Journal of Iraqi Al-Khwarizmi (JIKh) Volume:9 Issue:1 Year: 2025 pages: 35-47

Orthogonal Generalized Higher Symmetric Reverse Bi-Derivations on Semiprime Γ- Rings

Authors Names	**ABSTRACT**
Jafar Salih Aneed *Salah Mehdi Salih* ***Article History*** Publication date: 1/ 4 /2025 ***Keywords:*** Symmetric Bi-Derivation, Symmetric Reverse Bi-Derivation, Higher Symmetric Reverse Bi-Derivations, Generalized Higher Symmetric Reverse Bi-Derivations and Orthogonal.	The purpose of this paper is to study the concept of orthogonal generalized higher symmetric reverse bi- derivation on semiprime Γ-ring. We study some lemmas and theorems of orthogonality on semiprime Γ-rings. We prove that if M is a 2-tortion free semiprime Γ-ring then D_n and G_n are orthogonal generalized higher symmetric reverse bi-derivations associated with higher symmetric reverse bi-derivations $d_n$and g_n for all n∈N. Then the following relations are hold for all a,b,c∈M , $\alpha \in \Gamma$ and n∈N: i. $D_n(a,b)\alpha G_n(b,c) = G_n(b,c)\alpha D_n(a,b) = 0$ hence $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b)$ =0 ii. d_n and G_n orthogonal and $d_n(a,b)\alpha G_n(b,c)$=$G_n(b,c)\alpha d_n(a,b) = 0$. iii. g_n and D_n orthogonal and $g_n(a,b)\alpha D_n(b,c)$=$D_n(b,c)\alpha g_n(a,b) = 0$. iv. $d_nG_n = G_nd_n = 0$ and $g_nD_n = D_ng_n = 0$ v. $G_nD_n = D_nG_n = 0$.

1. Introduction

If M and Γ be two additive abelian groups, then M is said to be Γ-ring if:
1. $a\alpha b \in M$ 2. $a\alpha(b + c) = a\alpha b + a\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $(a\alpha b)\beta c = a\alpha(b\beta c)$ for any $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Nobusawa 1964 [11] was the first to present the idea of Γ-ring and generalized by Barnes 1966 [3] as above. The Γ-ring M is said to be prime if aΓMΓb=0 implies that a=0 or b=0 and its called semiprime if aΓMΓa=0 implies that a=0 for each a,b∈M, also M is called n-torsion free (n-tf) if na =0 for all a∈M implies that a=0 where n is positive integer [7]. Ozturk et al. [12] introduced his definition as follows: The mapping d:M×M⟶M is called symmetric if d(a,b)=d(b,a) for all a,b ∈M. Jing in [6] introduced his definition as follows: The additive mapping d:M⟶M is called derivation on Γ-ring M if d(aαb)=d(a)αb+aαd(b) for each a,b∈M and α∈Γ.

Ozturk et al. [12] its definintion is as follows: A symmetric bi-additive mapping d:M×M⟶M is called symmetric bi-derivation on M if d(aαb,c)=d(a,c)bα+aαd(b,c) for each a,b,c∈M and α∈Γ. C. Jaya Subba Reddy in [4] its definintion is as follows: The reverse bi-additive mapping d:R×R⟶R is called symmetric reverse bi-derivation on prime ring R if satisfies the identity d(ab,c)=d(b,c)a+bd(a,c) and d(a,bc)=d(a,c)b+cd(a,c) for all a,b,c∈R. In [5] Ceven and Ozturk their definition as follows: Let D be additive mapping of M, then D is called generalized derivation on M if there exists a derivation d from M into M such that D(aαb)=D(a)αb+aαd(b) for every a , b ∈M and α∈Γ. Marir and Salih in [8] are introduced their concept as follows: If d = $(d_i)_{i\in N}$ be a family of bi-additive mapping on M×M into M is called higher bi-derivation if $d_n(a\alpha b, c\alpha d) = \sum_{i+j=n} d_i(a,c)\alpha d_j(b,d)$ for every a,b,c ∈ M and α∈Γ. .H.Majeed and S.M.Salih in [1] are introduced the definition on a generalized higher derivation A as follows: Let M be a Γ-ring, $D_n = (D_i)_{i\in N}$ be a family of additive mappings of M such that for all

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$n \in N$, $a, b \in M$ and $\beta \in \Gamma$, we have: $D_n(a\beta b) = \sum_{i+j=n} D_i(a)\beta d_j(b)$

Ashraf M. and Jamal M. in [2] introduced the definition of orthogonal derivation on Γ-ring as follows: Let d and g be two derivations from M into M, then d and g are said to be orthogonal if $d(a)\Gamma M\Gamma g(b) = (0) = g(b)\Gamma M\Gamma d(a)$ for all a,b,c∈M.

Majeed A.H. and Suliman N.N. in [10] present the definition of orthogonal generalized derivation on $\Gamma -$ ring M as follows: Let D and G be two generalized derivations from M into M are said to be orthogonal if: $D(a)\Gamma M\Gamma G(b) = (0) = G(b)\Gamma M\Gamma D(a)$ for all a,b ∈ M

Salih S.M. and Aneed J.S. [13]are presented and studied the concept of orthogonal on semiprime $\Gamma -$ ring M as follows: Let D and G are two generalized higher symmetric reverse bi-derivations(ghsrb-d) and generalized higher symmetric bi-derivations(ghsb-d) on M, then D and G are called orthogonal if for all a,b,c ∈M and n∈N:

$D_n(a,b)\Gamma M\Gamma G_n(b,c)=(0)=G_n(b,c)\Gamma M\Gamma D_n(a,b)$

2. Orthogonal Generalized Higher Symmetric Reverse Bi-Derivations on Semiprime Γ-ring

We will introduce the definition of orthogonality and some lemmas related to inductance and help prove some theorems.

Definition(2.1):

Let $D = (D_i)_{i\in N}$ and $G = (G_i)_{i\in N}$ are two generalized higher symmetric reverse bi-derivations(ghsrb-d) on a semiprime $\Gamma -$ring M, then D and G are called orthogonal if for all $a, b, c \in M$ and n∈N, then

$$D_n(a, b)\Gamma M\Gamma G_n(b, c) = (0)= G_n(b, c)\ \Gamma M\ \Gamma\ D_n(a, b)$$

where $D_n(a, b)\Gamma M\Gamma G_n(b, c) = \sum_{i=1}^{n} D_i(a, b)\alpha m\beta G_i(b, c)$ for all m ∈M and α, β∈Γ

Example(2.2):

Let d_n and g_n are two higher symmetric reverse bi-derivations (hsrb-d) on a Γ-ring M for all n ∈N. Let $M' = M\times M$ and $\Gamma' = \Gamma\times\Gamma$ we define d'_n and g'_n by $d'_n: M' \times M' \rightarrow M'$ and $g'_n: M' \times M' \rightarrow M'$ such that $d'_n((a, b), (c, d)) = (d_n(a, b),0)$ and $g'_n((a, b), (c, d)) = (0, g_n(c, d))$ for all $(a, b), (c, d) \in M'$. Then d'_n and g'_n are (hsrb-d). Moreover, if (D_n , d_n) and (G_n , g_n) are two (ghsrb-d) on M, we defined D'_n and G'_n on M' such that $D'_n((a, b), (c, d)) = (D_n(a, b),0)$ and $G'_n((a, b), (c, d)) = (0, G_n(c, d))$ for all $(a, b), (c, d) \in M$. Then (D'_n, d'_n) and (G'_n , g'_n) are two (ghsrb-d) on M' such that D'_n and G'_n are orthogonal.

Lemma(2.3): [5]

Let M be a 2-tfsp Γ-ring and a,b ∈ M, then the following conditions are equivalent for each α,β∈Γ:

1. $a\alpha M\beta b = 0$
2. $b\alpha M\beta a = 0$
3. $a\alpha M\beta b + b\alpha M\beta a = 0$

If one of these conditions is fulfilled, then $a\alpha b=b\alpha a=0$

Lemma(2.4): [4]

Let M be a 2-tfsp Γ-ring and a,b ∈M such that $a\alpha M\beta b+b\alpha M\beta a=0$ for every α,β∈Γ, then $a\alpha M\beta b=b\alpha M\beta a=0$.

Lemma(2.5):

Assume that D_n and G_n are bi-additive mappings on a semiprime Γ-ring M satisfies $D_n(a, b)\Gamma M\Gamma G_n(a, b) = (0)$, then $D_n(a, b)\Gamma M\Gamma G_n(b, c) = (0)$ for every a, b, c∈M and n∈N.

Proof:

By assumption $D_n(a, b)\Gamma M\Gamma G_n(a, b) = (0)$ then

$D_n(a,b)\Gamma M\Gamma G_n(a,b) = \sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(a,b) = 0$ (1)

Replace a by $a + c$ in (1) for every c∈M then

$$\sum_{i=1}^{n} D_i(a+c,b)\alpha m\beta G_i(a+c,b) = 0$$

$$\sum_{i=1}^{n}(D_i(a,b) + D_i(c,b))\alpha m\beta(G_i(a,b) + G_i(c,b)) = 0$$

$$\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(a,b) + D_i(a,b)\alpha m\beta G_i(c,b) + D_i(c,b)\alpha m\beta G_i(a,b) + D_i(c,b)\alpha m\beta G_i(c,b) = 0$$

By equation (1) we get $\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b) + D_i(c,b)\alpha m\beta G_i(a,b) = 0$

$$\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b) = -\sum_{i=1}^{n} D_i(c,b)\alpha m\beta G_i(a,b) \qquad (2)$$

Multiplication the equation (2) by $\gamma t\delta \sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b)$ for each t ∈M and $\gamma, \delta \in \Gamma$ we get

$\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b)\, \gamma t\delta \sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b) = 0$

Since M is semiprime Γ-ring, we get

$$\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(c,b) = 0 \qquad (3)$$

Replace $G_i(c,b)$ by $G_i(b,c)$ in (3) we get $\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(b,c) = 0$

Hence $D_n(a,b)\, \Gamma M\Gamma\, G_n(b,c) = (0)$

Lemma (2.6)

Let M be a 2-tfsp Γ-ring. If D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for all n∈N, then D_n and G_n are orthogonal if and only if $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = (0)$ for all a, b, c∈M , n∈N and α, β∈Γ.

Proof:

Suppose that $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = (0)$

$$\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c) + G_i(b,c)\alpha D_i(a,b) = 0 \qquad (1)$$

Replace a by wβa in (1) for all w ∈ M we get

$$\sum_{i=1}^{n} D_i(w\beta a,b)\alpha G_i(b,c) + G_i(b,c)\alpha D_i(w\beta a,b) = 0$$

$$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha G_i(b,c) + G_i(b,c)\alpha D_i(a,b)\beta d_i(w,b) = 0 \qquad (2)$$

Replace $\alpha D_i(a,b)\beta d_i(w,b)$ by $\beta d_i(w,b)\alpha D_i(a,b)$ in (2) we get

$$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha G_i(b,c) + G_i(b,c)\beta d_i(w,b)\alpha D_i(a,b) = 0$$

By lemma (2.4) we get

$$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha G_i(b,c) = \sum_{i=1}^{n} G_i(b,c)\beta d_i(w,b)\alpha D_i(a,b) = 0 \qquad (3)$$

Replace $d_i(w,b)$ by m in (3) for all m∈M we get

$D_n(a,b)\, \Gamma M\Gamma\, G_n(b,c) = (0) = G_n(b,c)\, \Gamma M\Gamma\, D_n(a,b)$

Thus D_n and G_n are orthogonal

Now, assume that D_n and G_n are orthogonal

$D_n(a,b)\, \Gamma M\Gamma\, G_n(b,c) = (0) = G_n(b,c)\, \Gamma M\Gamma\, D_n(a,b)$

$\sum_{I=1}^{n} D_i(a,b)\alpha m\beta G_i(b,c) = 0 = \sum_{i=1}^{n} G_i(b,c)\alpha m\beta D_i(a,b)$ $\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(b,c) + G_i(b,c)\alpha m\beta D_i(a,b) = 0$

By lemma (2.3) we get $\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c) = \sum_{i=1}^{n} G_i(b,c)\alpha D_i(a,b) = 0$

$$\sum_{I=1}^{n} D_i(a,b)\alpha G_i(b,c) + \sum_{i=1}^{n} G_i(b,c)\alpha D_i(a,b) = 0$$

Thus $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = 0$

Lemma(2.7)

If M be a 2-tfsp Γ-ring. Let D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for all $n\epsilon N$. Then D_n and G_n orthogonal iff $D_n(a,b)\alpha G_n(b,c) = (0)$ or $G_n(b,c)\alpha D_n(a,b) = 0$ for all $a,b,c\epsilon M$ and $\alpha,\beta\epsilon\Gamma$.

Proof :

Suppose that $D_n(a,b)\alpha G_n(b,c) = (0)$

$$D_n(a,b)\alpha G_n(b,c) = \sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c) = 0 \qquad (1)$$

Replace a by wβa in (1) for all $w \in M$ we get $\sum_{i=1}^{n} D_i(w\beta a,b)\alpha G_i(b,c) = 0$

$$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha G_i(b,c) = 0 \qquad (2)$$

Replace $d_i(w,b)$ by m for all $m\in M$ in (2) we get $\sum_{i=1}^{n} D_i(a,b)\beta m\alpha G_i(b,c) = 0$

Then D_n and G_n orthogonal

Now, if $G_n(b,c)\alpha D_n(a,b) = (0)$. Then D_n and G_n orthogonal.

Conversely, assume that D_n and G_n orthogonal

Then $D_n(a,b)\,\Gamma M\Gamma\, G_n(b,c) = (0)$ implies that $\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(b,c) = 0$

By lemma (2.3) we get $\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c) = 0$

Hence $D_n(a,b)\alpha G_n(b,c) = 0$

And by $G_n(b,c)\Gamma M\Gamma D_n(a,b) = \sum_{i=1}^{n} G_i(b,c)\alpha m\beta D_i(a,b) = 0$

By lemma(2.3) we get $\sum_{i=1}^{n} G_i(b,c)\alpha D_i(a,b) = 0$

Hence $G_n(b,c)\alpha D_n(a,b) = 0$

Lemma (2.8)

If M be a 2-tfsp Γ-ring. Let D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for all $n\epsilon N$.Then D_n and G_n orthogonal iff $D_n(a,b)\alpha g_n(b,c) = 0$ or $d_n(a,b)\alpha G_n(b,c) = 0$ for all $a,b,c\epsilon M$ and $\alpha,\beta\epsilon\Gamma$.

Proof :

Suppose that $D_n(a,b)\alpha g_n(b,c) = (0)$

$$D_n(a,b)\alpha g_n(b,c) = \sum_{i=1}^{n} D_i(a,b)\alpha g_i(b,c) = 0 \qquad (1)$$

Replace a by wβa in (1) for all $w \in M$ we get

$$\sum_{i=1}^{n} D_i(w\beta a,b)\alpha g_i(b,c) = 0$$

$$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha g_i(b,c) = 0 \qquad (2)$$

Replace $g_i(b,c)$ by $G_i(b,c)$ and $\beta d_i(w,b)\alpha$ by $\alpha m\beta$ for all $m\in M$ in (2) we get
$\sum_{i=1}^{n} D_i(a,b)\alpha m\beta G_i(b,c)=0$
By lemma(2.3) we get $\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c)=0$
$D_n(a,b)\alpha G_n(b,c)=0$
By lemma(2.7) we get D_n and G_n orthogonal .
Similarly way if $d_n(a,b)\alpha G_n(b,c)=0$ we get D_n and G_n are orthogonal
Conversely, assume that D_n and G_n are orthogonal.
By lemma(2.7) we get $D_n(a,b)\alpha G_n(b,c)=0$

$$\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c)=0 \qquad (3)$$

Replace a by $w\beta a$ in (3) we get $\sum_{i=1}^{n} D_i(w\beta a,b)\alpha G_i(b,c)=0$
$\sum_{i=1}^{n} D_i(a,b)\beta d_i(w,b)\alpha G_i(b,c)=0$ (4)
Replace $\beta d_i(w,b)\alpha G_i(b,c)$ by $\alpha d_i(w,b)\beta g_i(b,c)$ in (4) we get
$\sum_{i=1}^{n} D_i(a,b)\alpha d_i(w,b)\beta g_i(b,c)=0$
By lemma(2.3) we get $\sum_{i=1}^{n} D_i(a,b)\alpha g_i(b,c)=0$
Hence $D_n(a,b)\alpha g_n(b,c)=(0)$
Also replace a by $a\beta w$ in (3) we get $\sum_{i=1}^{n} D_i(a\beta w,b)\alpha G_i(b,c)=0$

$$\sum_{i=1}^{n} D_i(w,b)\beta d_i(a,b)\alpha G_i(b,c)=0 \qquad (5)$$

Left multiplication the equation (5) by $d_i(a,b)\alpha G_i(b,c)\delta$ for all $\delta\in\Gamma$ we get
$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c)\delta D_i(w,b)\beta d_i(a,b)\alpha G_i(b,c)=0$
Since M is semiprime, then $\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c)=0$
Hence $d_n(a,b)\alpha G_n(b,c)=(0)$

Lemma(2.9)

Let M be a 2-tfsp Γ-ring. If D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for every $n\epsilon N$. Then D_n and G_n orthogonal iff $D_n(a,b)\alpha G_n(b,c)=d_n(a,b)\alpha G_n(b,c)=0$ for all $a,b,c\epsilon M$ and $\alpha\epsilon\Gamma$.

Proof:

Assume that D_n and G_n orthogonal
From lemma(2.7) we get $D_n(a,b)\alpha G_n(b,c)=0$ (1)
And by lemma (2.8) we get $d_n(a,b)\alpha G_n(b,c)=0$ (2)
From (1) and (2) we get $D_n(a,b)\alpha G_n(b,c)=d_n(a,b)\alpha G_n(b,c)=0$
Conversely, suppose that $D_n(a,b)\alpha G_n(b,c)=0$
By lemma(2.7) we get D_n and G_n are orthogonal
Now, if $d_n(a,b)\alpha G_n(b,c)=0$
By lemma(2.8) we get D_n and G_n are orthogonal

3. Main Theorems

We will present and study some basic theorems for orthogonality on Γ-ring M.

Theorem(3.1):

If M be a 2-tfsp Γ-ring. Let D_n and G_n are orthogonal associated with two (hsrb-d) d_n and g_n respectively for every $n\epsilon N$.Then the following relations are hold for every $a,b,c\epsilon M$ and $\alpha,\beta\epsilon\Gamma$

i. $D_n(a,b)\alpha G_n(b,c)=G_n(b,c)\alpha D_n(a,b)=0$

hence $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = 0$

ii. d_n and G_n orthogonal and $d_n(a,b)\alpha G_n(b,c)$=$G_n(b,c)\alpha d_n(a,b) = 0$.

iii. g_n and D_n orthogonal and $D_n(a,b)\alpha g_n(b,c) = g_n(b,c)\alpha D_n(a,b) = 0$.

iv. d_n and g_n are orthogonal (hsrb-d).

v. $d_n G_n = G_n d_n = 0$ and $g_n D_n = D_n g_n = 0$

vi. $G_n D_n = D_n G_n = 0$.

Proof: (i)

Since D_n and G_n orthogonal and by lemma(2.7) we get

$D_n(a,b)\alpha G_n(b,c) = 0$ and $G_n(b,c)\alpha D_n(a,b) = 0$

$D_n(a,b)\alpha G_n(b,c) = G_n(a,b)\alpha D_n(b,c) = 0$

Hence $D_n(a,b)\alpha G_n(b,c) + G_n(a,b)\alpha D_n(b,c) = 0$

Proof: (ii)

Since D_n and G_n orthogonal, then

By lemma(2.8) we get $d_n(a,b)\alpha G_n(b,c) = 0$ (1)

$$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c) = 0 \quad (2)$$

Replace a by $w\beta a$ in (2) for all $w \epsilon$M and $\beta \in \Gamma$ we get

$\sum_{i=1}^{n} d_i(w\beta a,b)\alpha G_i(b,c) = 0$

$\sum_{i=1}^{n} d_i(a,b)\beta d_i(w,b)\alpha G_i(b,c) = 0$ (3)

Replace $d_i(w,b)$ by m in (3) for all mϵM we get

$\sum_{i=1}^{n} d_i(a,b)\beta m\alpha G_i(b,c) = 0$ (4)

From (i) we have $G_n(b,c)\alpha D_n(a,b) = 0$

$\sum_{i=1}^{n} G_i(b,c)\alpha D_i(a,b) = 0$ (5)

Replace a by $a\beta w$ in (5) we get $\sum_{i=1}^{n} G_i(b,c)\alpha D_i(a\beta w,b) = 0$

$\Rightarrow \sum_{i=1}^{n} G_i(b,c)\alpha D_i(w,b)\beta d_i(a,b) = 0$

By lemma (2.3) we get $\sum_{i=1}^{n} G_i(b,c)\alpha d_i(a,b) = 0$

$G_n(b,c)\alpha d_n(a,b) = 0$ (6)

In the equation $\sum_{i=1}^{n} G_i(b,c)\alpha d_i(a,b) = 0$ replace a by $a\beta w$ we get

$\sum_{i=1}^{n} G_i(b,c)\alpha d_i(a\beta w,b) = 0$

$$\sum_{i=1}^{n} G_i(b,c)\alpha d_i(w,b)\beta d_i(a,b) = 0 \quad (7)$$

Replace $\alpha d_i(w,b)\beta$ by $\beta d_i(w,b)\alpha$ in (7) we get

$\sum_{i=1}^{n} G_i(b,c)\beta d_i(w,b)\alpha d_i(a,b) = 0$ (8)

Replace $d_i(w,b)$ by m in (8) for every m$\in M$, then

$\sum_{i=1}^{n} G_i(b,c)\beta m\alpha d_i(a,b) = 0$ (9)

From (4) and (9) we get d_n and G_n orthogonal

From (1) and (6) we get $d_n(a,b)\alpha G_n(b,c) = G_n(b,c)\alpha d_n(a,b) = 0$

Proof: (iii) In the same way as used in proof (2)

Proof: (iv)

From (i) we have $D_n(a,b)\alpha G_n(b,c) = 0$

$\sum_{i=1}^{n} D_i(a,b)\alpha G_i(b,c) = 0$ (1)

$Replacing\ a$ by $a\beta w$ in (1) we get $\sum_{i=1}^{n} D_i(a\beta w\ ,b)\alpha G_i(b,c)=0$

$$\sum_{i=1}^{n} D_i(w\ ,b)\beta d_i(a\ ,b)\alpha G_i(b,c)=0 \qquad (2)$$

Replacing a by $s\gamma a$ and $G_i(b,c)$ by $g_i(b,c)$ in (2) for each s ϵM we get

$$\sum_{i=1}^{n} D_i(w\ ,b)\beta d_i(s\gamma a\ ,b)\alpha g_i(b,c)=0$$

$$\sum_{i=1}^{n} D_i(w,b)\beta d_i(a,b)\gamma d_i(s,b)\alpha g_i(b,c)=0 \qquad (3)$$

Replace $d_i(s,b)$ by m in (3) for all m∈ M

$$\sum_{i=1}^{n} D_i(w,b)\beta\ d_i(a,b)\gamma m\alpha g_i(b,c)=0 \qquad (4)$$

Left multiplication (4) by $d_i(a,b)\gamma m\alpha g_i(b,c)\delta$ for all $\delta\in\Gamma$ we get

$\sum_{i=1}^{n} d_i(a,b)\gamma m\alpha g_i(b,c)\delta D_i(w\ ,b)\beta d_i(a\ ,b)\gamma m\alpha g_i(b,c)=0$

Since M is semiprime then $\sum_{i=1}^{n} d_i(a,b)\gamma m\alpha g_i(b,c)=0$

Thus $d_n(a,b)\Gamma M\Gamma g_n(b,c)=0$

Similarly from (i) we have $G_n(b,c)\alpha D_n(a,b)=0$ we get $g_n(b,c)\Gamma M\Gamma d_n(a,b)=0$

Hence d_n and g_n orthogonal (hsrb-d).

Proof: (v)

From (ii) we have $d_n(a,b)\alpha G_n(b,c)=0$

$d_n(d_n(a,b)\alpha G_n(b,c),m)=0$

$$\sum_{i=1}^{n} d_i(d_i(a,b)\alpha G_i(b,c),m)=0 \qquad (1)$$

Replace $a\ by\ w\beta a$ in (1) for all $w\epsilon$M and $\beta\in\Gamma$ we get

$\sum_{i=1}^{n} d_i(d_i(w\beta a\ ,b)\alpha G_i(b,c),m)=0$

$$\sum_{i=1}^{n} d_i(d_i(a\ ,b)\beta d_i(w\ ,b)\alpha G_i(b,c),m)=0$$

$$\sum_{i=1}^{n} d_i(G_i(b,c),m)\beta d_i(d_i(w\ ,b),m)\alpha d_i(d_i(a,b),m)=0 \qquad (2)$$

Replace $d_i(a,b)$ by $G_i(b,c)$ in (2) we get

$$\sum_{i=1}^{n} d_i(G_i(b,c),m)\beta d_i(d_i(w\ ,b),m)\alpha d_i(G_i(b,c),m)=0$$

Since M is semiprime, then $\sum_{i=1}^{n} d_i(G_i(b,c),m)=0$

Thus $d_nG_n=0 \qquad (3)$

And by (ii) $G_n(b,c)\alpha d_n(a,b)=0$

$G_n(G_n(b,c)\alpha d_n(a,b),m)=0$

$$\sum_{i=1}^{n} G_i(G_i(b,c)\alpha d_i(a,b),m)=0 \qquad (4)$$

Replace $a\ \ by\ \ a\delta w$ in (4) for all $\delta\in\Gamma$ we get

$\sum_{i=1}^{n} G_i(G_i(b,c)\alpha d_i(a\delta w,b),m) = 0$
$\sum_{i=1}^{n} G_i(G_i(b,c)\alpha d_i(w,b)\delta d_i(a,b),m) = 0$

$$\sum_{i=1}^{n} G_i(d_i(a,b),m)\alpha g_i(d_i(w,b),m)\delta g_i(G_i(b,c),m) = 0 \qquad (5)$$

Replace $g_i(G_i(b,c),m)$ by $G_i(d_i(a,b),m)$ in (5) we get
$\sum_{i=1}^{n} G_i(d_i(a,b),m)\alpha g_i(d_i(w,b),m)\delta G_i(d_i(a,b),m) = 0$
Since M is semiprime, then $\sum_{i=1}^{n} G_i(d_i(a,b),m) = 0$
Thus $G_n d_n = 0$ (6)
From equations (3) and (6) we get $G_n d_n = d_n G_n = 0$
In the same way we prove that $D_n g_n = g_n D_n = 0$

Proof: (vi)
Since D_n and G_n orthogonal, then $D_n(a,b)\Gamma M\Gamma G_n(b,c)$ =0
$D_n(D_n(a,b)\Gamma M\Gamma G_n(b,c),s) = (0)$ for all $s\epsilon$M
$\sum_{i=1}^{n} D_i(D_i(a,b)\alpha m\beta G_i(b,c),s) = 0$
$\sum_{i=1}^{n} D_i(G_i(b,c),s)\alpha d_i(m,s)\beta d_i(D_i(a,b),s) = 0$ (1)
Replace $d_i(D_i(a,b),s)$ by $D_i(G_i(b,c),s)$ in (1) we get
$\sum_{i=1}^{n} D_i(G_i(b,c),s)\alpha d_i(m,s)\beta D_i(G_i(b,c),s) = 0$
Since M is semiprime we get $\sum_{i=1}^{n} D_i(G_i(b,c),s) = 0$
Thus $D_n G_n = 0$ (2)
And by $G_n(b,c)\Gamma M\Gamma D_n(a,b) = 0$
$G_n(G_n(b,c)\Gamma M\Gamma D_n(a,b),s) = 0$ for all $s\epsilon$M
$\sum_{i=1}^{n} G_i(G_i(b,c)\alpha m\beta D_i(a,b),s) = 0$
$\sum_{i=1}^{n} G_i(D_i(a,b),s)\alpha g_i(m,s)\beta g_i(G_i(b,c),s) = 0$ (3)
Replace $g_i(G_i(b,c),s)$ by $G_i(D_i(a,b),s)$ in (3) we get
$\sum_{i=1}^{n} G_i(D_i(a,b),s)\alpha g_i(m,s)\beta G_i(D_i(a,b),s) = 0$
Since M is semiprime we get $\sum_{i=1}^{n} G_i(D_i(a,b),s) = 0$
Thus $G_n D_n = 0$ (4)
By equations (2) and (4) we get $G_n D_n = D_n G_n = 0$

Theorem (3.2):
Let M be a 2-tfsp Γ-ring. If D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for each nϵN. Then the following relations are equivalent for every $a,b,c\epsilon M$ and $\alpha,\beta\epsilon\Gamma$.
i. D_n and G_n orthogonal.
ii. $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = 0$.
iii. $d_n(a,b)\alpha G_n(b,c) + g_n(b,c)\alpha D_n(a,b) = 0$.
Proof: $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{ii})$
Assume that D_n and G_n orthogonal

By theorem(3.1) (i) we have $D_n(a,b)\alpha G_n(b,c) = G_n(b,c)\alpha D_n(a,b) = 0$
Hence $D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = 0$
Conversely, $let\ D_n(a,b)\alpha G_n(b,c) + G_n(b,c)\alpha D_n(a,b) = 0$
By lemma(2.6) we get D_n and G_n orthogonal

Proof: $(\boldsymbol{i}) \Leftrightarrow (\boldsymbol{iii})$
Let $D_n\ and\ G_n$ orthogonal
Then, by lemma(2.8) we get $d_n(a,b)\alpha G_n(b,c) = 0$ (1)
Since D_n and G_n orthogonal, then $G_n(b,c)\Gamma M\Gamma D_n(a,b) = 0$

$$\sum_{i=1}^{n} G_i(b,c)\alpha m\beta D_i(a,b) = 0 \quad (2)$$

By lemma (2.3) we get $\sum_{i=1}^{n} G_i(b,c)\alpha D_i(a,b) = 0$ (3)
Replace $G_i(b,c)\ by\ g_i(b,c)$ in (3) we get
$\sum_{i=1}^{n} g_i(b,c)\alpha D_i(a,b) = 0$
Thus $g_n(b,c)\alpha D_n(a,b) = 0$ (4)
From (1) and (4) we get $d_n(a,b)\alpha G_n(b,c) + g_n(b,c)\alpha D_n(a,b) = 0$
Conversely, suppose that $d_n(a,b)\alpha G_n(b,c) + g_n(b,c)\alpha D_n(a,b) = 0$
$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c) + g_i(b,c)\alpha D_i(a,b) = 0$ (5)
Replacing $a\ by\ t\gamma a$ in (5) we get
$\sum_{i=1}^{n} d_i(t\gamma a,b)\alpha G_i(b,c) + g_i(b,c)\alpha D_i(t\gamma a,b) = 0$

$$\sum_{i=1}^{n} d_i(a,b)\gamma d_i(t,b)\alpha G_i(b,c) + g_i(b,c)\alpha D_i(a,b)\gamma d_i(t,b) = 0 \quad (6)$$

Replace $d_i(a,b)$ by $D_i(a,b)$ and $g_i(b,c)$ by $G_i(b,c)$ in (6) we get
$\sum_{i=1}^{n} D_i(a,b)\gamma d_i(t,b)\alpha G_i(b,c) + G_i(b,c)\alpha D_i(a,b)\gamma d_i(t,b) = 0$ (7)
Replacing $\alpha D_i(a,b)\gamma d_i(t,b)$ by $\gamma d_i(t,b)\alpha D_i(a,b)$ in (7) we get
$\sum_{i=1}^{n} D_i(a,b)\gamma d_i(t,b)\alpha G_i(b,c) + G_i(b,c)\gamma d_i(t,b)\alpha D_i(a,b) = 0$ (8)
By Lemma (2.4) we get
$\sum_{i=1}^{n} D_i(a,b)\gamma d_i(t,b)\alpha G_i(b,c) = \sum_{i=1}^{n} G_i(b,c)\gamma d_i(t,b)\alpha D_i(a,b) = 0$
Hence $D_n(a,b)\Gamma M\Gamma G_n(b,c) = (0) = G_n(b,c)\Gamma M\Gamma D_n(a,b)$
Hence $D_n\ and\ G_n$ orthogonal

<u>Theorem(3.3):</u>

Let M be a 2-tfsp Γ-ring. If D_n and G_n are two (ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for all nϵN.Then D_n and G_n orthogonal iff $D_n(a,b)\alpha G_n(b,c) = 0$ and $d_n G_n = d_n g_n = 0$ for all a,b,c ∈M and α,βϵΓ.

Proof:

Assume that D_n and G_n orthogonal
By lemma (2.7) we get $D_n(a,b)\alpha G_n(b,c) = 0$ (1)
And by Theorem (3.1) (ii) we get $G_n(b,c)\alpha d_n(a,b) = 0$
$d_n(G_n(b,c)\alpha d_n(a,b), m) = 0$ for all m∈ M
$\sum_{i=1}^{n} d_i(G_i(b,c)\alpha d_i(a,b), m) = 0$ (2)
Replace $a\ by\ a\beta t$ in (2) for all $t\epsilon M$ we get

$$\sum_{i=1}^{n} d_i(G_i(b,c)\alpha d_i(a\beta t,b), m) = 0$$

$\sum_{i=1}^{n} d_i(G_i(b,c)\alpha d_i(t,b)\beta d_i(a,b), m) = 0$

$$\sum_{i=1}^{n} d_i(d_i(a,b),m)\alpha d_i(d_i(t,b),m)\beta d_i(G_i(b,c),m) = 0 \qquad (3)$$

Replace $d_i(a,b)$ by $G_i(b,c)$ in (3) we get

$$\sum_{i=1}^{n} d_i(G_i(b,c),m)\alpha d_i(d_i(t,b),m)\beta d_i(G_i(b,c),m) = 0$$

Since M is semiprime we get $\sum_{i=1}^{n} d_i(G_i(b,c),m) = 0$

Thus $d_nG_n = 0$ (4)

Also, by Theorem (3.1) (iv) we get $g_n(b,c)\Gamma M\Gamma d_n(a,b) = 0$

$d_n(g_n(b,c)\Gamma M\Gamma d_n(a,b),m_1) = 0$ for all $m_1 \epsilon M$

$\sum_{i=1}^{n} d_i(g_i(b,c)\alpha m\beta d_i(a,b),m_1) = 0$ for all $m_1 \epsilon M$

$\sum_{i=1}^{n} d_i(d_i(a,b),m_1)\alpha d_i(m,m_1)\beta d_i(g_i(b,c),m_1) = 0$ (5)

Replace $d_i(a,b)$ by $g_i(b,c)$ in (5) we get

$\sum_{i=1}^{n} d_i(g_i(b,c),m_1)\alpha d_i(m,m_1)\beta d_i(g_i(b,c),m_1) = 0$

Since M is semiprime, then $\sum_{i=1}^{n} d_i(g_i(b,c),m_1) = 0$

Thus $d_ng_n = 0$ (6)

From (1), (4) and (6) we get $D_n(a,b)\alpha G_n(b,c) = (0)$ *and* $d_nG_n = d_ng_n = 0$

Conversely, suppose that $D_n(a,b)\alpha G_n(b,c) = 0$ (7)

And $d_nG_n = 0$

$(d_nG_n)(b\alpha a,c) = 0$

$\sum_{i=1}^{n} d_i(G_i(b\alpha a,c),m) = 0$ for all mϵM

$$\sum_{i=1}^{n} d_i(G_i(a,c)\alpha g_i(b,c),m) = 0$$

$$\sum_{i=1}^{n} d_i(g_i(b,c),m)\alpha d_i(G_i(a,c),m) = 0 \qquad (8)$$

Replace $d_i(g_i(b,c),m)$ by $d_i(a,b)$ and $d_i(G_i(a,c),m)$ by $G_i(b,c)$ in (8) we get

$$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c) = 0$$

Hence $d_n(a,b)\alpha G_n(b,c) = 0$ (9)

From (7) and (9) we get $D_n(a,b)\alpha G_n(b,c) = d_n(a,b)\alpha G_n(b,c) = 0$

By lemma(2.9) we get D_n and G_n orthogonal.

<u>Theorem(3.4):</u>

Let D_n be a (ghsrb-d) associated with (hsrb-d) d_n of a 2-tfsp Γ-ring M for all nϵN. If $D_n(a,b)\alpha D_n(b,c) = 0$ then $D_n = d_n = 0$ for all a,b,c ∈M and α,βϵΓ.

Proof:

By assumption $D_n(a,b)\alpha D_n(b,c) = \sum_{i=1}^{n} D_i(a,b)\alpha D_i(b,c) = 0$ (1)

Right multiplication (1) by $\delta D_i(a,b)$ we get

$$\sum_{i=1}^{n} D_i(a,b)\alpha D_i(b,c)\delta D_i(a,b) = 0$$

Since M is semiprime we get $\sum_{i=1}^{n} D_i(a,b) = 0$
Then $D_n = 0$ (2)
Replace a by tβa in (1) for every t∈M, then
$\sum_{i=1}^{n} D_i(t\beta a\ ,b)\alpha D_i(b,c) = 0 \Longrightarrow \sum_{i=1}^{n} D_i(a,b)\beta d_i(t,b)\alpha D_i(b,c) = 0$ (3)
Replace $d_i(t,b)\alpha D_i(b,c)$ by $D_i(b,c)\alpha d_i(t,b)$ in (3) we get

$$\sum_{i=1}^{n} D_i(a,b)\beta D_i(b,c)\alpha d_i(t,b) = 0$$

By Lemma (2.3) we get $\sum_{i=1}^{n} D_i(a,b)\beta d_i(t,b) = 0$ (4)
Left multiplication (4) by $d_i(t,b)\delta$ for all $\delta\epsilon\Gamma$ we get

$$\sum_{i=1}^{n} d_i(t,b)\delta D_i(a,b)\beta d_i(t,b) = 0$$

Since M is semiprime, then $\sum_{i=1}^{n} d_i(t,b) = 0$
Thus $d_n = 0$ (5)
From (2) and (5) we get $D_n = d_n = 0$

Theorem(3.5)

If M be a 2-tfsp Γ- ring. Let D_n and G_n are two(ghsrb-d) associated with two (hsrb-d) d_n and g_n respectively for all nϵN. Then D_n and g_n as well as G_n and d_n orthogonal iff $D_n = d_n = 0\ or\ G_n = g_n = 0$ for all $a,b,c\epsilon M$ and $\alpha,\beta\epsilon\Gamma$.

Proof:
Suppose that $D_n\ and\ g_n$ as well as $G_n\ and\ d_n$ are orthogonal
By theorem (3.1)(iii) we have got $D_n(a,b)\alpha g_n(b,c) = 0$
$\sum_{i=1}^{n} D_i(a,b)\alpha g_i(b,c) = 0$ (1)
Right multiplication (1) by $\beta D_i(a,b)$ for all $\beta\epsilon\Gamma$ we get
$\sum_{i=1}^{n} D_i(a,b)\alpha g_i(b,c)\beta D_i(a,b) = 0$
Since M is semiprime we get $\sum_{i=1}^{n} D_i(a,b) = 0$
Hence $D_n = 0$ (2)
And by theorem (3.1)(ii) we get $d_n(a,b)\alpha G_n(b,c) = 0$
$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c) = 0$ (3)
Right multiplication (3) by $\beta d_i(a,b)$ we get
$\sum_{i=1}^{n} d_i(a,b)\alpha G_i(b,c)\beta d_i(a,b) = 0$
Since M is semiprime we get $\sum_{i=1}^{n} d_i(a,b) = 0$
Thus $d_n = 0$ (4)
Now, also by theorem (3.1)(iii) we get $g_n(b,c)\alpha D_n(a,b) = 0$
$\sum_{i=1}^{n} g_i(b,c)\alpha D_i(a,b) = 0$ (5)
Right multiplication (5) by $\beta g_i(b,c)$ we get
$\sum_{i=1}^{n} g_i(b,c)\alpha D_i(a,b)\beta g_i(b,c) = 0$
Since M is semiprime, then $\sum_{i=1}^{n} g_i(b,c) = 0$
Hence g_n=0 (6)
And by theorem(3.1)(ii) we have got $G_n(b,c)\alpha d_n(a,b) = 0$
$\sum_{i=1}^{n} G_i(b,c)\alpha d_i(a,b) = 0$ (7)
Right multiplication (7) by $\beta G_i(b,c)$ we get
$\sum_{i=1}^{n} G_i(b,c)\alpha d_i(a,b)\beta G_i(b,c) = 0$

Since M is semiprime, then $\sum_{i=1}^{n} G_i(b,c)=0$

Hence $G_n=0$ (8)

From (2), (4), (6) and (8) we get $D_n=d_n=0$ *or* $G_n=g_n=0$

Conversely, assume that $D_n=d_n=0$ or $G_n=g_n=0$

$D_n(c\alpha a,b)=0$ for all $\alpha\epsilon\Gamma$

$g_n(D_n(c\alpha a,b),m)=0$ for all $m\epsilon$M

$\sum_{i=1}^{n} g_i(D_i(c\alpha a,b),m)=0$

$$\sum_{i=1}^{n} g_i(D_i(a,b)\alpha d_i(c,b),m)=0$$

$$\sum_{i=1}^{n} g_i(d_i(c,b),m)\alpha g_i(D_i(a,b),m)=0 \qquad (9)$$

Replace $g_i(d_i(c,b),m)$ by $D_i(a,b)$ and $g_i(D_i(a,b),m)$ by $g_i(b,c)$ in (9) we get

$$\sum_{i=1}^{n} D_i(a,b)\alpha g_i(b,c)=0$$

$D_n(a,b)\alpha g_n(b,c)=0$

By theorem(3.1) (iii) we get D_n and g_n are orthogonal

Similarly, if $G_n=g_n=0$ we get G_n and d_n are orthogonal.

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