Strongly Supplement Extending Modules

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Abstract

The main purpose of this paper is to discuss various concepts and properties of modules theory. As well we introduce a new concept which is called a strongly supplement extending module if, each submodule of H is essential in a stable supplement of H. Also, this work goes further to study the relationships between the presenting concept and other provided concepts. Where is the concept presented is a proper generalization of strongly extending modules and strong than supplement extending modules. Additionally, studies the possibility of its inheritance. It also provided numerous descriptions and example.

Keywords: Strongly Supplement Extending, Strongly Extending and Supplement Extending.

المقاسات التوسعة المكملة بقوة

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الخلاصة

الغرض الرئيسي من هذه العمل هو مناقشة المفاهيم والخصائص المختلفة لنظرية المقاسات بالإضافة إلى أننا نقدم مفهومًا جديدًا يسمى المقاسات المكملة بقوة والتي فيها كل مقاس جزئي من H هو مقاس جوهري في مقاس جزئي مكمل مستقر في H، يذهب هذا العمل أبعد من ذلك لدراسة العلاقات بين المفهوم المقدم ومفاهيم الأخرى. حيث يتم تقديم المفهوم على انه تعميم مناسب للوحدات الموسعة بقوة واقوى من مقاسات التوسع المكملة. فضلا عن ذلك دراسة إمكانية توريثها. كما قدم العديد من الأوصاف والأمثلة.

الكلمات المفتاحية: المقاسات المكلمة الموسعة بقوة، المقاسات الموسعة بقوة ومقاسات التوسع التكميلية.

* البحث مستل من رسالة ماجستير للباحث الأول.

Introduction

Throughout this paper all rings have an identity and modules are unitary. A submodule S of H an R-module is said to be essential in H and denote by $S \leq_e H$ if $S \cap D \neq (0)$, $\forall (0) \neq D \leq H$ (Goodeal, 1976). Also, a submodule S of H an R-module is closed in H and denoted by $S \leq_c H$ if $S \leq_e D \leq H$ then $S = D$, (Goodeal, 1976). In addition, that H is called extending module if every submodule of H is essential in a direct summand of H (Harada, 1982). Furthermore, that S is called a supplement submodule of N in H and denote by (S $\leq_{sup}$ H), if S+N = H and S∩N≪ S. (Kasch and Wallace, 1982), in anther hand, H is called a supplement extending module if every submodule of H is essential in a supplement submodule in H (Yaseen and Tawfeek, 2015). Also, in (Tawfeeq, 2015) H is said to be supplement simple if (0), H are only supplement in H.

(Abbas, 1991), introduced a submodule S of an R-module H is called stable if, g(S)⊆ S for each g∈ HomR (S, H). If each submodule of an R-module H is stable, then H is said to be fully stable (Abaas, 1991). Later (Al-saadi, 2007) gave the definition of a strongly extending module if, every submodule of H is essential in stable direct summand of H.R. Wisbauer mentioned that module H is defined as local module if it has a proper submodule which contains all other proper submodules of H. Equivalently, H is called local if it is a hollow and has a unique maximal submodule (Tawfeeq, 2015), also, every local module is lifting, H is said to be lifting if for every submodule S of H there exists a submodule A of S such that H = A⊕ D and S∩ D≪D, where D submodule of H (Clark, *et al.*, 2006) .This work aims to provides several examples and remarks to clarify the relationships between these different module types. It is shown that every strongly extending module is also a strongly supplement extending module, and every uniform module is a strongly supplement extending module. Additionally, it is proven that every supplement simple module is strongly supplement extending if and only if it is uniform.

Finally, the paper gives some sufficient conditions for certain types of modules to be equivalent. For instance, it is shown that a supplement simple module is uniform if and only if it is strongly supplement extending, and that a lifting local module is strongly supplement extending if and only if it is uniform.

Overall, this paper provides a thorough exploration of strongly supplement extending modules and their relationship to other module types, as well as some useful conditions for determining when certain types of modules are equivalent.

Materials and Methods

This study examines strongly supplement extending modules and their properties, including their relationship with other module concepts. Essential and closed submodules, supplement submodules, stable submodules, and extending modules are defined to introduce the concept of strongly supplement extending modules. The study provides examples and remarks to illustrate the properties of these modules and contrasts them with other module concepts. To establish equivalence with certain module concepts. The study develops new results and theorems to establish the properties of strongly supplement extending modules and their relationship with other module concepts.

Strongly Supplement Extending Module

Definition (2.1)

An R-module H is called strongly supplement extending module if each submodule of H is essential in a stable supplement of H.

Result and Discussion

In this section the mean results of this work will be presenting as follow:

Examples and Remarks (2.2)

1. Every strongly extending modules is strongly supplement extending modules. As; H is strongly extending module, then for each S ≤H is essential in stable direct summand of H therefor by every summand is supplement, we get H is strongly supplement extending.
2. Every strongly supplement extending module is supplement extending module.

Proof

Let H be a strongly supplement extending module. Then for every submodule S of H is essential in stable supplement this leads H is supplement extending module. In general, the contrast cannot be considered valid, as V=F(2) the the vector space over a field F. Let $\grave{L_1} = \{(\beta, 0) | (\beta \in F\}$ and $\grave{L_2} = \{(0, \eta) | \ \eta \in F\}$, then $\grave{L_1}$ and $\grave{L_2}$ are subspaces of V. They are spanned by the vectors (1,0), (0,1) respectively. Thus, each of them is of dimension one. $\grave{L_1} \cap \grave{L_2} = (0) \ll \grave{L_2}$ these yields that $\grave{L_1} + \grave{L_2} = \grave{L_1} \oplus \grave{L_2}$ then dimension (V) =dimension $(\grave{L_1})$ +dimension $(\grave{L_2}) = 1 + 1 = 2$, hence, $V = \grave{L_1} \oplus \grave{L_2}$. If $f: \grave{L_2} \longrightarrow V$ such that $f\ ((0, a) = (a, 0))$ for all $a \in F$ thus $f(\grave{L_2}) \nsubseteq \grave{L_2}$ observe that $\grave{L_2}$ isn't stable submodule of V Thus V isn't strongly supplement extending modules.

3. Every H is uniform module H is strongly supplement extending module.

Proof

Since every uniform is strongly extending, we get H is strongly supplement extending by (2).

4. Contrasted of (3) is not valid in generally. As, Z-modules Z_6 is strongly supplement extending module but cannot be uniform.
5. Every H is fully stable semi-simple modules is strongly supplement extending modules. In fact, H is strongly extending module then by (2) we get H is strongly supplement extending module.
6. The contrasted of (5) need not be valid in general. Such as, the Q as Z-module is strongly supplement extending modules since is uniform (by (3)), but it is not semisimple by uniformity of Z_Q and also it is not fully stable.

Now we will set some sufficient conditions to make the concepts are equivalent.

Proposition (2.3)

Let H is a supplement simple, then H is uniform if and only if H is strongly supplement extending modules.

Proof

(⇒) Obviously by [Example and Remark (2.2) (3)]. H is strongly supplement extending module.

(⇐). Lets $S \leq H$. By hypothesis, S is essential in a stable supplement of H. But (0) and (H) are the only supplement of H (since H is supplement simple), thus $S \leq_e H$. Hence, H is uniform R-module.

Proposition (2.4)

Let H a supplement simple R-module, then H is supplement extending module if and only if H is strongly supplement extending modules.

Proof

⇒ Let H be supplement extending module and by hypothesis we get His uniform by (Tawfeeq, 2015) Therefore by (Example and Remark (2.2), (3)). H is strongly supplement extending module
⇐ Let H be strongly supplement extending module, then by (Remarks and Examples (2.2) (2). We get His supplement extending module.

Proposition (2.5)

Let H a fully stable, then H is supplement extending module if and only if H is strongly supplement extending modules.

Proof: ⇒ Obvious
⇐ By [Remark and Example (2.2), 2].

Proposition (2.6)

If H a lifting module, then H is strongly extending module if H is strongly supplement extending module.

Proof

⇒ By [Remark and Example (2.2), 1]
⇐ Obvious; by (Yaseen and Tawfeek, 2015).

In next proposition we will give another description of Strongly supplement extending module.

Proposition (2.7)

An R-module H is strongly supplement extending module if and only if every closed submodule of H is a stable supplement of H.

Proof

⇒ Let $T \leq_c H$ where H is strongly supplement extending modules, then there exists S stable supplement of H, such that $T \leq_e S$ However, $T \leq_c H$, hence, $T = S$ (i.e.) T is a stable supplement of H. (i.e.) T is a stable supplement of H.
(⟸) Let T ≤ H where H be an R-module, there exists a closed submodule S of H and $T \leq_e S$ by (Dung, *et al.*, 1994). Since $S \leq_c H$, thus by hypothesis, S is a stable supplement of H. Then T is essential in a stable supplement of H. Therefore, H is strongly supplement extending module.

The following result gives us other characterization of strongly supplement extending module

Theorem (2.8)

An R-module H. The next statements are equivalent:

1. H is strongly supplement extending module.
2. Every closed submodule of H is a stable supplement.
3. If D is a direct summand of E (H), therefor $D \cap H$ is a stable supplement of H.

Proof

(1)⇒(2) Let H be strongly supplement extending module then by [proposition (2.7)], as requested
(2)⇒(3) Let $D_1 \leq_{\oplus} E(H)$, (i.e.) $E(H) = D_1 \oplus S$ for some $S \leq E(H)$ to show that $D_1 \cap H \leq_c H$ let $D_1 \cap H \leq_e K$ so that that K ≤ H and let k ∈ K. Consequently $k = d + s$, where $d \in D_1$ while $s \in$ S. That $k \notin D_1$, then $s \neq 0$. But $H \leq_e E(H)$ by (Lambek, 1976) Theorem (3.30), and $0 \neq s \in S \leq E(H)$ therefore there exists $r \in R$, such that $0 \neq rs \in H$. Now, rk = rd + rs and $rd = rs - rk \in D_1 \cap H \leq K$. Thus, $rs = rk - rd \in K \cap S$. But $D_1 \cap H \leq_e K$, so $0 = \big((D1 \cap H) \cap S\big) \leq_e K$ and hence $K \cap S = (0)$ then $rs = 0$ which is a contradiction. Thus, $D_1 \cap H \leq_c H$ and hence by (2) we get $D_1 \cap H$ is a stable supplement in H.
(1)⇒(3). Let $D_1 \leq_{\oplus} H$ and let S be relative complement of D_1 in H, by (Goodeal, 1976) then by (Goodeal, 1976). $D_1 \oplus S \leq_e H$. But $H \leq_e E(H)$,

therefore, $D_1 \oplus S \leq_e E(H)$ by (Anderson and Fuller, 1973) thus $E(D_1) \oplus E(S) = E(D_1 \oplus S) = E(H)$. Now, therefore $E(D_1) \leq_{\oplus} E(H)$. Therefore, by using (3), we get $E(D_1) \cap H$ is a stable supplement in H. But $D_1 \leq_e E(D_1)$ and $H \leq_e H$, then $D1 = D1 \cap H \leq_e E(D1) \cap H$ by (Goodeal, 1976), proposition (1.1). Therefore, H is strongly supplement extending module.

Remark (2.9)

If $S \leq H$ an R-module is either stable or supplement but not need both in same time, for instance, $V = F^2$ the vector space over a field F. Let $\grave{L_1} = \{(\beta, 0) | (\beta \in F\}$ and $\grave{L_2} = \{(0, \eta) | \eta \in F\}$, then $\grave{L_1}$ and $\grave{L_2}$ are subspaces of V. They are spanned by the vectors $(1,0), (0,1)$ respectively. Thus, each of them is of dimension one $\grave{L_1} \cap \grave{L_2} = (0) \ll \grave{L_2}$ this yields that $\grave{L_1} + \grave{L_2} = \grave{L_1} \oplus \grave{L_2}$, then dimension $(V) =$ dimension $(\grave{L_1}) +$ dimension $(\grave{L_2}) = 1 + 1 = 2$, hence $\mathrm{V} = \grave{L_1} \oplus \grave{L_2}$ which is supplement. In fact, if $g: \grave{L_2} \longrightarrow V$ such that $g\left((0,s) = (s,0)\right)$ for all $s \in F$, thus $g\left(\grave{L_2}\right) \nsubseteq \grave{L_2}$ not stable.

As mention before every strongly supplement extending module is supplement module in (Remarks and Examples (2.2), (3)). Also, every quasi-injective module is supplement extending module by (Tawfeeq, 2015). Now, the question is there existing relation between strongly supplement extending module and quasi-injective. In actuality, they are distinct ideas. Since the Z-module Z is it is uniform, so, by (Remarks and Examples (2.2), (4)), so it is strongly supplement extending module but not quasi-injective. However, $V = F^2$ a quasi-injective. But not strongly supplement extending module.

In the following results we consider conditions under which quasi-injective is strongly supplement extending module.

Proposition (2.10)

Every multiplication quasi-injective module is strongly supplement extending module.

Proof

Let H be a multiplication quasi-injective module let $S \leq_{\mathrm{C}} H$. Since H is multiplication, thus $S = IH$ for some ideal I of R. By quasi-injectivity of H then we have every closed is supplement by (Dung, *et al*., 1994) then that it is enough to show that $S \leq^{\mathrm{ST}} H$ Let $f \in Hom\,(S, H)$ for some $s \in \mathrm{S}$ such that $S = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{r_i}\,\mathrm{h_I}$ where $\mathrm{r_i} \in \mathrm{I}$ and $\mathrm{H_i} \in H$. So $f(\mathrm{S}) = f(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{r_i}\,\mathrm{h_i}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{r_I}\, f(\mathrm{h_i}) \in IH = S$, thus, H is strongly supplement extending module.

If R is strongly supplement extending ring. Then H may not be strongly supplement extending module. As $Z \oplus Z_p$ Z-module. We observed that Z is strongly supplement ring. Since Z is uniform, in while $H = Z + Z_P$ not SSU-CS-module. Since the only supplement of H is $0 + \bar{0}, Z + \bar{0}, 0 + Z_P$ and let $g \in Hom$ (S, H) define as $g: Z + \bar{0}\, H\; by\; g\,(a, \bar{0})\,(\bar{0}, a), \forall\,(a, \bar{0}) \in Z + \bar{0}$. But in fact, $g(a, \bar{0}) \nsubseteq Z + \overline{0}$. Therefore is not stable in H. Now we give some condition to make strongly supplement extending ring is strongly supplement extending module.

Proposition (2.11)

Let H be a finitely generated faithful multiplication R- module over a commutative ring R. If R is strongly supplement extending ring, then H is strongly supplement extending module.

Proof

Let $S \leq_c H$ since H is faithful multiplication R–module. So, $S = [S{:}H]H$ where $[S{:}H] = \{r\,R{:}\,rH \subseteq S\}$ be an ideal of R but S is closed submodule in H. So, by ((Ahmed, 1992), proposition (3,41) $[S{:}H]$ closed ideal in R, but R is SUP-CS when consider as H an R-module. Hence, $[S{:}H]$ is supplement in R therefor there exists an ideal I of R where $[S{:}H] + I = R$ and $[S{:}H] \cap I << [S{:}H]$. Now, $H = RH = ([S{:}H] + I)H = [S{:}H]H + IH = S + IH.$ To show that $([S{:}H]H \cap IH) << [S{:}H]H$. Let $([S{:}H]H \cap IH)DH = [S{:}H]H$

Since H is multiplication module, $([S{:}H] \cap I) + D]H = [S{:}H]H$ But, H is finitely generated faithful multiplication R–module. So, by (El-Bast and Smith, 1988), (Theorem 3.1). $([S{:}H] \cap I) + D] = [S{:}H]$ then $D = [S{:}H]$ and clear that $DH = [S{:}H]H$. Hence, $([S{:}H]H \cap IH) << [S{:}H]H$. Therefore, $S \leq_{sup} H$. Now, let $f \in \mathrm{Hom}(S, H)$ and $s \in S = [S{:}H]H$. Then, $s = \sum_{i=1}^{n} r_i h_I$, where $r_i \in [S{:}H]$ and $H_i \in H$. So $f(S) = f(\sum_{i=1}^{n} r_i h_i) = \sum_{i=1}^{n} r_I\, f(h_i) \in [S{:}H]H = S$, hence $f(S) \subseteq S$ (i.e.) S is stable. Therefore, H is strongly supplement extending module.

We do not know in general whether Strongly supplement extending module property is inherent by submodule. The next results are partial answering the question: When do submodules inherit the strongly supplement extending module property?

Proposition (2.12)

Every closed submodule (and hence direct summand) of strongly supplement extending module is strongly supplement extending module.

Proof

Let $S \leq_c H$, and let $D \leq_c S$ then by (Clark, *et al.*, 2006). we get $D \leq_c H$, but H is strongly supplement extending module. Therefore, D is stable supplement submodule in H and since $D \leq S$, so by (Clark, *et al.*, 2006). we get $D \leq_{SUP} S$. In addition, we assumption $D \leq^{ST} S$. Let $g \in HomR(D, S)$ and consider the sequence $D \xrightarrow{g} S \xrightarrow{I} H$ such that i map inclusion. Therefore, $(i \circ g){:}\, D \to H$ and, since. $D \leq^{ST} H$ therefor $(i \circ g)(D) \subseteq D$. So $g(D) \subseteq D$ then $D \leq^{ST} S$ Thus, S is strongly supplement extending module.

Proposition (2.13)

Each submodule S of strongly supplement extending module H with the property that intersection of S with any stable supplement of H is stable supplement of S, is strongly supplement extending module.

Proof

Let $A \leq S$ since H is strongly supplement extending module and $A \leq H$ then there is a stable supplement K of H such that $A \leq_e K$ since $A \leq S$ and $A \leq K$ then $A \leq S \cap K$, thus by (Goodeal, 1976), proposition (1.1), we get $A \leq e K \cap S$ and by hypothesis $K \cap S$ is a stable supplement of S. thus, S is strongly supplement extending module.

A direct sum of strongly supplement extending module, it is not necessary to be strongly supplement extending module. As, $Q \oplus Z_2$ as Z-module is not strongly supplement extending module, since it is not supplement. So, for this we obtain a necessary and sufficient situation for a direct sum of two strongly supplement extending module, to be again strongly supplement extending module.

Proposition (2.14)

Let $H = H_1 \oplus H_2$ where H_1 and H_1 are fully stable and strongly supplement extending module, if $annH_1 + annH_2 = R$, then H is strongly supplement extending modules.

Proof

Let $S \leq_c$ H therefore $S \leq_{sup}$ H by (Tawfeeq, 2015), Proposition (2.2.4) but H is fully stable by (Abaas, 1991), proposition (4.2). Thus, H is strongly supplement extending module.

Proposition (2.15)

let H be fully stable simesimple modules. Then the next statements are equivalent:

1. H is strongly extending module.
2. H is strongly supplement extending module.
3. H is extending module.
4. H is supplement extending module.

Proof

(1) ⇒ (2) by (Examples and Remarks (2.2), (2)).

(2) ⇒ (3) let $S \leq_e$ H hence H is Strongly supplement extending module thus $S \leq_{sup} H$ and since H is semisimple then $S \leq_\oplus$ H therefore H is CS-module.

(3) ⇒ (4) by (Tawfeeq, 2015).

(4) ⇒ (1) let $S \leq_c$ H hence H is supplement extending module thus $S \leq_{sup} H$, and since H is semisimple then $S \leq_\oplus H$ and since H is fully stable therefore H is S-CS-module.

We obvious that, every strongly supplement extending modules are supplement extending module the antithesis is not true in generally. The next result gives a weaker condition from one in proposition (2.5) which ensures that the antithesis is true.

Proposition (2.16)

An $R - module\ H$ every supplement of H is stable. Therefor H is Strongly supplement extending module if H is supplement extending module.

Conclusion

Through this paper, we reached the next conclusions: every strongly extending is strongly supplement extending module. Every strongly supplement extending module is supplement extending module. Every fully stable semisimple module is strongly supplement extending module. Every uniform is strongly supplement extending module. But the convers of each of the statements does not generally hold. The strongly supplement extending module is inherited by closed (Direct Summand) submodule.

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