Journal of Iraqi Al-Khwarizmi (JIKh) Volume:8 Issue:2 Year: 2024 pages: 178-196

Solving Fuzzy System of Volterra Integral Equations by using Adomian Decomposition Method

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Publication date:10/ 12/2024

Keywords: *Fuzzy Systems of Volterra integral equations; Adomian decomposition method; Adomian polynomials; Trapezoidal Rule; Simpson Rule.*

ABSTRACT

The aim of this research is to propose a fuzzy system for solving linear and non-linear Volterra integral equations using the Adomian decomposition method, along with a corresponding numerical scheme. We examine various numerical aspects, including convergence and error analysis, and illustrate the method's effectiveness through examples solved using both the classic Adomian analysis method and numerical solutions with Trapezoidal and Simpson rules. Our findings demonstrate the method's clarity and effectiveness.

Finally, the accuracy and applicability of the method is implemented along with comparisons using some numerical examples.

1. Introduction

As we know the fuzzy differential and integral equations are one of the important parts of the fuzzy analysis theory that play major role in numerical analysis and to model the dynamical systems whose characteristic is biased on vagueness or uncertainty due to partial information about the problem **[1].**

Zadeh was the pioneer in a fuzzy numbers and arithmetic operations he was creator in this field **[2,3]** It's fascinating to observe the growing interest in fuzzy systems of integral equations over the recent years. These systems have been widely applied in many areas, such as physics, biology, chemistry, and engineering, among others. Fuzzy control has been a key focus of research in this field, and mathematical models have been developed to tackle complex problems. Integral equations play a crucial role in these models, providing a powerful framework for understanding and solving intricate mathematical problems. As such, the use of integral equations has become increasingly important in diverse fields, highlighting the far-reaching impact of fuzzy systems of integral equations, the linear Volterra equation of the second kind **[4]**. In 1981 George Adomian introduced The ADM method and developed **[5,6].**

This method has been applied to solve differential and integral equations for linear and non-linear problem. The main advantage of this method is greatly capable reducing the size of computational work while still maintaining high accuracy of the numerical solution. In **[7]** presented a reliable approach for convergence of the Adomian method when applied to a class of nonlinear Volterra integral equations.

In [8] proposed discrete decomposition method to solves the two-dimensional Burgers nonlinear difference equations. In [9] obtained the analytical solutions by using the modified aplace Adomian decomposition method On handle nonlinear integro differential

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equations of the first and the second kind. In [10] studied a simple modification on ADM and HPM and applied to solve the systems of Volterra integral equations of the second kind.

In [11] used the Combined Sumudu Transform-Adomian Decomposition Method (CST-ADM) to solve systems of non-linear Volterra integral equations. In [12] used the Standard Adomian Decomposition Method (ADM) and Modified Technique (MT) to solve the fuzzy systems of linear and non-linear four Volterra integro-differential equations (VIDEs) with a comparison between the both techniques.

The primary aim of this study is to utilize the (ADM) in solving both linear and non-linear Fuzzy systems of Volterra integral equations. To evaluate the effectiveness of the method, a comparison was made with the numerical solution of the system of Fuzzy Integral Equations (SFIEs) using the Simpson Rule (ADM-SIMP) and the Trapezoidal Rule (ADM-TRAP), with the goal of achieving the minimum amount of computation required for accurate results. This analysis provides valuable insights into the efficiency and precision of the ADM in solving complex Fuzzy systems, which can be of significant benefit in various fields, including engineering, finance, and science.

2. Preliminaries

Definition (1): (Fuzzy set). A fuzzy set is defined as a set of ordered pairs $\tilde{A} = \left\{\left(x, M_{\tilde{A}(x)}\right), x \in X\right\}$ where X is the universe of discourse, and $M_{\tilde{A}(x)}$ is the membership function of fuzzy set A. The membership function assigns a degree of membership to each element x in the universe of discourse. The value of $M_{\tilde{A}(x)}$ represents the degree of membership of x in the fuzzy set $\tilde{A}$ **[2]**.

Definition (2):): ($\boldsymbol{\alpha}$ -cut set) [13]: The α -cut set A^{α} is made up of members whose membership is not less than α.

$$A^{\alpha} = \{x \in X: \mu_A(x) \geq \alpha\}$$

note that α is arbitrary. This α -cut set is a crisp set.

Definition 3 : (Fuzzy number). **[13]:** If a fuzzy set is **convex** and **normalized**, and its membership function is defined in $\mathbb{R}$ and **piecewise** continuous, it is called as "**fuzzy number**". So fuzzy number (fuzzy set) represents a real number interval whose boundary is fuzzy.

Definition (3) [2]: An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of functions $\left(\underline{u}, \overline{u}\right)$ of the functions $\left(\underline{u}(\alpha), \overline{u}(\alpha)\right)$, which satisfy the following requirements

i. $\underline{u}(\alpha)$ is bounded left continuous increasing function over [0,1].
ii. $\overline{u}(\alpha)$ is a bounded left continuous decreasing function over [0,1].
iii. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$ for all $0 \leq \alpha \leq 1$.

Definition (4): For an arbitrary $\tilde{u}(x,\alpha) = \left(\underline{u}(x,\alpha), \overline{u}(x,\alpha)\right), \tilde{v}(x,\alpha) = \left(\underline{v}(x,\alpha), \overline{v}(x,\alpha)\right), 0 \leq \alpha \leq 1$ and scalar k, we define addition, subtraction, scalar product by k and multiplication are respectively as follows **[14,15].**

1. $\underline{(u+v)}(\alpha) = \left(\underline{u}(\alpha) + \underline{v}(\alpha)\right), \quad \overline{(u+v)}(\alpha) = \left(\overline{u}(\alpha) + \overline{v}(\alpha)\right),$

2. $\underline{(u-v)}(\alpha) = \left(\underline{u}(\alpha) - \overline{v}(\alpha)\right)$, $\overline{(u-v)}(\alpha) = \left(\overline{u}(\alpha) - \underline{v}(\alpha)\right)$,
3. $k\breve{u} = \begin{cases} \left(k\underline{u}(\alpha), k\,\overline{u}(\alpha)\right), k \geq 0 \\ \left(k\,\overline{u}(\alpha), k\underline{u}(\alpha)\right), k < 0 \end{cases}$,
4. $\tilde{u}.\tilde{v} = \begin{cases} \underline{uv}(\alpha) = \max\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\,\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\} \\ \overline{uv}(\alpha) = \min\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\,\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\} \end{cases}$.

Definition (7): (The fuzzy Riemann integral) We say that **[16].**

1- $\tilde{f}(x)$ is a fuzzy valued function if $\tilde{f}: X \to \mathcal{F}$,
2- $\tilde{f}(x)$ is a closed fuzzy valued function if $\tilde{f}: X \to \mathcal{F}_{cl}$,
3- $\tilde{f}(x)$ is a bounded fuzzy valued function if $\tilde{f}: X \to \mathcal{F}_{b}$.

We denote $A_\alpha = \left[\int_a^b \tilde{f}_\alpha^L(x)dx, \int_a^b \tilde{f}_\alpha^U(x)dx\right]$. If $\tilde{f}(x)$ is a closed and bounded fuzzy valued function on $[a,b]$, then the fuzzy Riemann integral $\int_a^b \tilde{f}(x)dx$, is a closed fuzzy number. Furthermore, the$[u]^\alpha$ set of $\int_a^b \tilde{f}(x)dx$ is

$$\left(\int_a^b \tilde{f}(x)dx\right)_\alpha = \left[\int_a^b \tilde{f}_\alpha^L(x)dx, \int_a^b \tilde{f}_\alpha^U(x)dx\right].$$

3. Fuzzy System Volterra Integral Equation

In this section, we present a system of fuzzy Volterra integral equations (SFVIEs) linear and non-linear and a numerical scheme to solve them using the Adomian decomposition method (ADM).

We consider the following system of fuzzy Volterra integral equations (SFVIEs) of the second kind **[17,18]:**

$$\begin{cases} \tilde{u}_1(x,\alpha) = \tilde{f}_1(x,\alpha) + \sum_{j=1}^{2} \lambda_{1j} \int_0^x K_{1j}(x,t)\tilde{F}_{1j}\left(\tilde{u}(t,\alpha)\right)dt\,, \\ \tilde{u}_2(x,\alpha) = \tilde{f}_2(x,\alpha) + \sum_{j=1}^{2} \lambda_{2j} \int_0^x K_{2j}(x,t)\tilde{F}_{2j}\left(\tilde{u}(t,\alpha)\right)dt\,, \end{cases} \tag{1}$$

Given SFVIEs (1) have unique solution $\tilde{u}(t,\alpha) = \left(\tilde{u}_1(t,\alpha), \tilde{u}_2(t,\alpha)\right)^T$, on [0,1], where $\lambda_{ij} \neq 0,\ i,j = 1,2$ are real constants, $\tilde{u}_i(x,\alpha), \tilde{f}_i(x,\alpha)$. and $K_{ij}(x,t)$ are unknown functions or analytical functions, , $\tilde{F}_{1j}\left(\tilde{u}(t,\alpha)\right)$ and $\tilde{F}_{2j}\left(\tilde{u}(t,\alpha)\right)$are functions of $\tilde{u}(t,\alpha)$, $0 \leq t \leq x$, $0 \leq x \leq 1$

The parametric form of the given SFVIEs (1) is presented below:

$$\begin{cases} \underline{u}_1(x,\alpha) = \underline{f_1}(x,\alpha) + \sum_{j=1}^{2} \lambda_{1j} \int_0^x K_{1j}(x,t) \underline{F}_{1j}\left(\underline{u}(t,\alpha)\right) dt\,, \\ \overline{u}_1(x,\alpha) = \overline{f}_1(x,\alpha) + \sum_{j=1}^{2} \lambda_{1j} \int_0^x K_{1j}(x,t) \overline{F}_{1j}\left(\overline{u}(t,\alpha)\right) dt\,, \\ \underline{u}_2(x,\alpha) = \underline{f_2}(x,\alpha) + \sum_{j=1}^{2} \lambda_{2j} \int_0^x K_{2j}(x,t) \underline{F}_{2j}\left(\underline{u}(t,\alpha)\right) dt\,, \\ \overline{u}_2(x,\alpha) = \overline{f}_2(x,\alpha) + \sum_{j=1}^{2} \lambda_{2j} \int_0^x K_{2j}(x,t) \overline{F}_{2j}\left(\overline{u}(t,\alpha)\right) dt\,, \end{cases} \tag{2}$$

4. Applied ADM to SFVIEs

The ADM gives the solution of the SFVIEs (2) as an infinite series usually converging to the closed form solution. To solve the SFVIEs (2) by the Adomian's technique, assume an infinite series solution for the unknowns functions $\tilde{u}_1(x,\alpha) = \left[\underline{u}_1(x,\alpha), \overline{u}_1(x,\alpha)\right]$ and $\tilde{u}_2(x,\alpha) = \left[\underline{u}_2(x,\alpha), \overline{u}_2(x,\alpha)\right]$ given by **[2,17, 18]** as follows

$$\begin{aligned} \underline{u}_1(x,\alpha) &= \sum_{n=0}^{\infty} \underline{u}_{1,n}(x,\alpha)\,, \quad \overline{u}_1(x,\alpha) = \sum_{n=0}^{\infty} \overline{u}_{1,n}(x,\alpha)\,, \\ \underline{u}_2(x,\alpha) &= \sum_{n=0}^{\infty} \underline{u}_{2,n}(x,\alpha)\,, \quad \overline{u}_2(x,\alpha) = \sum_{n=0}^{\infty} \overline{u}_{2,n}(x,\alpha)\,, \end{aligned} \tag{3}$$

and decomposing the non-linear functions

$\tilde{F}_{1j}\left(\tilde{u}(t,\alpha)\right) = \left[\underline{F}_{1j}\left(\underline{u}(t,\alpha)\right), \overline{F}_{1j}\left(\overline{u}(t,\alpha)\right)\right]$, and

$\tilde{F}_{2j}\left(\tilde{u}(t,\alpha)\right) = \left[\underline{F}_{2j}\left(\underline{u}(t,\alpha)\right), \overline{F}_{2j}\left(\overline{u}(t,\alpha)\right)\right]$, as

$$\begin{aligned} \underline{F}_{1j}\left(\underline{u}(t,\alpha)\right) &= \sum_{n=0}^{\infty} \underline{A}_{1j,n}\,, \quad \overline{F}_{1j}\left(\overline{u}(t,\alpha)\right) = \sum_{n=0}^{\infty} \overline{A}_{1j,n}\,, \\ \underline{F}_{2j}\left(\underline{u}(t,\alpha)\right) &= \sum_{n=0}^{\infty} \underline{A}_{2j,n}\,, \quad \overline{F}_{2j}\left(\overline{u}(t,\alpha)\right) = \sum_{n=0}^{\infty} \overline{A}_{2j,n}\,, \end{aligned} \tag{4}$$

where $\tilde{A}_{1j,n} = \left[\underline{A}_{1j,n}, \overline{A}_{1j,n}\right]$ and $\tilde{A}_{2j,n} = \left[\underline{A}_{2j,n}, \overline{A}_{2j,n}\right]$ [61] are polynomials (so called Adomian polynomials) of $\tilde{u}_{i,0}(t,\alpha), \tilde{u}_{i,1}(t,\alpha), \dots, \tilde{u}_{i,n}(t,\alpha)$ given by

$$\left.\begin{aligned}
\underline{A}_{1j,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F\left(\sum_{k=0}^{n}\lambda^i\underline{u}_{1,k}(t,\alpha)\right)\right]_{\lambda=0}, \\
\overline{A}_{1j,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F\left(\sum_{k=0}^{n}\lambda^i\overline{u}_{1,k}(t,\alpha)\right)\right]_{\lambda=0}, \\
\underline{A}_{2j,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F\left(\sum_{k=0}^{n}\lambda^i\underline{u}_{2,k}(t,\alpha)\right)\right]_{\lambda=0}, \\
\overline{A}_{2j,n} &= \frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F\left(\sum_{k=0}^{n}\lambda^i\overline{u}_{2,k}(t,\alpha)\right)\right]_{\lambda=0},
\end{aligned}\right\} \quad n \geq 0 \qquad (3.5)$$

Applying the **Adomian's technique** as in **[19, 20]** the SFVIEs (2) can be written as

$$\begin{cases}
\underline{u}_1(x,\alpha) = \underline{f}_1(x,\alpha) + \sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\underline{N}_{1j}dt, \\
\overline{u}_1(x,\alpha) = \overline{f}_1(x,\alpha) + \sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\overline{N}_{1j}dt, \\
\underline{u}_2(x,\alpha) = \underline{f}_2(x,\alpha) + \sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\underline{N}_{2j}dt, \\
\overline{u}_2(x,\alpha) = \overline{f}_2(x,\alpha) + \sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\overline{N}_{2j}dt,
\end{cases} \qquad (6)$$

where $\underline{N}_{1j} = \underline{F}_{1j}\left(\underline{u}(t,\alpha)\right), \overline{N}_{1j} = \overline{F}_{1j}\left(\overline{u}(t,\alpha)\right)$ and $\underline{N}_{2j} = \underline{F}_{2j}\left(\underline{u}(t,\alpha)\right), \overline{N}_{2j} = \overline{F}_{2j}\left(\overline{u}(t,\alpha)\right)$ are the non-linear operators.

Using (3) and (4) into the SFVIEs (6) it follows that

$$\begin{cases}
\sum_{n=0}^{\infty}\underline{u}_{1,n}(x,\alpha) = \underline{f}_1(x,\alpha) + \sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\sum_{n=0}^{\infty}\underline{A}_{1j,n}\,dt, \\
\sum_{n=0}^{\infty}\overline{u}_{1,n}(x,\alpha) = \overline{f}_1(x,\alpha) + \sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\sum_{n=0}^{\infty}\overline{A}_{1j,n}\,dt, \\
\sum_{n=0}^{\infty}\underline{u}_{2,n}(x,\alpha) = \underline{f}_2(x,\alpha) + \sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\sum_{n=0}^{\infty}\underline{A}_{2j,n}\,dt, \\
\sum_{n=0}^{\infty}\overline{u}_{2,n}(x,\alpha) = \overline{f}_2(x,\alpha) + \sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\sum_{n=0}^{\infty}\overline{A}_{2j,n}\,dt.
\end{cases} \qquad (7)$$

Using the ADM, according to (7), the lower iterations (L) are then determined in the following recursive way:

$$\begin{cases}\underline{u}_{1,0}(x,\alpha)=\underline{f_1}(x,\alpha),\\ \underline{u}_{2,0}(x,\alpha)=\underline{f_2}(x,\alpha),\end{cases}$$

$$\begin{cases}\underline{u}_{1,n+1}(x,\alpha)=\sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\underline{A}_{1j,n}dt\,,\\ \underline{u}_{2,n+1}(x,\alpha)=\sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\underline{A}_{2j,n}dt\,,\end{cases}\qquad n\geq 0 \qquad (8)$$

and the upper iterations (U) are

$$\begin{cases}\overline{u}_{1,0}(x,\alpha)=\overline{f}_1(x,\alpha),\\ \overline{u}_{2,0}(x,\alpha)=\overline{f}_2(x,\alpha),\end{cases}$$

$$\begin{cases}\overline{u}_{1,n+1}(x,\alpha)=\sum_{j=1}^{2}\lambda_{1j}\int_0^x K_{1j}(x,t)\overline{A}_{1j,n}dt\,,\\ \overline{u}_{2,n+1}(x,\alpha)=\sum_{j=1}^{2}\lambda_{2j}\int_0^x K_{2j}(x,t)\overline{A}_{2j,n}dt\,.\end{cases}\qquad n\geq 0 \qquad (9)$$

Thus, all components $\left(\tilde{u}_{i,n}(x,\alpha)=\left[\underline{u}_{i,n}(x,\alpha),\overline{u}_{i,n}(x,\alpha)\right],i=1,2\right)$ of $\tilde{u}_i(x,\alpha)=\left[\underline{u}_i(x,\alpha),\overline{u}_i(x,\alpha)\right]$ can be calculated once the $\tilde{A}_{1j,n}$ and $\tilde{A}_{2j,n}$ are given. We then define the n-term approximants to the solution $\tilde{u}_i(x,\alpha)=\left[\underline{u}_i(x,\alpha),\overline{u}_i(x,\alpha)\right]$ by $\phi_{i,n}\left[\underline{u}_i(x,\alpha)\right]=\sum_{k=0}^{n-1}\underline{u}_{i,k}(x,\alpha)$ with $\lim_{n\to\infty}\phi_{i,n}\left[\underline{u}_i(x,\alpha)\right]=\underline{u}_i(x,\alpha)$ and by $\phi_{i,n}[\overline{u}_i(x,\alpha)]=\sum_{k=0}^{n-1}\overline{u}_{i,k}(x,\alpha)$ with $\lim_{n\to\infty}\phi_{i,n}[\overline{u}_i(x,\alpha)]=\overline{u}_i(x,\alpha)$.

5. Applications and Numerical Results

In this section, we apply ADM to obtain approximate-exact solutions for linear/non-linear SFVIEs which are presented in the following two problems. To show the high accuracy of the solution results compared with the exact solutions, the norm errors are defined by $\boldsymbol{L}_\infty[\mathbf{0},\mathbf{1}],\boldsymbol{L}_{\mathbf{2},\sum}[\mathbf{0},\mathbf{1}],\boldsymbol{L}_{\mathbf{2},\int}[\mathbf{0},\mathbf{1}]$ and we give the error residual. Moreover, we present figures in the contour plot **2D** on the $(x,\alpha)-$ plane.

Problem 1. Consider the linear SFVIEs of the second kind

$$\begin{cases}\tilde{u}_1(x,\alpha)=\tilde{f}_1(t,\alpha)+\lambda_1\int_{0^+}^{x}[\tilde{u}_1(t,\alpha)+x\tilde{u}_2(t,\alpha)]dt\,,\\ \tilde{u}_2(x,\alpha)=\tilde{f}_2(t,\alpha)+\lambda_2\int_{0^+}^{x}[t\tilde{u}_1(t,\alpha)+t\tilde{u}_2(t,\alpha)]dt\,,\end{cases} \qquad (10)$$

where $\tilde{f}_1(x,\alpha)=\left[\underline{f_1}(x,\alpha),\overline{f}_1(x,\alpha)\right]$ and $\tilde{f}_2(x,\alpha)=\left[\underline{f_2}(x,\alpha),\overline{f}_2(x,\alpha)\right]$ are given by

$$\underline{f_1}(x,\alpha)=-x\alpha+\left(\frac{1}{50}x\alpha-\frac{3}{2}x^2\alpha-\frac{1}{20000}\alpha+\left(\frac{1}{25}x\alpha-\frac{3}{50}x\right)\ln(5)+\left(\frac{1}{25}x\alpha-\frac{3}{50}x\right)\ln(2)-\frac{3}{100}x+(2x^2\alpha-3x^2)\ln(x)+\frac{1}{10000}+2x^2\right)\lambda_1+2x,$$

$$\overline{f}_1(x,\alpha) = x\alpha + \left(-\frac{1}{50}x\alpha + \frac{3}{2}x^2\alpha + \frac{1}{20000}\alpha + \left(-\frac{1}{25}x\alpha + \frac{1}{50}x\right)\ln(2) + \frac{1}{100} + \left(-\frac{1}{25}x\alpha + \frac{1}{50}x\right)\ln(5) + (-2x^2\alpha + x^2)\ln(x) - x^2\right)\lambda_1,$$

$$\underline{f_2}(x,\alpha) = \left(\frac{1}{3}x^3\alpha + \frac{149}{3000000}\alpha - \frac{1}{2}x^2\alpha + \left(\frac{1}{5000}\alpha - \frac{3}{10000}\right)\ln(5) + \left(x^2\alpha - \frac{3}{2}x^2\right)\ln(x) + \left(\frac{1}{5000}\alpha - \frac{3}{10000}\right)\ln(2) + \frac{3}{4}x^2 - \frac{2}{3}x^3 - \frac{223}{3000000}\right)\lambda_2 + (-2\alpha+3)\ln(x),$$

$$\overline{f}_2(x,\alpha) = \left(-\frac{149}{3000000}\alpha - \frac{1}{3}x^3\alpha + \frac{1}{2}x^2\alpha + \left(-\frac{1}{5000}\alpha + \frac{1}{10000}\right)\ln(5) + \left(-x^2\alpha + \frac{1}{2}x^2\right)\ln(x) + \left(-\frac{1}{5000}\alpha + \frac{1}{10000}\right)\ln(2) + \frac{1}{40000} - \frac{1}{4}x^2\right)\lambda_2 + (2\alpha-1)\ln(x).$$

The exact solutions of the SFVIEs (3.10) are

$\underline{u}_{1E}(x,\alpha) = (2-\alpha)x\,,\qquad \overline{u}_{1E}(x,\alpha) = \alpha x,$

$\underline{u}_{2E}(x,\alpha) = (3-2\alpha)lnx,\quad \overline{u}_{2E}(x,\alpha) = (2\alpha-1)lnx$

The parametric form of the given SFVIEs (3.10) can be written as:

$$\begin{cases} \underline{u_1}(x,\alpha) = \underline{f_1}(x,\alpha) + \lambda_1\displaystyle\int_{0^+}^{x}\left[\underline{u_1}(t,\alpha) + x\underline{u_2}(t,\alpha)\right]dt\,, \\ \overline{u_1}(x,\alpha) = \overline{f_1}(x,\alpha) + \lambda_1\displaystyle\int_{0^+}^{x}[\overline{u}_1(t,\alpha) + x\overline{u}_2(t,\alpha)]dt\,, \\ \underline{u_2}(x,\alpha) = \underline{f_2}(x,\alpha) + \lambda_2\displaystyle\int_{0^+}^{x}\left[t\underline{u_1}(t,\alpha) + t\underline{u_2}(t,\alpha)\right]dt\,, \\ \overline{u_2}(x,\alpha) = \overline{f_2}(x,\alpha) + \lambda_2\displaystyle\int_{0^+}^{x}[t\overline{u}_1(t,\alpha) + t\overline{u}_2(t,\alpha)]dt\,. \end{cases} \tag{11}$$

Operating by the same way proceeding (3)–(7) as above on the SFVIEs (11), and applying the ADM, the lower iterations (L) are then determined in the following recursive way:

$$\begin{cases} \underline{u}_{1,0}(x,\alpha) = \underline{f_1}(x,\alpha), \\ \underline{u}_{2,0}(x,\alpha) = \underline{f_2}(x,\alpha), \end{cases}$$

$$\begin{cases} \underline{u}_{1,n+1}(x,\alpha) = \lambda_1\displaystyle\int_{0^+}^{x}\left[\underline{u}_{1,n}(t,\alpha) + x\underline{u}_{2,n}(t,\alpha)\right]dt \\ \underline{u}_{2,n+1}(x,\alpha) = \lambda_2\displaystyle\int_{0^+}^{x}\left[t\underline{u}_{1,n}(t,\alpha) + t\underline{u}_{2,n}(t,\alpha)\right]dt \end{cases}, \; n \geq 0 \tag{12}$$

and the upper iterations (U) are

$$\begin{cases} \overline{u}_{1,0}(x,\alpha) = \overline{f}_1(x,\alpha), \\ \overline{u}_{2,0}(x,\alpha) = \overline{f}_2(x,\alpha), \end{cases}$$

$$\begin{cases} \overline{u}_{1,n+1}(x,\alpha) = \lambda_1 \int_0^x \left[\overline{u}_{1,n}(t,\alpha) + x\overline{u}_{2,n}(t,\alpha)\right] dt \\ \overline{u}_{2,n+1}(x,\alpha) = \lambda_2 \int_0^x \left[t\overline{u}_{1,n}(t,\alpha) + t\overline{u}_{2,n}(t,\alpha)\right] dt \end{cases}, n \geq 0 \qquad (13)$$

Thus, the approximate solutions in a series form are:

$$\underline{\phi}_{1,7}(x,\alpha) = \sum_{i=0}^{6} \underline{u}_{1,i}(x,\alpha), \quad \overline{\phi}_{1,7}(x,\alpha) = \sum_{i=0}^{6} \overline{u}_{1,i}(x,\alpha)$$

$$\underline{\phi}_{2,7}(x,\alpha) = \sum_{i=0}^{6} \underline{u}_{2,i}(x,\alpha), \quad \overline{\phi}_{2,7}(x,\alpha) = \sum_{i=0}^{6} \overline{u}_{2,i}(x,\alpha).$$

Tables 3.1–3.4 display a comparison with the exact solutions ($\tilde{u}_{iE}(x,\alpha), i = 1,2$), the numerical results applying the ADM $\left(\tilde{\phi}_{i,7}(x,\alpha)\right)$ and the numerical solution of SFVIEs (12), (13) with the Simpson rule (ADM-SIMP) and the Trapezoidal rule (ADM-TRAP) on the interval [0,1]. Twenty points have been used in the Simpson and trapezoidal methods. In Tables 5–8 we present the maximum errors on the interval [0,1], where n represents the number of iterations. The error residual is given in Table 9.

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,7}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	1_L	0.34	0.340000000	0.339995000	0.339999999
	1_U	0.06	0.059999999	0.060000833	0.060000000
0.4	1_L	0.68	0.680000002	0.679865134	0.679999987
	1_U	0.12	0.119999999	0.120022477.	0.120000001
0.6	1_L	1.02	1.020000266	1.019376800	1.020000097
	1_U	0.18	0.017999992	0.180103836	0.179999954
0.8	1_L	1.36	1.360008727	1.358301639	1.360007872
	1_U	0.24	0.239997547	0.240282039	0.239997689
1.0	1_L	1.70	1.700140778	1.696524926	1.700137916
	1_U	0.30	0.299959035	0.300561236	0.299959511

Table 1. Numerical results for $\tilde{u}_1(x,\alpha)$ (Problem 1) when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,7}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	2_L	-3.862650989	-3.862650989	-3.862646815	-3.862650989
	2_U	0.643775164	0.643775164	0.643774712	0.643775164
0.4	2_L	-2.199097756	-2.199097755	-2.199016263	-2.199097753
	2_U	0.366516292	0.366516292	0.366509272	0.366516292
0.6	2_L	-1.225981497	-1.225981323	-1.225668933	-1.225981309
	2_U	0.204330249	0.204330201	0.204308518	0.204330199

0.8	2_L	-0.535544523	-0.535537477	-0.534785365	-0.535537423
	2_U	0.089257420	0.089255439	0.089213434	0.089255430
1.0	2_L	0.0	0.000130023	0.001581141	0.000130170
	2_U	0.0	-0.000037876	-0.000103020	-0.000037901

Table 2. Numerical results for $\tilde{u}_2(x,\alpha)$ (Problem 1) when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,7}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	1_L	0.22	0.220000000	0.219997500	0.219999999
	1_U	0.18	0.180000000	0.179998333	0.179999999
0.4	1_L	0.44	0.440000000	0.439932566	0.439999993
	1_U	0.36	0.360000000	0.359955044	0.359999995
0.6	1_L	0.66	0.660000120	0.659688387	0.660000035
	1_U	0.54	0.540000072	0.539792249	0.540000015
0.8	1_L	0.88	0.880003936	0.879150382	0.880003508
	1_U	0.72	0.720002338	0.719433297	0.720002053
1.0	1_L	1.10	1.100062888	1.098254773	1.100061457
	1_U	0.90	0.900036925	0.898831389	0.900035970

Table 3. Numerical results for $\tilde{u}_1(x,\alpha)$ (Problem 1) when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,7}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	2_L	-1.9313254949	-1.931325494	-1.931323303	-1.931325494
	2_U	-1.2875503299	-1.287550329	-1.287548799	-1.287550329
0.4	2_L	-1.0995488782	-1.099548877	-1.099505319	-1.099548876
	2_U	-0.7330325854	-0.733032585	-0.733001671	-0.733032584
0.6	2_L	-0.6129907485	-0.612990669	-0.612821454	-0.612990663
	2_U	-0.4086604990	-0.408660452	-0.408538961	-0.408660447
0.8	2_L	-0.2677722615	-0.267769084	-0.267357308	-0.267769057
	2_U	-0.1785148410	-0.178512953	-0.178214622	-0.178512935
1.0	2_L	0.0	0.000058066	0.000859358	0.000058139
	2_U	0.0	0.000034080	0.000618763	0.000034129

Table 4. Numerical results for $\tilde{u}_2(x,\alpha)$ (Problem 1) when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	1_L	4.406E-01	5.766E-01	1.494E-01
	1_U	1.268E-01	1.608E-01	4.082E-02

4	1_L	1.526E-01	1.818E-01	4.393E-02
	1_U	4.087E-02	4.796E-02	1.140E-02
5	1_L	4.127E-02	4.649E-02	1.053E-02
	1_U	1.055E-02	1.179E-02	2.640E-03
6	1_L	9.190E-03	9.984E-03	2.134E-03
	1_U	2.276E-03	2.461E-03	5.210E-04
7	1_L	8.944E-04	9.417E-04	1.871E-04
	1_U	2.646E-04	2.780E-04	5.482E-04
8	1_L	1.407E-04	1.459E-04	2.754E-05
	1_U	4.096E-05	4.241E-05	7.954E-06

Table 5. Norm Error for $\tilde{u}_1(x,\alpha)$ (Problem 1) when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	2_L	3.450E-01	4.328E-01	1.091E-01
	2_U	1.015E-01	1.239E-01	3.058E-02
4	2_L	1.279E-01	1.486E-01	3.503E-02
	2_U	3.461E-02	3.968E-02	9.204E-03
5	2_L	3.595E-02	3.986E-02	8.830E-03
	2_U	9.243E-02	1.017E-02	2.226E-03
6	2_L	8.202E-03	8.818E-03	1.845E-03
	2_U	2.037E-03	2.181E-03	4.523E-04
7	2_L	8.163E-04	8.543E-04	1.667E-04
	2_U	2.420E-04	2.527E-04	4.893E-04
8	2_L	1.300E-04	1.342E-04	2.490E-05
	2_U	3.787E-05	3.905E-05	7.202E-06

Table 6. Norm Error for $\tilde{u}_2(x,\alpha)$ (Problem 1) when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	1_L	1.974E-01	2.606E-01	6.790E-02
	1_U	1.163E-01	1.553E-01	4.074E-02
4	1_L	6.969E-02	8.336E-02	2.021E-02
	1_U	4.204E-02	5.054E-02	1.231E-02
5	1_L	1.906E-02	2.151E-02	4.890E-03
	1_U	1.165E-02	1.318E-02	3.007E-03

6	1_L	4.275E-03	4.650E-03	9.962E-04
	1_U	2.637E-03	2.872E-03	6.169E-04
7	1_L	3.976E-04	4.190E-04	8.346E-05
	1_U	2.320E-04	2.447E-04	4.889E-05
8	1_L	6.288E-05	6.523E-05	1.232E-05
	1_U	3.692E-05	3.832E-05	7.257E-06

Table 7. Norm Error for $\tilde{u}_1(x,\alpha)$ (Problem 1) when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
3	2_L	1.536E-01	1.942E-01	4.929E-02
	2_U	8.981E-02	1.147E-01	2.933E-02
4	2_L	5.828E-02	6.795E-02	1.607E-02
	2_U	3.505E-02	4.104E-02	9.757E-03
5	2_L	1.658E-02	1.841E-02	4.091E-03
	2_U	1.012E-02	1.127E-02	2.512E-03
6	2_L	3.813E-03	4.104E-03	8.609E-04
	2_U	2.350E-03	2.532E-03	5.326E-04
7	2_L	3.627E-04	3.799E-04	7.430E-05
	2_U	2.115E-04	2.217E-04	4.349E-05
8	2_L	5.806E-05	5.997E-05	1.114E-05
	2_U	3.408E-05	3.521E-05	6.558E-06

Table 8. Norm Error for $\tilde{u}_2(x,\alpha)$ (Problem 1) when $\alpha = 0.9$

		$\tilde{u}_1(x,\alpha)$			$\tilde{u}_2(x,\alpha)$		
x	i	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
0.2	L	1.253E-11	9.074E-12	5.616E-12	2.720E-12	1.970E-12	1.219E-12
	U	3.604E-12	1.466E-13	3.311E-12	7.822E-13	3.163E-14	7.189E-13
0.4	L	5.078E-08	3.688E-08	2.298E-08	2.240E-08	1.627E-08	1.014E-08
	U	1.408E-08	1.827E-10	1.371E-08	6.211E-09	7.854E-11	6.054E-09
0.6	L	3.493E-06	2.537E-06	1.581E-06	2.238E-06	1.625E-06	1.013E-06
	U	9.677E-07	1.167E-08	9.443E-07	6.200E-07	7.574E-09	6.049E-07
0.8	L	7.074E-05	5.128E-05	3.182E-05	5.628E-05	4.080E-05	2.531E-05
	U	2.006E-05	6.069E-07	1.885E-05	1.598E-05	4.955E-07	1.499E-05
1.0	L	7.536E-04	5.441E-04	3.347E-04	6.863E-04	4.955E-04	3.047E-04

U	2.237E-04	1.429E-05	1.951E-04	2.041E-04	1.332E-05	1.775E-04

Table 9. Error residual for (Problem 1)

In the following Figures 1–4, we present the contour plot in $2D$ on the $(x,\alpha)-$ plane for the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,7}(x,\alpha)\right)$. We represent the exact solutions with a continuous line and the ADM with the symbol $\circ$.

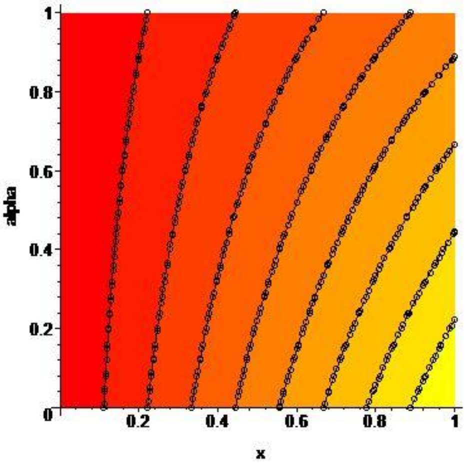

Fig. 1. The contour plot in $2D$ $(x,\alpha)-$ plane for the exact solution $\underline{u}_{1E}(x,\alpha)$ and the ADM $\underline{\phi}_{1,7}(x,\alpha)$

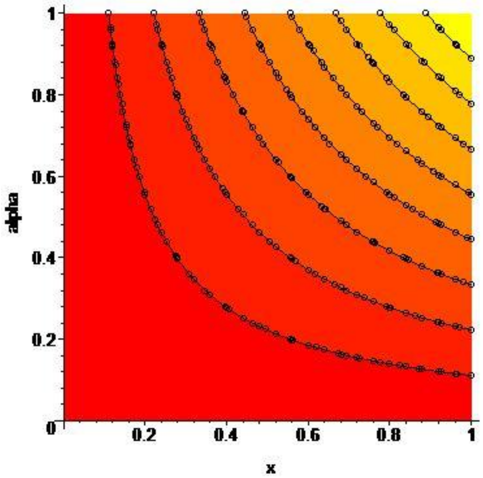

Fig. 2. The contour plot in $2D$ $(x,\alpha)-$ plane for the exact solution $\overline{u}_{1E}(x,\alpha)$ and the ADM $\overline{\phi}_{1,7}(x,\alpha)$

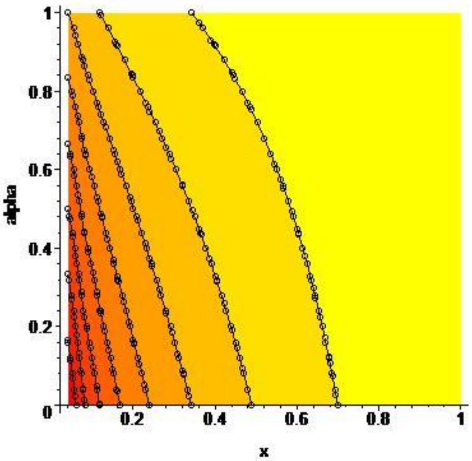

Fig. 3. The contour plot in $2D$ $(x,\alpha)-$ plane for the exact solution $\underline{u}_{2E}(x,\alpha)$ and the ADM $\underline{\phi}_{2,7}(x,\alpha)$

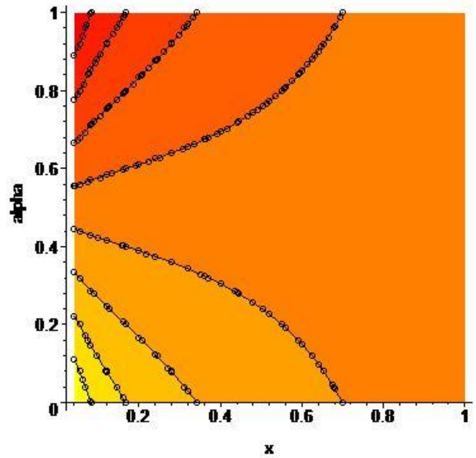

Fig. 4. The contour plot in $2D$ $(x,\alpha)-$ plane for the exact solution $\overline{u}_{2E}(x,\alpha)$ and the ADM $\overline{\phi}_{2,7}(x,\alpha)$

Problem 2. Let us consider the non-linear SFVIEs of the second kind:

$$\begin{cases} \tilde{u}_1(x,\alpha) = \tilde{f}_1(x,\alpha) + \lambda_1 \displaystyle\int_0^x [\tilde{u}_1^2(t,\alpha) + \tilde{u}_2(t,\alpha)]dt\,, \\ \tilde{u}_2(x,\alpha) = \tilde{f}_2(x,\alpha) + \lambda_1 \displaystyle\int_0^x \tilde{u}_1(t,\alpha)\tilde{u}_2(t,\alpha)dt, \end{cases} \tag{14}$$

where $\tilde{f}_1(x,\alpha) = \left[\underline{f_1}(x,\alpha), \overline{f}_1(x,\alpha)\right]$ and $\tilde{f}_2(x,\alpha) = \left[\underline{f_2}(x,\alpha), \overline{f}_2(x,\alpha)\right]$ are given by

$$\underline{f_1}(x,\alpha)=\frac{1}{10}x\alpha+\left(\frac{1}{12}x^3\alpha^3-\frac{1}{300}x^3\alpha^2+\frac{11}{150}x^3\alpha-\frac{41}{50}x^3\right)\lambda_1+\frac{11}{10}x,$$
$$\overline{f}_1(x,\alpha)=\frac{3}{10}x\alpha+\left(-\frac{49}{300}x^3\alpha^2-\frac{7}{50}x^3\alpha-\frac{109}{300}x^3\right)\lambda_1+\frac{7}{10}x,$$
$$\underline{f_2}(x,\alpha)=-\frac{1}{4}x^2\alpha^3+\left(-\frac{1}{160}x^4\alpha^4+\frac{11}{160}x^4\alpha^3+\frac{1}{32}x^4\alpha-\frac{11}{32}x^4\right)\lambda_2+\frac{5}{4}x^2,$$
$$\overline{f}_2(x,\alpha)=\frac{2}{5}x^2\alpha^2+\left(-\frac{3}{100}x^4\alpha^3-\frac{7}{100}x^4\alpha^2-\frac{9}{200}x^4\alpha-\frac{21}{200}x^4\right)\lambda_2+\frac{3}{5}x^2.$$

The exact solutions of the SFVIEs (14) are

$$\underline{u}_{1E}(x,\alpha)=(1.1-0.1\alpha)x,\qquad \overline{u}_{1E}(x,\alpha)=(0.3\alpha+0.7)x,$$
$$\underline{u}_{2E}(x,\alpha)=(2.5-0.5\alpha^3)x^2,\quad \overline{u}_{2E}(x,\alpha)=(0.4\alpha^2+0.6)x^2.$$

The parametric form of the given SFVIEs (14) can be written as:

$$\begin{cases}\underline{u}_1(x,\alpha)=\underline{f_1}(x,\alpha)+\lambda_1\int_0^x\left[\underline{u}_1^2(t,\alpha)+\underline{u}_2(t,\alpha)\right]dt,\\ \overline{u}_1(x,\alpha)=\overline{f}_1(x,\alpha)+\lambda_1\int_0^x\left[\overline{u}_1^2(t,\alpha)+\overline{u}_2(t,\alpha)\right]dt,\\ \underline{u}_2(x,\alpha)=\underline{f_2}(x,\alpha)+\lambda_2\int_0^x\left[\underline{u}_1(t,\alpha)\underline{u}_2(t,\alpha)\right]dt,\\ \overline{u}_2(x,\alpha)=\overline{f}_2(x,\alpha)+\lambda_2\int_0^x[\overline{u}_1(t,\alpha)\overline{u}_2(t,\alpha)]dt.\end{cases}\tag{15}$$

Operating by the same way proceeding (3)–(7) as above on the SFVIEs (15), and applying the ADM, the lower iterations (L) are then determined in the following recursive way:

$$\begin{cases}\underline{u}_{1,0}(x,\alpha)=\underline{f_1}(x,\alpha),\\ \underline{u}_{2,0}(x,\alpha)=\underline{f_2}(x,\alpha),\end{cases}$$
$$\begin{cases}\underline{u}_{1,n+1}(x,\alpha)=\lambda_1\int_0^1\left[\underline{A}_{1,n}+\underline{u}_{2,n}(x,\alpha)\right]dt\\ \underline{u}_{2,n+1}(x,\alpha)=\lambda_2\int_0^1\underline{B}_{1,n}dt\end{cases},\ n\geq 0\tag{16}$$

and the upper iterations (U) are

$$\begin{cases}\overline{u}_{1,0}(x,\alpha)=\overline{f}_1(x,\alpha),\\ \overline{u}_{2,0}(x,\alpha)=\overline{f}_2(x,\alpha),\end{cases}$$
$$\begin{cases}\overline{u}_{1,n+1}(x,\alpha)=\lambda_1\int_0^1\left[\overline{A}_{1,n}+\overline{u}_{2,n}(t,\alpha)\right]dt\\ \overline{u}_{2,n+1}(x,\alpha)=\lambda_2\int_0^1\overline{B}_{1,n}dt\end{cases},\ n\geq 0\tag{17}$$

For the non-linear terms defined by

$$\underline{u}_{1,n}^2(t,\alpha)=\underline{N}_1=\sum_{n=0}^{\infty}\underline{A}_{1,n},$$

$$\overline{u}_{1,n}^{2}(t,\alpha) = \overline{N}_1 = \sum_{n=0}^{\infty} \overline{A}_{1,n},$$

$$\underline{u}_{1,n}(t,\alpha)\underline{u}_{2,n}(t,\alpha) = \underline{N}_2 = \sum_{n=0}^{\infty} \underline{B}_{1,n},$$

$$\overline{u}_{1,n}(t,\alpha)\overline{u}_{2,n}(t,\alpha) = \overline{N}_2 = \sum_{n=0}^{\infty} \overline{B}_{1,n},$$

the corresponding Adomian polynomials $[\underline{A}_{1,n}, \overline{A}_{1,n}]$ and $[\underline{B}_{1,n}, \overline{B}_{1,n}]$ are given by

$$\underline{A}_{1,n} = \sum_{i=0}^{n} \underline{u}_{1,i}\underline{u}_{1,n-i}, \quad n \geq i, \quad n \geq 0$$

$$\overline{A}_{1,n} = \sum_{i=0}^{n} \overline{u}_{1,i}\overline{u}_{1,n-i}, \quad n \geq i, \quad n \geq 0$$

$$\underline{B}_{1,n} = \sum_{i=0}^{n} \underline{u}_{1,i}\underline{u}_{2,n-i}, \quad n \geq i, \quad n \geq 0$$

$$\overline{B}_{1,n} = \sum_{i=0}^{n} \overline{u}_{1,i}\overline{u}_{2,n-i}, \quad n \geq i, \quad n \geq 0$$

We take $\lambda_1 = \lambda_2 = 1$, the approximate solutions in a series form are

$$\underline{\phi}_{1,5}(x,\alpha) = \sum_{i=0}^{4} \underline{u}_{1,i}(x,\alpha), \quad \overline{\phi}_{1,5}(x,\alpha) = \sum_{i=0}^{4} \overline{u}_{1,i}(x,\alpha)$$

$$\underline{\phi}_{2,5}(x,\alpha) = \sum_{i=0}^{4} \underline{u}_{2,i}(x,\alpha), \quad \overline{\phi}_{2,5}(x,\alpha) = \sum_{i=0}^{4} \overline{u}_{2,i}(x,\alpha).$$

Tables 10–13 display a comparison with the exact solutions $(\tilde{u}_{iE}(x,\alpha), i = 1,2)$, the numerical results applying the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ and the numerical solution of SFVIEs (12), (13) with the Simpson rule (ADM-SIMP) and the Trapezoidal rule (ADM-TRAP) on the interval [0,1]. Twenty points have been used in the Simpson and trapezoidal methods. In Tables 14–17, we present the maximum errors on the interval [0,1], where n represents the number of iterations. The error residual is given in Table 18.

x	i	**Exact** $\tilde{u}_{1E}(x,\alpha)$	**ADM** $\tilde{\phi}_{1,5}(x,\alpha)$	**ADM-TRAP**	**ADM-SIMP**
0.2	1_L	0.214	0.213999999	0.213999999	0.214007960
	1_U	0.158	0.157999999	0.158004200	0.157999999
0.4	1_L	0.428	0.427999894	0.427999894	0.428063575
	1_U	0.316	0.315999988	0.316033590	0.315999988
0.6	1_L	0.642	0.641982135	0.642196538	0.641982133
	1_U	0.474	0.473997903	0.474111248	0.473997903

0.8	1_L	0.856	0.855385173	0.855877872	0.855385167
	1_U	0.632	0.631923594	0.632190220	0.631923593
1.0	1_L	1.070	1.061475697	1.062265693	1.061475640
	1_U	0.790	0.788851485	0.789346052	0.788851475

Table 10. Numerical results for $\tilde{u}_1(x,\alpha)$ (Problem 2) when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	2_L	0.04973	0.049729999	0.049731330	0.049729999
	2_U	0.02544	0.025439999	0.025440502	0.025439999
0.4	2_L	0.19892	0.198919973	0.198941256	0.198919973
	2_U	0.10176	0.101759997	0.101768036	0.1017599978
0.6	2_L	0.44757	0.447563273	0.447670792	0.4475632733
	2_U	0.22896	0.228959444	0.229000122	0.2289594448
0.8	2_L	0.79568	0.795371625	0.795702233	0.7953716211
	2_U	0.40704	0.407013042	0.407140755	0.4070130418
1.0	2_L	1.24325	1.237908315	1.238585883	1.2379082594
	2_U	0.63600	0.635493789	0.635791895	0.6354937821

Table 11. Numerical results for $\tilde{u}_2(x,\alpha)$ (Problem 2) when $\alpha = 0.3$

x	i	Exact $\tilde{u}_{1E}(x,\alpha)$	ADM $\tilde{\phi}_{1,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	1_L	0.202	0.2019999999	0.2020069594	0.2019999999
	1_U	0.194	0.1939999999	0.1940062163	0.1939999999
0.4	1_L	0.404	0.4039999325	0.4040556064	0.4039999325
	1_U	0.388	0.3879999531	0.3880496823	0.3879999531
0.6	1_L	0.606	0.6059884801	0.6061760439	0.6059884799
	1_U	0.582	0.5819919385	0.5821595386	0.5819919384
0.8	1_L	0.808	0.8075980891	0.8080323937	0.8075980849
	1_U	0.776	0.7757157221	0.7761056375	0.7757157190
1.0	1_L	1.010	1.0043168772	1.0050466677	1.0043168359
	1_U	0.970	0.9659189854	0.9665935038	0.9659189537

Table 12. Numerical results for $\tilde{u}_1(x,\alpha)$ (Problem 2) when $\alpha = 0.9$

x	i	Exact $\tilde{u}_{2E}(x,\alpha)$	ADM $\tilde{\phi}_{2,5}(x,\alpha)$	ADM-TRAP	ADM-SIMP
0.2	2_L	0.04271	0.0427099999	0.0427110784	0.0427099999
	2_U	0.03696	0.0369599999	0.0369608962	0.0369599999

0.4	2_L	0.17084	0.1708399844	0.1708572387	0.1708399844
	2_U	0.14784	0.1478399901	0.1478543302	0.1478399901
0.6	2_L	0.38439	0.3843860200	0.3844732337	0.3843860199
	2_U	0.33264	0.3326374537	0.3327099631	0.3326374536
0.8	2_L	0.68336	0.6831749954	0.6834450382	0.6831749925
	2_U	0.59136	0.5912403521	0.5914658623	0.5912403502
1.0	2_L	1.06775	1.0644817386	1.0650595188	1.0644817015
	2_U	0.92400	0.9218540025	0.9223499247	0.9218539768

Table 13. Numerical results for $\tilde{u}_2(x,\alpha)$ (Problem 2) when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
4	1_L	7.593E-02	8.412E-02	1.866E-02
	1_U	1.879E-02	2.065E-02	4.521E-03
6	1_L	1.762E-02	1.853E-02	3.682E-03
	1_U	2.598E-03	2.715E-03	5.292E-04
8	1_L	4.067E-03	4.170E-03	7.526E-04
	1_U	3.578E-04	3.655E-04	6.440E-05
10	1_L	9.375E-04	9.495E-04	1.574E-04
	1_U	4.927E-05	4.978E-05	8.030E-06

Table 14. Norm Error for $\tilde{u}_1(x,\alpha)$ (Problem 2) when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
4	2_L	4.207E-02	4.556E-02	9.709E-03
	2_U	7.199E-03	7.755E-03	1.633E-03
6	2_L	9.620E-03	1.000E-02	1.921E-03
	2_U	9.861E-04	1.021E-03	1.926E-03
8	2_L	2.214E-03	2.258E-03	3.954E-04
	2_U	1.356E-04	1.379E-04	2.362E-05
10	2_L	5.101E-04	5.153E-04	8.316E-04
	2_U	1.867E-05	1.883E-05	2.961E-06

Table 15. Norm Error for $\tilde{u}_2(x,\alpha)$ (Problem 2) when $\alpha = 0.3$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
4	1_L	5.730E-02	6.335E-02	1.400E-02
	1_U	4.547E-02	5.019E-02	1.106E-02

6	1_L	1.198E-02	1.258E-02	2.489E-03
	1_U	8.724E-03	9.148E-03	1.802E-03
8	1_L	2495E-03	2555E-03	4585E-04
	1_U	1667E-03	1706E-03	3048E-04
10	1_L	5.189E-04	5.252E-04	8.648E-05
	1_U	3.184E-04	3.221E-04	5.277E-05

Table 16. Norm Error for $\tilde{u}_1(x,\alpha)$ (Problem 2) when $\alpha = 0.9$

n	i	$L_\infty[0,1]$	$L_{2,\sum}[0,1]$	$L_{2,\int}[0,1]$
4	2_L	2.901E-02	3.138E-02	6.669E-03
	2_U	2.100E-02	2.269E-02	4.813E-03
6	2_L	5.993E-03	6.228E-03	1.190E-03
	2_U	3.989E-03	4.142E-03	7.891E-04
8	2_L	1.244E-03	1.269E-03	2.209E-04
	2_U	7.612E-04	7.755E-04	1.344E-04
10	2_L	2.588E-04	2.613E-04	4.191E-05
	2_U	1.453E-04	1.467E-04	2.341E-05

Table 17. Norm Error for $\tilde{u}_2(x,\alpha)$ (Problem 2) when $\alpha = 0.9$

		$\tilde{u}_1(x,\alpha)$			$\tilde{u}_2(x,\alpha)$		
x	i	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
0.2	L	1.689E-09	1.459E-09	1.143E-09	1.754E-10	1.491E-10	1.092E-10
	U	2.555E-10	4.529E-10	8.314E-10	1.873E-11	3.549E-11	7.293E-11
0.4	L	3.061E-06	2.652E-06	2.089E-06	6.356E-07	5.419E-07	3.991E-07
	U	4.789E-07	8.415E-07	1.528E-06	7.023E-08	1.318E-07	2.680E-07
0.6	L	2.146E-04	1.869E-04	1.485E-04	6.676E-05	5.723E-05	4.252E-05
	U	3.562E-05	6.162E-05	1.098E-04	7.832E-06	1.447E-05	2.888E-05
0.8	L	3.729E-03	3.274E-03	2.637E-03	1.543E-03	1.334E-03	1.005E-03
	U	6.776E-04	1.144E-03	1.982E-03	1.984E-04	3.582E-04	6.943E-04
1.0	L	2.837E-02	2.522E-02	2.070E-02	1.458E-02	1.276E-02	9.815E-03
	U	5.870E-03	9.588E-03	1.594E-02	2.144E-03	3.740E-03	6.952E-03

Table 18. Error residual for (Problem 2)

In the following Figures 5–8, we present plot of the exact solutions $\left(\tilde{u}_{iE}(x,\alpha)\right)$ and the ADM $\left(\tilde{\phi}_{i,5}(x,\alpha)\right)$ when $\alpha = 0.3$. We represent the exact solutions with a continuous line and the ADM with the symbol ∘.

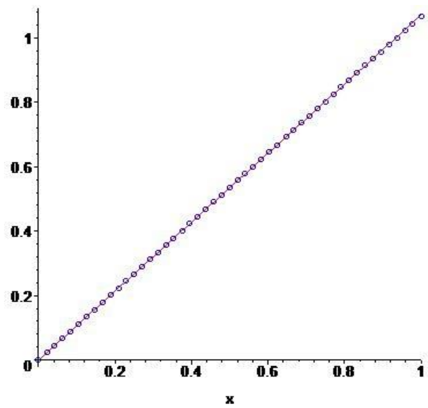

Fig. 5. Plot the exact solution $\underline{u}_{1E}(x,\alpha)$ and the ADM $\underline{\phi}_{1,5}(x,\alpha)$ when $\alpha = 0.3$

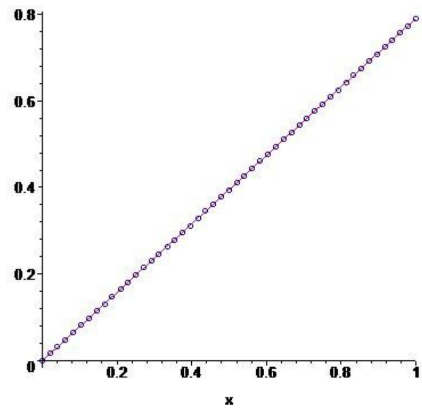

Fig. 6. Plot the exact solution $\overline{u}_{1E}(x,\alpha)$ and the ADM $\overline{\phi}_{1,5}(x,\alpha)$ when $\alpha = 0.3$

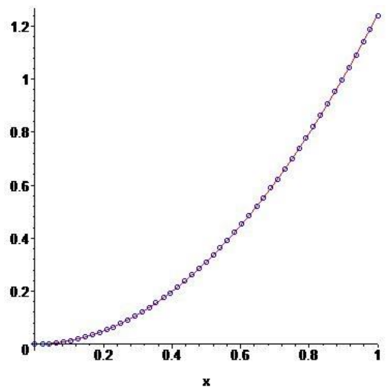

Fig. 7. Plot the exact solution $\underline{u}_{2E}(x,\alpha)$ and the ADM $\underline{\phi}_{2,5}(x,\alpha)$ when $\alpha = 0.3$

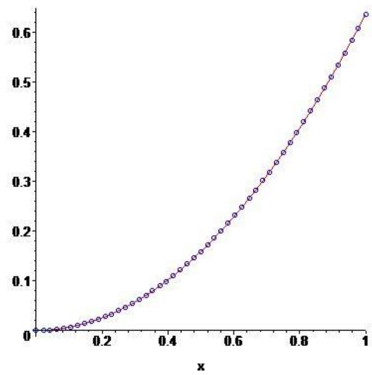

Fig. 8. Plot the exact solution $\overline{u}_{2E}(x,\alpha)$ and the ADM $\overline{\phi}_{2,5}(x,\alpha)$ when $\alpha = 0.3$

Conclusions

The ADM has been successfully used to solve the FSVIEs giving it wider applicability. The results obtained in all studied cases demonstrate the reliability and the efficiency of this method. This was confirmed by the numerical results and graphical. The results obtained from the two given examples, have been shown that the ADM is a powerful and efficient technique in finding an approximate solution of both linear and non-linear FSVIEs. According to comparison of the numerical results, mentioned in the tables. Therefore, increasing the number of iterations in the ADM, makes the approximate solution tends to the exact solution. The increased accuracy of the results can also lead to better predictions and more effective decision-making. Finally, we conclude that the accuracy of ADM is advantageous.

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