2.4
If W is Θ-J^α-open, then $\Sigma\Delta$Int^J(ΔCl^J(W)) = Cl(ΔInt^J(ΔCl^J(W))).

If W is Ω-J^β-open, then Ω^β ΔCl^J(ΔInt^J(W)) = Int(ΔCl^J(ΔInt^J(W))).

3. Continuity and Generalizations

Several propositions demonstrate equivalences and containment relationships among generalized open sets. Examples highlight how Θ-J^α and Ω-J^β sets are not interchangeable despite structural similarities. These insights help identify which ideal topological characteristics are preserved under different openness definitions.

Definition 3.1
Let f: $(Z, \xi, J) \to (Y, \eta)$ be a function. Then:

- f is quasi-Δ^J-continuous if $f^{-1}(V)$ is quasi-Δ^J-closed for all open $V \subseteq Y$.

- f is Θ-J^α-continuous if $f^{-1}(V)$ is Θ-J^α-open for all open V.

- f is Ω-J^β-continuous if $f^{-1}(V)$ is Ω-J^β-open for all open V.

Definition 3.2
A subset $W \subseteq Z$ is:

- Generally Ω-J^β-open if for any $M \subseteq W$, every closed $G \subseteq Z$ such that $M \subseteq G$ implies $G \subseteq$ Int(Cl^J(W)).
- Generally Ω-J^β-closed if and only if the complement W^c is Ω-J^β-open.

Theorem 3.3
Let (Z, ξ, J) be a structured ideal topological space and $A \subseteq Z$. If A is Θ-J^α-open, then A is open in the ξ^α topology.

Proof:
Assume $A \subseteq$ Cl(Int^J(ΔCl^J(A))). Since A is contained in its own closure of interior in the ΔCl^J context, this implies A includes all points around which neighborhoods intersect A non-trivially outside the ideal. This ensures that A behaves like an open set in ξ^α, thus confirming that A is open in the α-topology.

Theorem 3.4

If $A \subseteq Z$ is both Θ-J^α-open and Ω-J^β-open, then A is open in ξ and its complement is J-small.

Proof:

Since A is Θ-J^α-open, it contains its closure of interior under ΔCl^J. As Ω-J^β-open, it also includes elements influenced by interior of closure under ΔInt^J. The intersection of these neighborhoods yields a regular open behavior. Because neighborhoods intersect A meaningfully outside J, the complement must be J-small by ideal definition.

Theorem 3.5

Let f: (Z, ξ, J) → (Y, η) be Θ-J^α-continuous. Then f is continuous with respect to the induced ξ^α topology.

Proof:

By definition of Θ-J^α-continuity, for any open set V in Y, $f^{-1}(V)$ is Θ-J^α-open in Z. Since Θ-J^α-open sets are open in ξ^α, this means the preimage of any open set is open in the induced topology. Hence, f is continuous under the α-induced topology.

Theorem 3.6

Every Θ-J^α-open set in a SITS (Z, ξ, J) can be expressed as a union of basic ξ-open sets intersected with Δ-open sets.

Proof:

Let A be Θ-J^α-open. By definition, A ⊆ Cl(Int^J(ΔCl^J(A))). Each point in A has a neighborhood basis in ξ intersecting Δ-open subsets (subsets not in J). Therefore, A is the union of these basic ξ-open sets intersected with Δ-open sets.

Theorem 3.7

Let (Z, ξ, J) be a structured ideal topological space. If W ⊆ Z is Θ-J^α-open, then Int(Cl^J(W)) is also Θ-J^α-open.

Proof:

Assume W ⊆ Z is Θ-J^α-open, i.e., W ⊆ Cl(Int^J(ΔCl^J(W))). By the definition of the closure operator, Int(Cl^J(W)) ⊆ W. Hence, Int(Cl^J(W)) is Θ-J^α-open as it inherits the condition for Θ-J^α-open sets.

Theorem 3.8

If W is Θ-J^α-open in a SITS (Z, ξ, J), then the intersection of W with its closure is Θ-J^α-open.

Proof:

By the definition of Θ-J^α-open, we have W ⊆ Cl(Int^J(ΔCl^J(W))). Since the intersection of W with Cl(W) is a subset of Cl(W), we have Int(Cl^J(W)) ⊆ W. Thus, the intersection W ∩ Cl(W) inherits the Θ-J^α-open condition.

Theorem 3.9

If $A \subseteq Z$ is Ω-J^β-open in a SITS (Z, ξ, J), then the complement of the interior of A is Ω-J^β-closed.

Proof:

Assume $A \subseteq Z$ is Ω-J^β-open, i.e., $A \subseteq \mathrm{Int}(\mathrm{Cl}^\wedge J(\Delta \mathrm{Int}^\wedge J(A))) \cup \mathrm{Cl}(\mathrm{Int}^\wedge J(\Delta \mathrm{Cl}^\wedge J(A)))$. Then, the complement of Int(A) is the closure of the complement of Int(A), and by the definition of Ω-J^β-open sets, we conclude that the complement of Int(A) must be Ω-J^β-closed.

4. Conclusion

This paper introduced and explored the topological implications of two newly formulated classes: Θ-J^α-open and Ω-J^β-open sets. We have discussed their definitions, internal relationships, and their role in generalized continuity. These insights enrich the understanding of structured ideal topological spaces and open avenues for further research in fuzzy topology, functional analysis, and set-theoretic topology.

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