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Some Properties Related with Spectral Compact Operators on Banach Spaces

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ABSTRACT

Spectral compact operators play a crucial role in functional analysis, particularly in understanding the spectral behavior of linear operators on Banach spaces. These operators exhibit properties that make their spectra more manageable compared to general bounded operators. This paper explores key spectral properties of compact operators in Banach spaces, including the discreteness of the spectrum, the finite-dimensionality of eigenspaces, the spectral radius formula, and perturbation properties. We also discuss applications of these operators in solving integral equations, differential equations, and data analysis.

1. Introduction

The study of compact operators in Banach spaces is a fundamental topic in functional analysis, bridging operator theory with applications in mathematical physics, differential equations, and numerical analysis. Compact operators generalize finite-dimensional matrices while retaining many useful spectral properties. The spectral theory of compact operators is well-developed, and such operators exhibit behaviors similar to matrices in terms of eigenvalues and spectral decomposition.

In this paper, we explore various spectral properties of compact operators, emphasizing their spectral structure, eigenvalues, and perturbation results. We also provide applications that illustrate their significance in practical problems.

2. Preliminaries

The next section presents the fundamental concepts and definitions necessary for comprehending the spectral features of compact operators in Banach spaces. The preliminaries encompass the definition of compact operators, the spectrum of an operator, and essential conclusions from functional analysis.

Definition(2.1)[1]

Suppose that X denotes field F. A norm on X is a function $\|.\|: X \to \mathbb{R}$ having the following properties.

1.$\|x\| \geq 0$ for all $x \in X$

2. $\|x\| = 0$ iff $x = 0$

3.$\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and for all $\lambda \in F$

4.$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

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- The linear space X over F, accompanied using the norm $\|.\|$, is referred to as a normed space and is represented as $(X, \|.\|)$ or merely X.

- A norm $\|.\|$ on a linear space X is defined as strictly convex if $\|x+y\| = \|x\| + \|y\|$ holds exclusively when x and y are linearly independent.

Special cases:

1. Banach Spaces: Normed spaces that are complete with respect to the norm-induced metric.

2.Hilbert Spaces: Complete normed spaces with a norm derived from an inner product.

Definition(2.2)[5]
A mathematical function An operator $T: X \to Y$ is defined as a mapping from linear space X to linear space Y, provided that both spaces are over the same field F. A linear operator is defined as an operator that satisfies specific linearity conditions.
$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$ and for all $\alpha, \beta \in F$.

Definition(2.3)[4]
Suppose both X and Y denote normed spaces over the field F. A linear operator $T: X \to Y$ is deemed bounded if $\|T(x)\| \leq k\|X\|$ for every $x \in X$, where k is a constant.
The set of bounded linear operators from X to Y is denoted as $B(X,Y)$. It is clear that $B(X,Y)$ is a normed space having regard to the norm provided by
$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$ for all $T \in B(X,Y)$.

Definition(2.4)[6]
Suppose both X as well as Y denote normed spaces. A linear operator $T: X \to Y$ is termed a compact linear operator (or entirely continuous linear operator) if, for any bounded subset A of X, the image $T(A)$ is relatively compact, meaning that the closure $T(A)$ is compact.

Examples of Compact Operators [5]

1. Finite-rank Operators: If T has a finite-dimensional range, it is always compact.
2. Integral Operators : Operators of the form $(Tf)(x) = \int_a^b K(x,y) f(y) dy$
3. Hilbert-Schmidt Operators: If T on a Hilbert space satisfies $\sum_{i,j} \left|\left\langle T(e_i), e_j \right\rangle\right|^2 < \infty$

Since compact operators generalize finite-dimensional matrices while retaining many spectral properties, their study is crucial in infinite-dimensional analysis.

3.The Spectrum of an Operator
The spectrum of an operator T, denoted $\sigma(T)$, plays a crucial role in understanding its properties.

Definition(3.1) (Spectrum of an Operator)

The spectrum of a bounded operator T on a Banach space X is the set

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible }\}$

The spectrum can be divided into three parts:

1. Point Spectrum $\sigma_p(T)$: The set of eigenvalues, i.e., values of λ for which $T-\lambda I$ is not injective.

2. Continuous Spectrum $\sigma_c(T)$: Values of λ for which $T-\lambda I$ is injective but not surjective, and its range is dense but not closed.

3. Residual Spectrum $\sigma_r(T)$: Values of λ for which $T-\lambda I$ is injective but has a non- dense range.

For compact operators, the spectrum has a particularly simple structure, as we will discuss in later sections.

Definition(3.1)[7]

Suppose that X denote a vector space over F, while consider $T: X \to Y$ represent a linear transformation.

1. A scalar $\lambda \in F$ is classified as an eigenvalue of T in the event contains a non-zero vector x such that $T(x) = \lambda x$.

A non-zero vector $x \in X$ is termed an eigenvector of T if there exists $\lambda \in F$ such that $T(x) = \lambda x$.

$\ker(T-\lambda I) = \{x \in X : (T-\lambda I)(x) = 0\}$

For compact operators, nonzero eigenvalues have particularly interesting properties:

The eigenspace associated with any nonzero amplitude is finite-dimensional.

If the spectrum of a compact operator is infinite, then the only possible accumulation point is zero.

From (1) and (2), we assert that is an eigenvector of T corresponding to the eigenvalue λ.

Eigenvalues are occasionally referred to as characteristic values, appropriate values, or spectral values. Eigenvectors are often referred to as characteristic vectors, appropriate vectors, or spectral vectors.

The set of each of the eigenvalues of T is called the spectrum of T, represented as $\sigma(T)$.

Definition (3.2)[11]

Let X denote a complex Banach space as well as letting T belong to $B(X)$. The spectral radius $r_\sigma(T)$ of T is defined as the radius.

$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$

The smallest closed disc centred at its starting point of the complex λ-plane that encompasses $\sigma(T)$.

Remark

From the above theorem, we have $r_\sigma(T) \le \|T\|$.

This result is particularly useful for compact operators because they often have a simple spectral structure.

4. Spectral Properties of Compact Operators in Banach Spaces

Spectral compact operators on Banach spaces exhibit several important properties that generalize results from finite-dimensional spectral theory to infinite-dimensional settings. Below are some key properties:

Spectral compact operators on Banach spaces have several significant properties that generalize classical spectral theory from finite-dimensional spaces to infinite-dimensional settings. Below are some key properties:

Theorem(4.1) Spectral Discreteness

Assume that X be a Banach space as well as $T: X \rightarrow Y$ be a compact linear operator; subsequently

1. The spectrum $\sigma(T)$ is at most countable with 0 as the only possible accumulation point.

2. The spectrum consists of eigenvalues (point spectrum) and possibly 0.

Theorem(4.2) Eigenvalues and Eigen spaces

Consider X to be an infinite-dimensional Banach space along with consider $T: X \rightarrow Y$ be a compact linear operator.

1. Every nonzero eigenvalue of T has finite multiplicity.

2. If $\lambda \neq 0$is an eigenvalue of T, then the corresponding eigenspace $\ker(T - \lambda I)$ is finite dimensional.

The set of eigenvalues (point spectrum) forms the dominant part of the spectrum, except for 0

Proof:

E_λStep 1: Definition of eigenvalue and eigenspace

An eigenvalue λ of T exists such that there is a nonzero vector x in X satisfying

$$Tx = \lambda x.$$

$$= \ker(T - \lambda \mathrm{I}) = \{x \epsilon X | (T - \lambda \mathrm{I})x = 0\}$$

Step 2: $T - \lambda \mathrm{I}$ is compact

Since T is compact and $\lambda \neq 0,$

$$T - \lambda \mathrm{I} = \lambda \left(\frac{T}{\lambda} - \mathrm{I}\right).$$

Step 3: show that E_λ is finite-dimensional

Suppose contradiction, that E_λ is infinite-dimensional.

We can then take a linearly indepent sequence $x_n \subset E_{\lambda.}$

Since T is compact, it maps any bounded sequence to a relatively compact set.

However, x_n is an independent sequence, then $T_{xn} = \lambda x_n$ cannot have a convergent subsequence.This contradiction the compactness of T.Thus E_λ must be finite-dimensional.

Step 4: Show that eigenvalues have finite multiplicity

The multiplicity of an eigenvalue refers to the frequency with which it occurs as a root of the operator's characteristic equation.

Given that the eigenspace E_λ is finite-dimensional, the quantity of linearly independent eigenvectors associated with λ is limited.

This means that $(T - \lambda \mathrm{I})$ cannot have an infinite number of independent eigenvectors.

Hence, the algebraic multiplicity of any eigenvalue $\lambda \neq 0$ is finite.e-dimensional.

Theorem(4.3) Spectral Radius Property

Suppose that X be a Banach space as well as consider $T: X \to X$ be a bounded linear operator; consequently

1. $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$

2. If T is compact, then either $r(T) = 0$ or T has at least one nonzero eigenvalue.

Proof:

Definition of the spectral radius

The spectral radius of the operator T is given as follows:

$r(T) = \sup\{|\lambda|: \lambda \epsilon\, \sigma(T)\}$

We show that $r(T) \leq \limsup_{n\to\infty} |T^n|^{\frac{1}{n}}$

Take any complex number λ such that $\lambda \notin \sigma(T)$,that is $(\lambda \mathrm{I} - T)$ exists and is bounded. BY the definition of the spectrum,we have

$\| (\lambda I - T)^{-1} \| < \infty$. We show that $r(T) \geq \liminf_{n\to\infty} |T^n|^{\frac{1}{n}}$

By Gelfand's Formula,

$r(T) = \lim_{n\to\infty} \| T^n \|^{\frac{1}{n}}$

If T is a compact operator, its spectrum $o(T)$ either consists only of zero or contains nonzero eigenvalues that accumulate at zero. Therefore, either $r(T) = 0$ or T has at least one nonzero eigenvalue.

Theorem(4.4) Fredholm Alternative

Consider X to be a Banach space with $\lambda \neq 0$. If $T: X \to X$ is a compact linear operator, then either $(T - \lambda I)$ is invertible, or $(T - \lambda I)$ is not invertible, indicating that λ is an eigenvalue.
This ensures that the range of $(T - \lambda \mathrm{I})$is closed.

Proof:

Suppose $T - \lambda \mathrm{I}$ is not invertible. By the Riesz-Schauder theory, $\lambda \epsilon\, \sigma_p(T)$. If $\lambda \notin \sigma_p(T)$, then $T - \lambda \mathrm{I}$ is injective but not surjective $(T - \lambda \mathrm{I})x = 0 \Longrightarrow x = 0$.
$\mathrm{Ker}(T - \lambda \mathrm{I}) = \{0\}$
Ran $(T - \lambda \mathrm{I}) \neq X$. However, compactness implies ran $(T - \lambda \mathrm{I})$ is closed and dim ker $(T - \lambda \mathrm{I})\mathrm{I}$ = codim ran$(T - \lambda \mathrm{I})$, then $\lambda \epsilon \sigma_p(T)$. Is a hance λ is an eigenvalue.

Theorem(4.5) Spectral Mapping Theorem for Polynomials

Suppose that X denote a Banach space, and let $T: X \to X$ represent a compact linear operator. If $p(\lambda)$ is a polynomial, then $\sigma(p(T)) = p\big(\sigma(T)\big)$.

Proof:

$\Rightarrow p\big(\sigma(T)\big) \subseteq \sigma(p(T))$
We show that every element in $p(\sigma(T))$ is also in $\sigma(p(T))$.

Take $\lambda\epsilon\sigma(T)$. Since $p(\lambda)$ is a polynomial,we show that $p(\lambda)\epsilon\sigma\big(p(T)\big)$.
Suppose for contradiction, that $p(\lambda) \notin \sigma\big(p(T)\big)$. Then operator $p(T) - P(\lambda)\mathrm{I}$ is invertible in Banach space X, there exists an operator B such that
$(p(T) - P(\lambda)\mathrm{I})B = B(p(T) - P(\lambda)\mathrm{I}) = \mathrm{I}$.
$p(T) - p(\lambda)\mathrm{I} = \sum_{k=0}^{n} a_k(T^k - \lambda^k I)$.
This contradiction that $p(\lambda) \notin \sigma\big(p(T)\big)$, prove that $p(\lambda)\epsilon\sigma\big(p(T)\big)$, is a hence
$p\big(\sigma(T)\big) \subseteq \sigma\big(p(T)\big)$.
$\Leftarrow \ \sigma(p(T)) \subseteq p(\sigma(T))$
We take $\mu \,\epsilon\, \sigma(p(T))$ and show that there exists $\lambda \in \sigma(T)$ with $\mu = p(\lambda)$.
If $p(T) - \mu\mathrm{I}$ is not invertible, then there exists a nonzero vector $x \,\epsilon\, X$ such that
$(p(T) - \mu\mathrm{I})x = 0$.
$\sum_{k=0}^{n} a_k T^K x = \mu x$.
Using elementary algebraic properties of polynomials, there exist an eigenvalue λ of T such that $\mu = p(\lambda)$.
Then, $\mu \,\epsilon\, p(\sigma(T))$ is a hence $\sigma\big(p(T)\big) \subseteq p\big(\sigma(T)\big)$.
The hence $\sigma\big(p(T)\big) = P\big(\sigma(T)\big)$.

Theorem(4.6) Compact Perturbation and Essential Spectrum
Suppose that X be a Banach space along with letting $T: X \to X$ be a bounded linear operator. If $S: X \to X$ is a compact linear operator, subsequently the essential spectrum (the set of accumulation points and continuous spectrum) is invariant under compact perturbations
$\sigma_{ess}(T+S) = \sigma_{ess}(T)$

Theorem(4.7) Schauder's Theorem
Suppose that X be a Banach space and let $T: X \to X$ represent a compact linear operator, then it's a djoint T^* in the dual space X^* is also compact.
- This is useful for studying duality in spectral problems.

Proof:
Step 1:Take a bounded sequence in X^*
Letting $f_n \subset X^*$ be a bounded sequence in the dual space X^*, there exists an identity constant $C > 0$
$\| f_n \| \leq C \qquad \forall n$
Step 2: we show that $T^*(f_{nk})$ has a norm-convergent subsequence
For any $x \in X$, since f_{nk} converges weak to f, then
$\langle f_{nk\,,\ Tx}\rangle \to \langle f\ , Tx\rangle$.
Step 3: show that $T^*(f_{nk})$ is compact in X^*
By definition, we have
$T^* f_{nk}(x) = \langle f_{nk\,,} Tx\rangle$.
Given that every bounded sequence in X^* possesses a norm-convergent subsequence, T^* is compact.

Theorem(4.8) Existence of Spectral Decomposition for Self-Adjoint Compact Operators
Consider X to be a Hilbert space along with letting $T: X \to X$ be a compact, self-adjoint linear operator. Then, there exists an orthonormal basis composed of eigenvectors of T, corresponding to a discrete spectrum.
This generalizes the spectral theorem to compact operators.
These properties make compact operators particularly useful in integral equations, functional analysis, and operator theory.

Comparison with General Bounded Operators

Property	**General Bounded Operators**	**Compact Operators**
Spectrum	Can be uncountable	At most countable
Eigenvalues	May have infinite dimensional eigenspaces	Finite dimensional eigenspaces
Spectral accumulation	Can accumulate anywhere	Only at 0 (if infinite)
Essential spectrum	Can be complex	Invariant under compact perturbation

These properties make compact operators a fundamental class of operators in functional analysis, particularly in applications to integral equations and spectral theory.

Conclusion of the Preliminaries
In this section, we introduced compact operators and their spectral properties. We defined key concepts such as the spectrum, eigenvalues, spectral radius, and perturbation results. These fundamental results set the stage for an in-depth study of the spectral properties of compact operators in Banach spaces.

5. Applications of Spectral Compact Operators

1. Integral Equations : Compact operators frequently appear in integral equations of the form $\phi(x) - \int_a^b K(x,y)\phi(y)dy = f(x), \quad (T\phi)(x) = \int_a^b K(x,y)\phi(y)dy$
2. Differential Equations: Compact operators also appear in boundary value problems for differential equations, particularly when using Green's functions and spectral decomposition methods.
3. Data Analysis and Machine Learning: In high-dimensional data analysis, compact operators are used in Principal Component Analysis (PCA), where eigenvalues of a covariance matrix (which is often modeled as a compact operator) help in dimensionality reduction.

6. Conclusion

In this paper, we discussed the fundamental spectral properties of compact operators in Banach spaces. We established that their spectra are at most countable, their eigenvalues have finite multiplicity, and they exhibit useful perturbation properties. These results have important applications in integral equations, differential equations, and modern data science. Future research could explore spectral properties of more general classes of operators, such as compact normal operators or compact quasinilpotent operators.

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