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Using the finite difference method and boundary value to calculate the numerical approximation to solve fractional differential equations

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Abstract

Fractional calculus is an important area of research and applications of fractional calculus in ℝ or ℂ. The forward and backward difference method for finding numerical approximations to solutions of simple differential equations was developed for solving fractional differential equations.It was discussed using finite differences, where examples were discussed to deduce the exact results, we want to obtain. The comparison was made based on tables calculated by MATLAB in these examples used to compare the absolute error of the approximate solution and the exact solution.

Keywords: Gamma function, Taylor's formula, finite difference method

استخدام طريقة الفروق المنتهية والقيمة الحدية لحساب التقريب العددي لحل المعادلات التفاضلية الكسرية

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الخلاصة:

يعد حساب التفاضل والتكامل الكسري مجالا مهما للبحث وتطبيقات حساب التفاضل والتكامل الكسري في R أو c.تم تطوير طريقة الفرق الأمامي والخلفي لإيجاد التقديرات التقريبية العددية لحلول المعادلات التفاضلية البسيطة لحل المعادلات التفاضلية الكسرية. تمت مناقشته باستخدام طريقة الفروقات المنتهية، حيث تمت مناقشة الأمثلة لاستنتاج النتائج الدقيقة التي نريد الحصول عليها. تم إجراء المقارنة بناء على الجداول المحسوبة بواسطة MATLAB في هذه الأمثلة المستخدمة لمقارنة الخطأ المطلق للحل التقريبي والحل الدقيق.

الكلمات المفتاحية: دالة جاما، صيغة تايلور، طريقة الفروق المنتهية.

1. Introduction

In this work, numerical approximations were used to solve differential equations containing fractions. Due to the emergence of fractional differential equations as well as fractional integral equations in many important issues, they are also used in modeling many practical problems such as electromagnetic waves, propagation equations, etc. The study was conducted using the known difference method. The research concluded with a discussion of numerical examples using this method

I also compared the exact solutions with the approximate solution, in addition to presenting the comparison using graphs and error tables.

2. Gamma function

$\forall z \in ℂ$ such that $Re(z) \succ 0$ then Gamma function is given by:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\ e^{-1}\, dt$$
…… (1)

An important property of the gamma function is the following regression relationship:

$$\Gamma(z+1) = z\,\Gamma(z)$$
…… (2)

For every natural number n, then: $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{4^n \times n!}$

$$\Gamma(1) = 1, \qquad \Gamma(0_+) = +\infty, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

The gamma function has no simple poles for points: $z = 0;\ -1;\ -2, -3;\ -4; \ldots$

3. Taylor's formula

Let g be a function that is differentiable $n+1$ times on the interval $[a,b]$ where $x_0 \in [a,b]$, $\forall\, x_0 \in [a,b]$ there exists an ξ between the numbers x, x_0 such that:

$$g(x) = g(x_0) + \frac{dg}{gx}(x_0)(x-x_0) + \frac{d^2g}{gx^2}(x_0)\frac{(x-x_0)^2}{2!} + \frac{d^3g}{gx^3}(x_0)\frac{(x-x_0)^3}{3!} + \cdots + \frac{d^ng}{gx^n}(x_0)\frac{(x-x_0)^n}{n!} + \frac{d^{n+1}g}{gx^{n+1}}(\xi)\frac{(x-x_0)}{(n+1)}$$

…… (3)

$$R(\xi) = \frac{d^{n+1}g}{gx^{n+1}}(\xi)\frac{(x-x_0)}{(n+1)}$$

4. Finite Difference Formula Using Taylor Series

Approximation of the first derivative of the function g by finite differences forward by three points by setting $x_j = x_0 + jh$ in (3) If it was $x_0 = x_j,\ x = x_j + 1$ then:

$$g(x_j+1) = g(x_j) + g^{(1)}(x_j)(x_{j+1} - x_j) + g^{(2)}(x_j)\frac{(x_{j+1}-x_j)^2}{2!} + g^{(3)}(\xi_1)\frac{(x_{j+1}-x_j)^3}{3!}$$
$$= g(x_j) + g^{(1)}(x_j)h + g^{(2)}(x_j)\frac{h^2}{2!} + g^{(3)}(\xi_1)\frac{h^3}{3!}$$

…… (4)

If it was $x_0 = x_j,\ x = x_j + 2$ then:

$$g(x_j+2) = g(x_j) + g^{(1)}(x_j)\,(x_{j+2} - x_j) + g^{(2)}\,(x_j)\frac{(x_{j+2}-x_j)^2}{2!} + g^{(3)}\,(\xi_2)\frac{(x_{j+2}-x_j)^3}{3!}$$
$$= g(x_j) + g^{(1)}(x_j)(2h) + g^{(2)}\,(x_j)\frac{(2h)^2}{2!} + g^{(3)}\,(\xi_2)\frac{(2h)^3}{3!}$$

…… (5)

Multiplying (4) by (−4) and adding with (5) we find:

$$g^{(1)}(x_j) = \frac{-3g(x_j) + 4g(x_{j+1}) - g(x_{j+2})}{2h} - \frac{2h^2}{3!}g^{(2)}\,(\xi_1) + \frac{4h^2}{3!}g^{(3)}\,(\xi_2)$$

…… (6)

Where the error is equal to:

$$-\frac{2h^2}{3!}g^{(2)}\,(\xi_1) + \frac{4h^2}{3!}g^{(3)}\,(\xi_2) = O(h^2)$$

Approximation of the second derivative of the function g by finite differences forward by three points if it was $x_0 = x_j,\ x = x_j + 1$ then:

$$g(x_j+1) = g(x_j) + g^{(1)}(x_j)\,(x_{j+1} - x_j) + g^{(2)}\,(x_j)\frac{(x_{j+1}-x_j)^2}{2!} + g^{(3)}\,(\xi_1)\frac{(x_{j+1}-x_j)^3}{3!}$$
$$= g(x_j) + g^{(1)}(x_j)h + g^{(2)}\,(x_j)\frac{h^2}{2!} + g^{(3)}\,(\xi_1)\frac{h^3}{3!}$$

…… (7)

If it was $x_0 = x_j,\ x = x_j + 2$ then we have through (1):

$$(x_j + 2) = g(x_j) + g^{(1)}(x_j)\,(x_{j+2} - x_j) + g^{(2)}\,(x_j)\frac{(x_{j+2} - x_j)^2}{2!} + g^{(3)}\,(\xi_2)\frac{(x_{j+2} - x_j)^3}{3!}$$
$$= g(x_j) + g^{(1)}(x_j)(2h) + g^{(2)}\,(x_j)\frac{(2h)^2}{2!} + g^{(3)}\,(\xi_2)\frac{(2h)^3}{3!}$$

…… (8)

Multiplying (5) by -2 and adding with (6) we find:

$$g^{(2)}(x_j) = \frac{g(x_j) - 2g(x_{j+1}) + g(x_{j+2})}{h^2} - \frac{8h}{3!}g^{(3)}\,(\xi_2) + \frac{2h}{3!}g^{(3)}\,(\xi_1)$$

…… (9)

Where the error is equal to:

$$\frac{8h}{3!}g^{(3)}\,(\xi_2) + \frac{2h}{3!}g^{(3)}\,(\xi_1) = O(h)$$

Approximation of the second derivative of the function g by finite differences back to three points if it was $x_0 = x_j,\ x = x_j + 1$ then:

$$g(x_j - 1) = g(x_j) + g^{(1)}(x_j)\,(x_{j-1} - x_j) + g^{(2)}\,(x_j)\frac{(x_{j-1} - x_j)^2}{2!} + g^{(3)}\,(\xi_1)\frac{(x_{j-1} - x_j)^3}{3!}$$
$$= g(x_j) + g^{(1)}(x_j)h + g^{(2)}\,(x_j)\frac{h^2}{2!} + g^{(3)}\,(\xi_1)\frac{h^3}{3!}$$

…… (10)

If it was $x_0 = x_j,\ x = x_j + 2$ then:

$$g(x_j + 2) = g(x_j) + g^{(1)}(x_j)\,(x_{j-2} - x_j) + g^{(2)}\,(x_j)\frac{(x_{j-2} - x_j)^2}{2!} + g^{(3)}\,(\xi_2)\frac{(x_{j-2} - x_j)^3}{3!}$$
$$= g(x_j) - g^{(1)}(x_j)(2h) + g^{(2)}\,(x_j)\frac{(2h)^2}{2!} - g^{(3)}\,(\xi_2)\frac{(2h)^3}{3!}$$

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…… (11)

By multiplying (10) by -2 and adding with (11), we find:

$$g^{(2)}(x_j) = \frac{g(x_{j-2}) - 2g(x_{j-1}) + g(x_j)}{h^2} + \frac{8h}{3!}g^{(3)}(\xi_2) - \frac{2h}{3!}g^{(3)}(\xi_1)$$

…… (12)

Where the error is equal to:

$$\frac{8h}{3!}g^{(3)}(\xi_2) - \frac{2h}{3!}g^{(3)}(\xi_1) = O(h)$$

Approximation of the second derivative of the function g by three-point central finite differences through (5) if it was $x_0 = x_j,\ x = x_j + 1$ then:

$$\begin{aligned} g(x_j+1) &= g(x_j) + g^{(1)}(x_j)(x_{j+1} - x_j) + g^{(2)}(x_j)\frac{(x_{j+1}-x_j)^2}{2!} \\ &+ g^{(3)}(x_j)\frac{(x_{j+1}-x_j)^3}{3!} + \frac{(x_{j+1}-x_j)^4}{4!}g^{(4)}(\xi_1) \\ &= g(x_j) + g^{(1)}(x_j)h + g^{(2)}(x_j)\frac{h^2}{2!} + g^{(3)}(x_1)\frac{h^3}{3!} + \frac{h^4}{4!}g^{(4)}(\xi_1) \end{aligned}$$

…… (13)

If it was $x_0 = x_j,\ x = x_j + 2$ then:

$$\begin{aligned} g(x_j-1) &= g(x_j) + g^{(1)}(x_j)(x_{j-1} - x_j) + g^{(2)}(x_j)\frac{(x_{j-1}-x_j)^2}{2!} \\ &+ g^{(3)}(x_j)\frac{(x_{j-1}-x_j)^3}{3!} + \frac{(x_{j-1}-x_j)^4}{4!}g^{(4)}(\xi_2) \\ &= g(x_j) + g^{(1)}(x_j)h + g^{(2)}(x_j)\frac{h^2}{2!} + g^{(3)}(\xi_1)\frac{h^3}{3!} + g^{(4)}(\xi_2)\frac{h^4}{4!} \end{aligned}$$

…… (14)

We add (13) and (14) and find:

$$g^{(2)}(x_j) = \frac{g(x_{j-1}) - 2g(x_j) + g(x_{j+1})}{h^2} - \frac{h^4}{4!}g^{(4)}(\xi_1) - \frac{h^4}{4!}g^{(4)}(\xi_2)$$

…… (15)

Where the error is equal to:

$$\frac{h^4}{4!}g^{(4)}(\xi_1) - \frac{h^4}{4!}g^{(4)}(\xi_2) = O(h^2)$$

5. Approximation of fractional derivative by finite difference method

Suppose the domain $[a,b]$ is divided into subdomains $[x_k\ ,x_{k+1}]$ with a step $h = \frac{b-a}{a}$

We use equally spaced nodes $x_k = a + kh,\ \forall\, k = 0,1,2,\dots,n$

The trapezoidal base consisting of n subdomains can be expressed by the relation:

$T(f,h) = \frac{h}{2}\sum_{k=1}^{n}[f(x_{k+1}) + f(x_k)]$
…… (16)

This is an approximation of the integral of the function $f(x)$ on the interval $[a,b]$ were

$\int_a^b f(x)dx \cong$
$T(f,h)$ …… (17)

That's why we have:

$${}_{\alpha}^{C}D\, y(t) = \frac{1}{\Gamma(1-a)}\int_0^t \frac{y'(x)}{(t-x)^{\alpha}}\,dx$$

$0 < \alpha < 1,\ t \geq 0$

Integrating by parts we conclude the following:

$${}_{\alpha}^{C}D\, y(t) = \frac{1}{(1-a)\Gamma(1-a)}\left(y'(0)t^{1-a} + \int_0^t (t-x)^{1-\alpha}\, y''(x)\,dx\right)$$

We use the compound trapezoidal rule (15), (14) to estimate the integral, we put:

$$f(x) = \int_0^t (t-x)^{1-\alpha}\, y''(x)\,dx$$

Among them:

$$\int_0^t (t-x)^{1-\alpha} y''(x)\,dx \approx \frac{h}{2}\sum_{j=1}^{n}\left[\left(t-x_{j-1}\right)^{1-\alpha} y''(x_{j-1}) + \left(t-x_j\right)^{1-\alpha} y''(x_j)\right]$$

$$\approx \frac{h}{2}\sum_{j=1}^{n}\left(t-x_{j-1}\right)^{1-\alpha} y''(x_{j-1}) + \frac{h}{2}\sum_{j=1}^{n}\left(t-x_j\right)^{1-\alpha} y''(x_j)$$

…… (18)

$$\approx \frac{h}{2}\left[(t-x_0)^{1-\alpha} y(0) + \left(t-x_n\right)^{1-\alpha} y(x_n) + 2\sum_{j=1}^{n}\left(t-x_j\right)^{1-\alpha} y''(x_j)\right]$$

Therefore:

$${}_{\alpha}^{C}D\, y(t) = \frac{h}{2}\frac{\left[(t-x_0)^{1-\alpha} y''(0) + (t-x_n)^{1-\alpha} y''(x_n) + 2\sum_{j=1}^{n-1}\left(t-x_j\right)^{1-\alpha} y''(x_j)\right]}{(1-a)\Gamma(1-a)}$$

…… (19)

$$+\frac{(t-x_0)^{1-\alpha} y'(0)}{(1-a)\Gamma(1-a)}$$

Putting $y(x_j) = y_j$ through (6), (12) and (15) we find:

$$y'_j \approx \frac{-3\,y_j + 4\,y_{j+1} - y_{j+2}}{2h}$$

$$y''_j \approx \frac{y_{j-2} - 2\,y_{j-1} + y_j}{h}$$

$$y''_j \approx \frac{y_{j-1} - 2\,y_j + y_{j+1}}{h}$$

…… (20)

Therefore:

$$y'_0 \approx \frac{-3\,y_0 + 4\,y_1 - y_2}{2h}$$

$$y''_0 \approx \frac{y_0 - 2\,y_1 + y_2}{h}$$

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$$y''_0 \approx \frac{y_{n-2} - 2y_{n-1} + y_n}{h}$$

…… (21)

Among them:

$$ {}_{\alpha}^{C}D\, y(t) = \frac{1}{(1-a)\Gamma(1-a)}\left[\frac{y_0 - 2y_1 + y_2}{2h}(t - x_0)^{1-\alpha} + \frac{y_{n-2} - 2y_{n-1} + y_n}{2h}(t - x_n)^{1-\alpha}\right] + \frac{1}{(1-a)\Gamma(1-a)}\sum_{j=1}^{n-1}\left[\frac{y_{j-1} - 2y_j + y_{j+1}}{h}(t - x_j)^{1-\alpha}\right] + \frac{-3y_0 + 4y_1 - y_2}{2h\,((1-a)\Gamma(1-a))}(t - x_0)^{1-\alpha}$$

…… (22)

Therefore:

$$ {}_{\alpha}^{C}D\, y(t) = \frac{1}{(1-a)\Gamma(1-a)}\left[M_1 + \frac{h}{2}(M_2 + 2M_3 + M_4)\right]$$

…… (23)

Such that:

$$M_1 = \frac{-3y_0 + 4y_1 - y_2}{2h}(t - x_0)^{1-\alpha}$$

$$M_2 = \frac{y_0 - 2y_1 + y_2}{2h}(t - x_0)^{1-\alpha}$$

$$M_3 = \sum_{j=1}^{n-1}\left[\frac{y_{j-1} - 2y_j + y_{j+1}}{h}(t - x_j)^{1-\alpha}\right]$$

$$M_4 = \frac{y_{n-2} - 2y_{n-1} + y_n}{h}(t - x_n)^{1-\alpha}$$

6. Examples:

Numerical examples with a comparison between the exact solution and the approximate solution using the forward and backward finite difference method and the limit value using different values and comparing the accuracy of the approximate solution with the exact solution where the MATLAB program was used to calculate some values.

Example 1. Let us have the following equation:

$${}_{\alpha}^{c}D\, u(t) = -u(t) + \frac{t^{4-\alpha}}{\Gamma(5-a)} \ ,0 < \alpha < 1\ , t > 0$$

$$u(0) = 0$$

The exact solution is:

$$u(t) = t^{4}\, E_{\alpha,5}\,(-\,t^{\alpha})$$

We will find the approximate solution by the finite difference method and compare the exact solution and the approximate solution of the equation for different values

$n = 100, 500, 1000$ at each t_i where $\alpha = 0.5$

to show the importance of the step length $h = 1/n$ in approaching the exact solution of the equation. and we will give the absolute error by the finite difference method.

Table 1. Numerical solution using the finite difference method where $\alpha = 0.5$ and $n = 100{,}500{,}1000$

t	Exact	$n = 100$		$n = 500$		$n = 1000$	
		value	Error	value	Error	value	Error
0,1	3.63E − 06	3.77E − 06	1.29E − 07	3.65E − 06	1.32E − 08	3.64E − 06	4.78E − 09
0.2	5.52E − 05	5.59E − 05	6.94E − 07	5.53E − 05	6.77E − 08	5.52E − 05	2.43E − 08
0.3	2.69E – 04	2.71E – 04	1.80E – 06	2.69E – 04	1.72E – 07	2.69E – 04	6.18E – 08
0.4	8.24E – 04	8.28E – 04	4.49E – 06	8.25E – 04	3.31E – 07	8.24E – 04	1.18E – 07
0.5	1.96E – 03	1.97E – 03	5.80E – 06	1.96E – 03	5.45E – 07	1.96E – 03	1.94E – 07
0.6	3.97E – 03	4.00E – 03	8.73E – 06	3.97E – 03	8.16E – 07	3.97E – 03	2.91E – 07
0.7	7.20E – 03	7.21E – 03	1.23E – 05	7.20E – 03	1.14E – 06	7.20E – 03	4.08E – 07
0.8	1.20E – 02	1.21E – 02	1.65E – 05	1.20E – 02	1.53E – 06	1.20E – 02	5.45E – 07

0.9	1.89E – 02	1.90E – 02	2.13E – 05	1.89E – 02	1.97E – 06	1.89E – 02	7.03E – 07
1	2.84E − 02	2.84E − 02	2.67E − 05	2.84E − 02	2.47E − 06	2.84E − 02	8.81E − 07

We notice that the absolute error at $n = 100$ is greater than at $n = 500$ and that the absolute error at $n = 500$ is greater than at $n = 1000$.

We also notice that the error tends to zero when n tends to infinity.

$Error \to 0 \; when \; n \; \to \infty \; i.e. \; . \quad \lim_{n\to\infty} Error \to 0$

This means that the exact solution approaches the absolute solution.

Example 2. In the following equation we present some numerical solutions to the nonlinear boundary value problem:

$$f_t \; = \frac{\partial^\alpha}{\partial x^\alpha} \mathrm{f}^v(x,t), 1 < \alpha \leq 2, v > 2, 0 \leq x \leq 5,$$

$$f(0,t) = 0 = f(5,t), \forall \; t > 0,$$

$$f(x,0) = \; f_0 \,(x), \forall \;\; 0 \leq x \leq 5,$$

Where initial data f_0 is the gaussian function given by

$$f_0 \,(x) = \frac{2}{(2\pi(0.3)^2)^{\frac{1}{2}}} e^{\,-(x-2.5)^2/2(0.3)^2}$$

Using algorithms in MATLAB we get the values as in the table (2):

Table 2. Values of the numerical solution v, the exact solution f, and the error $E = v - f$, for $\alpha = 2$ and $\nu = 1$ in the problem at $t = 1$

x	v	f	E
0,0	0,00000000	0,00000000	0,00000000
0,5	0,13650936	0,14492708	−0,00841772
1,0	0,28017859	0,28987258	−0,00969399
1,5	0,41875250	0,41990747	−0,00115497
2,0	0,52474526	0,51329497	0,01145029
2,5	0,56558688	0,54801760	0,01756928
3,0	0,52629428	0,51462415	0,01167013
3,5	0,42130602	0,42221621	0,00208981
4,0	0,28306465	0,29267907	−0,00961442
4,5	0,13934058	0,14787562	−0,00853504
5,0	0,00000000	0,00000000	0,00273680

As the numerical solution is very close to the exact solution.

Fig 1. Numerical solutions when $v = 1.0$ with $\alpha = 1.2, \alpha = 1.4, \alpha = 1.6, \alpha = 1.8, \alpha = 2.0$

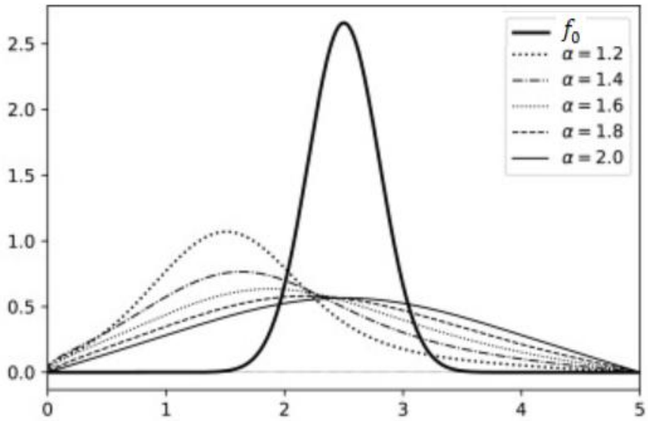
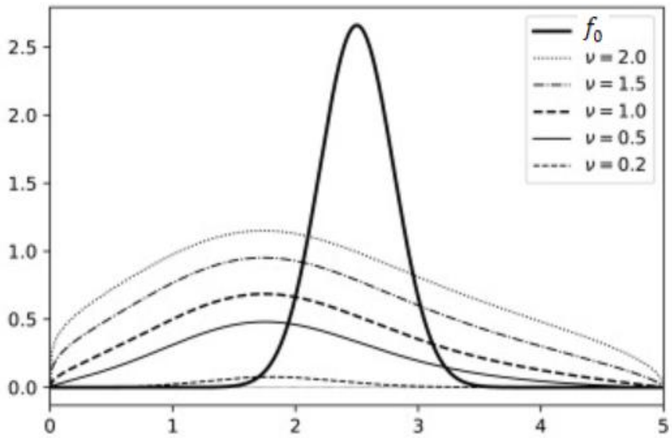


Fig 2. Numerical solutions when $\alpha = 1.5$ with $v = 0.2, v = 0.5, v = 1.0, v = 1.5, v = 2.0$



Conclusions

By searching for the possibility of finding an acceptable approximate solution by studying numerical examples using the forward and backward difference method and comparing the results, I noticed the effectiveness of this method, especially in example (1), where the results were acceptable and reliable and the error tended to zero when n tended to infinity, which means that the exact solution approaches the absolute solution.

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