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Pseudo-Random Hypergraphs

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Abstract
We study the interplay between notions of quasi-randomness for hypergraphs. In particular, we show a strong connection between discrepancy-type measures of quasirandomness in the hypergraph setting. Exploiting this connection, we provide a long list of disparate quasirandom properties and show that these properties are all equivalent (in the sense of Chung, Graham, and Wilson) with polynomial bounds on their interdependences.
Keywords: Graph, Hypergraph, Quasirandom, Cayley hypergraphs.

The research is extracted from the thesis of the first researcher.

1. Introduction

Pseudo-random features can be informally thought of as random certificates for the object in question. Given a class of composite objects, such as 3-uniform graphs or hypergraphs, we say that a (definite) property of these objects is a pseudo-random property if it satisfies two conditions: a uniform random object of that class, this satisfies the property with high probability.

Any object that satisfies this property will behave like a random object in other ways. In such cases, just knowing that an object has some quasi-random properties gives a lot of information about its behavior in many ways. Then such objects are called pseudo-random. The concept of pseudorandomness was originally introduced in the setting of graphs in a paper by Chang, Graham, and Wilson [10].

After the introduction of pseudorandom graphs, Chang and Graham [7, 8] and Kohayakawa, Roedel, and Skokan [19] undertook the task of generalizing such concepts to cloud graphs. They considered uniform k-supergraphs (k-graphs) that mimic the random supergraph $G^{(k)}$ (n, p), where each set of k elements in $[n]:=\{1,2,\dots,n\}$ is chosen as an independent edge to be with probability p.

This lack of correlation in the presence of edges leads to strong uniformity properties that make it easy to work with stochastic hypergraphs. It was then shown that some of these good features form an approximate equivalence class of similar stochastic features. The main concept in Chang and Graham's work was the deviation of a super graph.

According to the K-graph H, we write v (H) and e (H) to represent the number of vertices and edges in H (respectively) and we write δ (H) to represent its edge density. The skewness of a hypergraph can be considered as a measure of the skewness of its edge distribution, which is supposed to mimic a random distribution. Officially by:

$$\mathrm{dev}_k(H) = \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)} \in V(H)^k} \prod_{\omega \in \{0,1\}^k} \left(H(x_1^{(\omega_1)}, \ldots, x_k^{(\omega_k)}) - \delta(H)\right),$$

Where H (x1, … , xk) represents the edge marker function 1[{x1, . . . , xk} ∈ E(H)] . It is always true that 0 ≤ deck (H) ≤ 1, and it can be shown that random hypergraphs have very small deviations with high probability.

In contrast, the central concept in the works of Kohayakawa, Rodel, and Skukan was the divergence of a hypergraph, which quantifies the distance from a uniform distribution of its edges when measured against low-order structures. In their paper, these low-order structures are represented by k-cuts of (k − 1) graphs, but here we will work with the slightly more general notion of (k − 1)-cuts. The variance of a k-graph H is then defined by:

$$\mathrm{disc}_{k-1}(H) = \max_{S_1,\ldots,S_k \subseteq V(H)^{k-1}} \left| \mathbb{E}_{\mathbf{x} \in V(H)^k} \left[(H(\mathbf{x}) - \delta(H)) \prod_{i=1}^{k} S_i((x_j)_{j \neq i}) \right] \right|,$$

Where we use the same symbol for a set S and its indicator function 1[x ∈ S]. It is not hard to show that random supergraphs have very small variances with high probability. Both deviation and variance can be viewed as pseudo-random measures.

Another statistic that can be accurately estimated in random hypergraphs is the count of different smaller hypergraphs that occur as subgraphs. Given two supergraphs F and H, denote the number of labeled copies of F in H by N_F (H). If H is a random supergraph $G^{(k)}$ (n, p), then the expected value is N_F(H).

$$p^{e(F)} n(n-1)\ldots(n - v(F) + 1) = p^{e(F)} n^{v(F)} + O(v(F)^2 n^{v(F)-1});$$

Moreover, N_F (H) is strongly concentrated around this expected value. It was shown by Chang and Graham [7, 8], and by Kohayakawa, Roedel, and Skokan [6] that large pregraphs H have a small deviation ((dev_k (H) = O (1))) or small difference.

($disc_{k-1}$ (H) = O (1)) It should contain approximately the expected number of all subgraphs of finite size.
$N_F(H) = \delta(H)^{e(F)}\, v(H)^{v(F)} + o(v(F)^2\, v(H)^{v(F)})$ A special role in their results is played by the octagon. K-Octagon $Oct^{(k)}$ is a graph of a complete k-part graph where each vertex class has exactly two vertices. Note that the deviation from a k-diagram H can be interpreted as the weighted average of the number of octagons $Oct^{(k)}$,
When the weight is given by the balanced indicator function H (x1, . . ., xk) − δ(H). In [19] it was shown that Oct(k) is complete for the pseudorandom concepts presented above: any k-graph H that has approximately the "correct" ratio of subgraphs isomorphic to $Oct^{(k)}$ - meaning the expected ratio In a random hypergraph with the same edge density - it will mean pseudo-random. In particular, it follows that H will also have approximately the correct proportion of any other fixed (finite-sized) k-graph F as subgraphs.
Despite these results, and in contrast to the simpler set of graphs, it turns out that there are several distinct equivalence classes of pseudorandom concepts for hypergraphs. These different classes and their interrelationships were studied by Chang [5], Kuhayakawa, Nagel, Rödel, and Schacht [18], Kenlon, Hahn, Person, and Schacht [11], Lenz and Mebei [20] and Tausner [21]. They got. Name a few.
Let d and k be integers with 1 ≤ d < k and let H be a K-uniform hypergraph. D-difference H is defined by:

$$\mathrm{disc}_d(H) = \max_{S_B \subseteq V(H)^d :\, B \in \binom{[k]}{d}} \left| \mathbb{E}_{\mathbf{x}\in V(H)^k} \left[(H(\mathbf{x}) - \delta(H)) \prod_{B\in\binom{[k]}{d}} S_B\big((x_j)_{j\in B}\big) \right] \right|,$$

Where the maximum is taken over all $\binom{k}{d}$ Sets of d-subsets of $V(H)^d$ indexed by d-subsets [k]. It is a measure that shows how far the edges of H are from the uniform distribution against structures of order d. If the difference d from H is small, we consider it pseudorandom of order d. More formally, we say that H is ε-pseudorandom of order d if (H) ≤ ε $disc_d$.
The concept of deviation can be extended to other arrangements as well. We define d as the deviation from H, denoted by $dev_d(H)$.

$$\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V(H)^d}\mathbb{E}_{y_{d+1},\ldots,y_k\in V(H)} \prod_{\omega\in\{0,1\}^d} \left(H(x_1^{(\omega_1)},\ldots,x_d^{(\omega_d)},y_{d+1},\ldots,y_k) - \delta(H)\right).$$

Just as k deviation $dev_k(H)$ can be viewed as a weighted number of $Oct^{(k)}$ octagons, so d deviation can be viewed as a weighted number of $Oct^{(k)}$ octagons, which by adding k − d additional vertices are formed by $Oct^{(d)}$ and their connections to each of the edges. Chang [5] showed that any k-diagram that has a small d deviation must also have a small (d-1)-difference. However, as observed by Lenz and Mbei [20], the other direction does not hold and the two concepts are distinct.

Kohayakawa, Nagel, Rödel, and Schacht [18] proved that pseudorandomness of order 1 (also called weak pseudorandomness) is sufficient to control the number of any linear hypergraph, that is, those in which every pair of edges intersects at most one vertex.

After that, Kenlon, Hahn, Person, and Schacht [11] showed that there exists a linear supergraph $M_1^{(k)}$ that is complete for the concept of weak pseudorandomness: if a supergraph H has approximately the expected number of isomorphic subgraphs $M_1^{(k)}$, then it is necessarily pseudorandom. It is weak. Lenz and Mbei [20] determined the semantics between several pseudo-random concepts.

These results were eventually extended by Taussner [21], who presented many equivalence classes of pseudorandom concepts—including all those previously studied—and obtained the interrelationships between those classes. He proved that pseudorandomness of order d is equivalent to having a suitable number of all d-linear supergraphs, i.e., those in which every pair of edges intersect at most d vertices, and also having a small deviation of a certain type. He also constructed a special linear d k-diagram $M_d^{(k)}$ that is perfect for pseudorandomness of order d.

We refer the reader to Chang's website [4] for a long list of references. In this paper, we will consider supergraphs and focusing on the relationships between the corresponding pseudo-random properties.

2. Preliminaries: Pseudorandom Concepts

Here we collect the necessary notation used throughout the paper and formally introduce pseudorandom concepts in hypergraphs. For a more in-depth explanation, we refer the reader to a recent survey [3].

2.1.Pseudorandom supergraph

For every integer $1 \le d < k$ there is a related equivalent class of pseudo-random concepts for k-diagrams that are roughly related to uncorrelation with structures of order d. We will examine several of these pseudorandom concepts below.

Difference

For us, the central pseudo-random concept of supergraph is related to the mismatch of its edge distribution along cuts of a certain order. Variance is a measure of how far the edges of the hypergraph are from a uniform distribution, and can be measured using the cutoff norm defined below:

Definition 2 (norm cutoff). Let k, d ≥ 1 be integers with d < k, and let V be a finite set. We define the d-cut norm of a function f: $V^k \to R$ by:

$$\|f\|_{\square_d^k} := \max_{S_B \subseteq V^B \ \forall B \in \binom{[k]}{d}} \left| \mathbb{E}_{\mathbf{x} \in V^{[k]}} \left[f(\mathbf{x}) \prod_{B \in \binom{[k]}{d}} S_B(\mathbf{x}_B) \right] \right|,$$

Where is the maximum over all sets of the set $(S_B)_{B \in \binom{[k]}{d}}$, that each S_B is a subset of V^B.

Intuitively, the more uniformly distributed the edges of a hypergraph, the value of $\|H - \delta(H)\|_{\square_d^k}$ is smaller. Since the edges of a random supergraph are usually uniformly distributed, we can consider the (soft) small-cut norm as a pseudo-random property for supergraphs. More accurate:

Definition 3 (difference). The difference d of a k-graph H is defined by:

$\delta(H) = e(H)/v(H)^k \quad \mathrm{disc}_d(H) := \|H - \delta(H)\|_{\square_d^k}$ Where:

Given ε > 0, we say that a k-graph H is ε-pseudorandom of order d if $\mathrm{disc}_d(H) \le \varepsilon$.

It is easy to understand from the definition of the cut norm that:

$$0 \le \|f\|_{\square_1^k} \le \|f\|_{\square_2^k} \le \cdots \le \|f\|_{\square_{k-1}^k} \le \|f\|_\infty,$$

Etc:

$$0 \le \mathrm{disc}_1(H) \le \mathrm{disc}_2(H) \le \cdots \le \mathrm{disc}_{k-1}(H) \le 1$$

Therefore, a k-graph that is ε-pseudorandom of order d will also be ε-pseudorandom of order ℓ for all ≤ d ≤ ℓ 1.

Notice The concept of mismatch used in the works of Chang [5] and Kuhayakawa, Roedel, and Skukan [19] was slightly different from the one presented above. We recall it below and call it the category difference. Given a -graph G, let $\mathcal{K}_k(G)$ denote the set of k-sets in G (ie, the set of k-sets of vertices whose d subsets are all edges of G). The dissimilarity d-category of a K-uniform supergraph H is defined as:

$$\frac{1}{v(H)^k} \max_{d\text{-graph } G} \left| |H \cap \mathcal{K}_k(G)| - \delta(H)|\mathcal{K}_k(G)| \right|,$$

Where the maximum is on all d-graphs of G in the same vertex set of H. This concept is formally very similar to $\|H - \delta(H)\|_{\square_d^k}$ pseudorandom our measure of order d (when we open all symbols), and it can be shown that the two quantities are polynomially related. We consider d-category difference and d-difference as one concept in different guises and for technical reasons we have chosen to use the latter.

Counting hypergraphs

An important statistic for having a large supergraph H is the number of different smaller supergraphs F that occur as a subgraph. A convenient way to count such copies with homomorphism density:

Definition 4 (homomorphism density). Let F and H be two k-diagrams. The homomorphism density of F in H, denoted by t(F, H), is the probability that a randomly chosen map φ: V (F) → V (H) preserves edges.

Equivalently, homomorphism density can be defined by the formula:

$$t(F, H) = \mathbb{E}_{\mathbf{x}\in V(H)^{V(F)}} \prod_{e\in E(F)} H(\mathbf{x}_e),$$

It makes sense when H is edge-weighted. This weight item will also be used later. Notice that:

$$N_F(H) = t(F, H)v(H)^{v(F)} \pm \binom{v(F)}{2} v(H)^{v(F)-1},$$

Where $N_F(H)$ is the total number of labeled subgraphs of H that are isomorphic to F, statements about the number of subgraphs in large hypergraphs can be translated into statements about homomorphism density and vice versa.

An important class of hypergraphs in our results is as follows:

Definition 5 (d-linear hypergraphs). If both edges of H intersect at most d vertices, a hypergraph H is said to be d-linear. We denote the set of all d-linear k-diagrams by $\mathcal{L}_d^{(k)}$

It was proved by Tausner that pseudorandomness of degree d is necessary and sufficient to control the number of any d-linear supergraph. See Theorem 3 below.

Deviation

Recall that the k-octet $\mathrm{Oct}^{(k)}$ is a complete k-partite graph where each vertex class has two vertices. They are generalized by a squashed octagon, which is defined as follows:

Definition 6 (squashed octagon). According to the integers ≤ d < k 1, the squashed octagon $\mathrm{OCT}_d^{(k)}$ is defined as graph k in the set of $\{x_1^{(0)}, x_1^{(1)}, \ldots, x_d^{(0)}, x_d^{(1)}, y_{d+1}, \ldots, y_k\}$vertices whose set of edges is given by:

$$E(\mathrm{OCT}_d^{(k)}) = \left\{ \left\{ x_1^{(\omega_1)}, \ldots, x_d^{(\omega_d)}, y_{d+1}, \ldots, y_k \right\} : \omega \in \{0,1\}^d \right\}$$

Following Chang [5], we define the deviation d of a k-graph by H:

$$\mathrm{dev}_d(H) = \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)} \in V(H)^d} \mathbb{E}_{y_{d+1},\ldots,y_k \in V(H)} \prod_{\omega \in \{0,1\}^d} \left(H(x_1^{(\omega_1)}, \ldots, x_d^{(\omega_d)}, y_{d+1}, \ldots, y_k) - \delta(H) \right)$$

Note that this is equal to the squashed octagon with weight number $t(\mathrm{OCT}_d^{(k)}, H - \delta(H))$.

Octagonal norms

Octagonal norms provide an alternative measure of strong pseudorandomness, based on a weighted count of octagons. Their definition is basically due to Gowers [15].

Definition 7 (octagon norm). According to the function f: $V^k \rightarrow R$, we define its (soft) octagonal norm by:

$$\|f\|_{\mathrm{OCT}^k} := \left(\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)} \in V^k} \prod_{\omega \in \{0,1\}^k} f(\mathbf{x}^{(\omega)}) \right)^{1/2^k},$$

Where we write $\mathbf{x}^{(\omega)} := (x_i^{(\omega_i)})_{i \in [k]}$.

It can be shown that the expectation on the right-hand side of (1) is non-negative for any real function f and indeed defines a norm $\|\cdot\|_{\mathrm{OCT}^k}$. An important property of the octagonal norm is that it has a generalized inner product denoted by $\mathrm{OCT}_d^{(k)}$, which for 2^k functions we define $f_\omega : V^k \rightarrow \mathbb{R},\ \omega \in \{0,1\}^k$ by:

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{\mathrm{OCT}^k} := \mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)} \in V^k} \prod_{\omega \in \{0,1\}^k} f_\omega(\mathbf{x}^{(\omega)}).$$

With this product, we have the inside $\|f\|_{\mathrm{OCT}^k}^{2^k} = \langle f, f, \ldots, f \rangle_{\mathrm{OCT}^k}$. A very useful property of octagonal norms and related inner products is that they satisfy a kind of Cauchy-Schwartz inequality. This result was first established by Gowers (albeit with a different notation), and is now known as the Gowers-Cauchy-Schwartz inequality:

Lemma 1 (Gowers-Cauchy-Schwartz inequality). For any set of functions fω: $V^k \rightarrow R$, ω ∈ $\{0, 1\}^k$, we have:

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{\mathrm{OCT}^k} \leq \prod_{\omega \in \{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}.$$

This lemma is proved through frequent applications of the Cauchy-Schwartz inequality. See for example [3, Section 4.5] **for proof**. As a result of the

Gowers-Cauchy-Schwartz inequality, it can be easily shown that octagonal norms are stronger than cut norms:

Lemma 2. For every function f: V k → R we have: $\|f\|_{\square^k_{k-1}} \leq \|f\|_{\mathrm{OCT}^k}$

Proof of the given functions $u_B : V^B \to [0,1],\ B \in \binom{[k]}{k-1}$, let be the $f_{\omega_B} : V^{[k]} \to \mathbb{R}$ function defined by $f_{\omega_B}(\mathbf{x}_{[k]}) = u_B(\mathbf{x}_B)$where ωB ∈ {0, 1} [k] is the index vector of the set B. Also show that f1 = f and fω ≡ 1 for all ω ∈ {0, 1} $^{[k]}$ \ {1} not in the set $\{\omega_B : B \in \binom{[k]}{k-1}\}$.

Using Gowers-Cauchy-Schwartz inequality we conclude that since clearly:

$$\left|\mathbb{E}_{\mathbf{x}\in V^{[k]}}\left[f(\mathbf{x}) \prod_{B\in\binom{[k]}{k-1}} u_B(\mathbf{x}_B)\right]\right| = \left|\mathbb{E}_{\mathbf{x}^{(0)},\mathbf{x}^{(1)}\in V^k} \prod_{\omega\in\{0,1\}^k} f_\omega(\mathbf{x}^{(\omega)})\right| \leq \prod_{\omega\in\{0,1\}^k} \|f_\omega\|_{\mathrm{OCT}^k}.$$

Since clearly $\|f_\omega\|_{\mathrm{OCT}^k} \leq \|f_\omega\|_\infty \leq 1$ for all $\omega \in \{0,1\}^{[k]} \setminus \{\mathbf{1}\}$, the last product $\|f\|_{\mathrm{OCT}^k}$is the maximum. Since this inequality is valid for all functions $u_B : V^B \to [0,1],\ B \in \binom{[k]}{k-1}$: [1, 0] → VB, the claim follows.

2.2. Hypergraph equivalence theorem

In the following, we will present Tausner's theorem, which relates multiple pseudorandom concepts for any order d ≥ 1. We begin by constructing hypergraphs that are complete for these concepts.

According to k-graph K-partite F with vertex partition Xk, . . ., X1 and a d-set of $I \in \binom{[k]}{d}$ indices, we define the I-doubling of F as a supergraph db_I (F) that takes two copies of F and identifies the corresponding vertices in classes Xi, for all i ∈ I it will be obtained. The set of vertices is exactly I-double.

$$V(\mathrm{db}_I(F)) = Y_1 \cup \cdots \cup Y_k \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in I, \\ X_i \times \{0,1\} & \text{if } i \notin I \end{cases}$$

Its edge set is the set of all k-sets of the form

$$\{x_i : i \in I\} \cup \{(x_j, a) : j \in [k] \setminus I\}$$

where {0, 1} ∈ a and {x I: i ∈ [k]} is an edge of F. It starts with a k-partite supergraph with k vertices and a single edge, and then sequentially applies db_I to each $I \in \binom{[k]}{d}$ (in arbitrary order). We obtain a k-diagram which we denote by.$\mathrm{M}_d^{(k)}$

Below we reproduce a quantitative version of Tausner's original result [21]. This version can be obtained through the methods presented in [1].

Theorem 3 (equivalence theorem for pseudorandomness of order d). Let ≤ d < k 1 be integers and let H be a uniform k-supergraph with edge density δ. Then the following polynomial properties are equivalent:
(i) H has a small d-difference: $disc_d(H) \leq c1$.
(ii) H correctly enumerates all linear hypergraphs of d:

$$t(F,H) = \delta^{e(F)} \pm e(F)c_2 \quad \forall F \in \mathcal{L}_d^{(k)}.$$

(i)H has a few copies of $M = M_d^{(k)}$: $t(M,H) \leq \delta^{e(M)} + c_3$
(ii) (v) H has a small deviation relative to $M = M_d^{(k)}$:

$$\prod_{e \in E(M)} \left(H(\mathbf{x}_e) - \delta\right) \leq c_4.$$

Examples to consider:
It may be helpful to have a concrete example in mind for each of the pseudorandom concepts we consider. In the following, we provide such examples by exploiting the well-known pseudo-random properties of quadratic residuals. Let p be a large prime number and denote by Q_p the set of quadratic residues of p:

$$Q_p = \{x \in \mathbb{F}_p : \text{ exists } y \in \mathbb{F}_p \text{ with } x = y^2\}.$$

This set has size (p + 1) / 2 and was shown by Foury, Kowalski, and Michel [12] to be as nearly pseudorandom as possible, in the sense that

$$\|Q_p - 1/2\|_{U^{d+1}} = O_d(p^{-1/2^{d+1}}) \quad \text{for all } d \geq 1.$$

This bound is of the same order as expected from a random function with value {1, 1−}, while for any function with value {1, 1−} we have f in Fp, $\|f\|_{U^{d+1}} \geq p^{-1/2^{d+1}}$.
We define a k-graph representation $\mathcal{P}^{(k)} = \mathcal{P}^{(k)}(p)$ as a hypergraph whose vertices are elements of F_p and where $\{x_1, ..., x_k\}$ is an edge if $x_1, x_2, ..., x_k \in Q_p$. From the properties of quadratic residues, we see that $P^{(k)}$ has edge density O(1) + 1.2 and satisfies

$$\|\mathcal{P}^{(k)} - 1/2\|_{\square_{k-1}^k} \leq \|\mathcal{P}^{(k)} - 1/2\|_{\mathrm{OCT}^k} = \|Q_p - 1/2\|_{U^k} = O_k(p^{-1/2^k}),$$

Where the inequality follows **from Lemma 2** and the first equality follows from a simple change of variables in the expression defining the norm U^k (norm) (see **Lemma 3** in the next section). This hypergraph was previously considered by Chang and Graham [7] as an example of a pseudo-random hypergraphs.

Now fix some integer 2 ≤ d < k, and let be $\mathcal{P}_d^{(k)} = \mathcal{P}_d^{(k)}(p)$ the k-categories encoding the supergraph in $\mathcal{P}^{(d)}(p)$. More precisely, the vertex set $\mathcal{P}^{(d)}(p)$is F_p. and a k-set $\{x_1, x_2, \ldots, x_k\} \subset$ Fp, is an edge if:

$$\sum_{i \in B} x_i \in Q_p \quad \text{for all } B \in \binom{[k]}{d}.$$

The edge density $\mathcal{P}_d^{(k)}$ is exactly the homomorphism density of k -categories in $\mathcal{P}_d^{(k)}$ which (with pseudo-randomness $\mathcal{P}_d^{(k)}$) is equal to $\|\mathcal{P}_d^{(k)} - 2^{-\binom{k}{d}}\|_{\square_{d-1}^k} = o(1)$.

It is easy to show $\|\mathcal{P}_d^{(k)} - 2^{-\binom{k}{d}}\|_{\square_{d-1}^k} = o(1)$, since any set of witnesses for (d-1)-large differences $\mathcal{P}^{(d)}$ can be turned into witnesses. (d - 1)-sets a large difference$\mathcal{P}_d^{(k)}$ (and these cannot exist). Finally, we mention this point:

$$\begin{aligned}\|\mathcal{P}_d^{(k)} - 2^{-\binom{k}{d}}\|_{\square_d^k} &\geq \mathbb{E}_{\mathbf{x} \in \mathbb{F}_p^k}\left[(\mathcal{P}_d^{(k)}(\mathbf{x}) - 2^{-\binom{k}{d}}) \prod_{B \in \binom{[k]}{d}} \mathcal{P}^{(d)}(\mathbf{x}_B)\right] \\ &= \mathbb{E}_{\mathbf{x} \in \mathbb{F}_p^k}\left[(\mathcal{P}_d^{(k)}(\mathbf{x}) - 2^{-\binom{k}{d}})\mathcal{P}_d^{(k)}(\mathbf{x})\right] \\ &= 2^{-\binom{k}{d}} - 2^{-2\binom{k}{d}} + o(1).\end{aligned}$$

It follows that $\mathcal{P}_d^{(k)}$ the pseudorandom is of order d - 1, but it is not pseudorandom of order d.

3. Uniformity and quasi-random connection

Our main technical result is that the monotonicity of a given degree d for a polynomial set A is equivalent to the pseudorandomness of the same order d for the $\Gamma_A^{(k)}$ Cayley supergraph, for any value k > d. This generalizes the theorem of Aigner Horev and H`an [2], which considers the special case where d = 1.

Recall that a k-k-graph H is ε-pseudorandom of order d if $\|H - \delta(H)\|_{\square_d^k} \leq \varepsilon$, and the set A ⊆G is ε-uniform of degree d if $\|A - \delta\|_{U^{d+1}} \leq \varepsilon$ where $\delta = |A|/|G|$. Our main result is as follows:

Theorem 4 (Theorem 1 restated). Let d ≥ 1 be an integer and ε ∈ (0, 1). Let G be a finite additive group and A ⊆ G be a subset.

(i) If A is ε-uniform of degree d, then for all k ≥ d + 1, the $\Gamma_A^{(k)}$ Cayley ε-pseudorandom supergraph is of order d.
(ii) Conversely, if $\Gamma_A^{(k)}$ ε is a -pseudorandom of order d for some k ≥ d + 1, then A 2εck,d is uniform of degree d. Here we can take $c_{k,d} = 2^{-(d+2)(2d+2)^k}$.
As might be expected from the bounds given in this statement, the proof of statement (ii) is much more than that of statement (i). Both proofs are rudimentary in the sense that they only use the triangle and Cauchy-Schwartz inequalities, but the Cauchy-Schwartz applications needed to prove proposition (ii) are somewhat triadic and require careful analysis.
Proof of statement (i). Let fA: = A - δ be the balanced indicator function of set A. Choose optimal functions uB: $G^B \to [0, 1]$, such that:

$$\|\Gamma_A^{(k)} - \delta\|_{\square_d^k} = \left| \mathbb{E}_{\mathbf{x}\in G^k} \left[f_A(\Sigma(\mathbf{x})) \prod_{B\in\binom{[k]}{d}} u_B(\mathbf{x}_B) \right] \right|.$$

We may separate the first d+1 variables x[d+1] from the others and write them [22]:

$$\|\Gamma_A^{(k)} - \delta\|_{\square_d^k} = \left| \mathbb{E}_{\mathbf{x}_{[k]\setminus[d+1]}} \mathbb{E}_{\mathbf{x}_{[d+1]}} \left[f_A(\Sigma(\mathbf{x}_{[d+1]}) + \Sigma(\mathbf{x}_{[k]\setminus[d+1]})) \prod_{B\in\binom{[k]}{d}} u_B(\mathbf{x}_B) \right] \right|,$$

where the first expectation is greater than $G^{[k]\setminus[d\ 1]}$ and the second expectation is greater than $G^{[d+1]}$.

Now we fix x[k]\[d + 1] ∈ G[k]\[d + 1] and consider the internal expectation in the last term. Writing y: = $\Sigma(x_{[k]\setminus[d\ 1]})$, this expression can be written as:

$$\mathbb{E}_{\mathbf{x}_{[d+1]}} \left[\mathrm{T}^y f_A \circ \Sigma(\mathbf{x}_{[d+1]}) \prod_{D\in\binom{[d+1]}{d}} v_D(\mathbf{x}_D) \right]$$

For some appropriate functions $v_D : G^D \to [0,1],\ D \in \binom{[d+1]}{d}$, and therefore has an absolute maximum value

$$\|\mathrm{T}^y f_A \circ \Sigma\|_{\square_d^{d+1}} = \|\Gamma_{\mathrm{T}^y A}^{(d+1)} - \delta\|_{\square_d^{d+1}} = \|\Gamma_A^{(d+1)} - \delta\|_{\square_d^{d+1}}.$$

Since the octagon norm is stronger than the cut norm (Lemma 2), this last expression is maximal $\|\Gamma_A^{(d+1)} - \delta\|_{\mathrm{OCT}^{d+1}} = \|A - \delta\|_{U^{d+1}}$ (where we used Lemma 3).
Averaging over $x_{[k]\setminus[d+1]} \in G^{[k]\setminus[d+1]}$ and using the triangle inequality we conclude that as desired $\|A - \delta\|_{U^{d+1}} \leq \varepsilon,\ \|\Gamma_A^{(k)} - \delta\|_{\square_d^k} \leq$
The rest of this section will be devoted to the proof of proposition (ii).

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3.1. Proof that pseudorandomness implies uniformity

Prove an additive group G and integers k, d ≥ 1 with k ≥ d + 1. We want to show that A ⊆G is uniform of degree d whenever $\Gamma_A^{(k)}$ the pseudorandom is of order d. Our proof will proceed through an iterative argument, where we construct and analyze several systems of linear forms defined in G.

3.1.1. Linear systems and norms

We consider only linear forms: G^V → G: φ whose coefficients are 0 or 1, where V is a finite index set for the variables. These forms can be characterized by their support: this is the subset supp(φ) ⊆ V such that [23]:

$$\phi(\mathbf{x}) = \sum_{v \in \mathrm{supp}(\phi)} x_v \quad \text{for all } \mathbf{x} \in G^V.$$

A set of linear forms Φ = {φ1, . . . , φm} is a linear system and its support is the union of the supports of all its constituent forms: $\mathrm{supp}(\Phi) = \bigcup_{i=1}^{m} \mathrm{supp}(\phi_i)$ following Green and Tao [17], we say that a linear system Φ = {φ1, . . . , φm} is of s-normal form if, for each i ∈ [m], there exists a subset σi ⊆ supp(φi) of size at most s + 1 that does not exist entirely in supp(φj) for any $j \neq i$. . The importance of this concept is illustrated by the next result, which is primarily due to Green and Tao. For proof see [17, Appendix C] or [16, Section 2].

Lemma 4. If Φ = {φ1, . . . , φm} is a linear system in s-normal form, then for all functions f1, . . . , fm : G → [−1, 1] we have.

A certain set V0: = {(0, 1), (0, 2), . . ., (0, k)} is one of the indices of the variables of our linear forms, which is considered separately from the others and will play an important role in our arguments. In general, this set of k variables represents the ones we care about, while the other variables are just to aid the analysis and are eventually replaced by appropriately chosen values. Considering the linear form of φ, we define its weight w0(φ) as the number of variables in V0 that it uses: w0(φ): = |supp(φ)∩V0|. The weight of a linear system Φ is the maximum weight One of its forms is: w0(Φ) = max{w0(φ): φ ∈ Φ}.

In our analysis, we need a cut-type semisoft associated with linear systems, which serves to bridge the gap between the U^{d+1} norm for additive sets and the d-cut norm for their Cayley supergraph.

Definition 11 (cut type norms). Let Φ be a linear system on G and denote V (Φ)=V_0 ∪ supp(Φ). According to the function f: G → R, we define:

$$\|f\|_{\square(\Phi)} := \max_{u_\phi: G \to [-1,1],\, \forall \phi \in \Phi} \mathbb{E}_{\mathbf{x} \in G^{V(\Phi)}} \left[f\left(\sum_{v \in V_0} x_v \right) \prod_{\phi \in \Phi} u_\phi(\phi(\mathbf{x})) \right].$$

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We can now give an outline of our proof of proposition (ii). We will proceed through an iterative algorithm, where at each step we have a linear system Φs characterized by its linear form support. We start with Φ0, which contains only the linear form supported by V0, which then has weights k ≥ d + 1. Whenever the considered system Φs has a form φ with a weight higher than d, we replace φ with $^{2d+1}$ − 1 'dual'. Forms of much lower weight, thus creating a Φs + 1 system. The important point here is that the cut-type norms of the Φs and Φs + 1 systems are related (by some Cauchy-Schwartz magic trick). We stop as soon as all shapes in Φs weigh most d. Since the weights are bounded by d, we can then bound the associated $\|\cdot\|_{\square(\Phi_s)}$ norm based on the usual d-cut norm. Furthermore, we show that the first norm $\|\cdot\|_{\square(\Phi_s)}$ is bounded from below by the norm Ud + 1. Then this statement is followed by applying the resulting norm inequalities to the balanced index function A - δ of the considered set A ⊆ G.

3.1.2. Dual line shapes

It remains to present the concept of dual linear forms for use in our algorithm, which is motivated by the concept of U $^{d+1}$ - dual function. For the function f: G → R, its dual function D $_{d+1}$, f is defined by:

$$\mathcal{D}_{d+1}f(y) = \mathbb{E}_{h_1,\ldots,h_{d+1}\in G} \prod_{\omega\in\{0,1\}^{d+1}\setminus\{0\}} f\left(y + \sum_{i=1}^{d+1} \omega_i h_i\right),$$

So that we can write $\|f\|_{U^{d+1}}^{2^{d+1}} = \mathbb{E}_{y\in G}\left[f(y)\mathcal{D}_{d+1}f(y)\right]$. Dual linear forms are meant to mimic this concept but with an assumed "heavy" form of φ. Let φ: GV → G be the linear form of the weight w0(φ) ≥ d + 1, and let Id be a set of d + 1 elements distinct from V. For each ω ∈ {0, 1} d+1\{0}, we define a $\mathfrak{D}^{\omega}\phi : G^{V\cup I_d} \to G$ linear form with $w_0(\mathfrak{D}^{\omega}\phi) < w_0(\phi)$ such that [25]:

$$\mathcal{D}_{d+1}f(\phi(\mathbf{x}_V)) = \mathbb{E}_{\mathbf{x}_{I_d}\in G^{I_d}} \prod_{\omega\in\{0,1\}^{d+1}\setminus\{0\}} f(\mathfrak{D}^{\omega}\phi(\mathbf{x}_{V\cup I_d})) \quad \text{for all } \mathbf{x}_V \in G^V.$$

This can be achieved in the following way:

Definition 12 (dual figures). Given the linear form φ with weight $w_0(\varphi) \geq d + 1$, let Id be a copy of the set {1, 2, . . ., d + 1} which is separate from supp(φ) ∪ V_0 and write

$$\text{supp}(\phi) \cap V_0 = \left\{(0, j_1), (0, j_2), \ldots, (0, j_{w_0(\phi)})\right\}.$$

For each ω ∈ {0, 1}, we define a $\mathfrak{D}^{\omega}\phi$ linear form (with coefficients 0 or 1) with:

$$\text{supp}(\phi) \setminus \text{supp}(\mathfrak{D}^{\omega}\phi) = \{(0, j_i) : 1 \leq i \leq d+1, \omega_i = 1\},$$
$$\text{supp}(\mathfrak{D}^{\omega}\phi) \setminus \text{supp}(\phi) = \{i \in I_d : 1 \leq i \leq d+1, \omega_i = 1\}.$$

In other words, $\mathfrak{D}^{\omega}\phi$ it is constructed from φ by replacing the variables indexed by (0, ji) with ωi = 1 by new variables. Note that it does not matter in which order we label the elements in supp (φ) ∩ V_0 or I_d, since the resulting forms will be equivalent to any labeling. Note also that this definition satisfies equation (3): writing V to support φ and performing a change of variables:

$$y = \sum_{v \in V} x_v \quad \text{and} \quad h_i = x_i - x_{(0,j_i)} \quad \text{for all } 1 \leq i \leq d+1,$$

We see it. $\mathfrak{D}^{\omega}\phi(\mathbf{x}_{V \cup I_d}) = y + \sum_{i=1}^{d+1} \omega_i h_i$

3.1.3. The main algorithm and its analysis

Consider the following algorithm:

System Cut. (k, d) algorithm

1.

$V_0 \leftarrow \{(0,1), (0,2), \ldots, (0,k)\}$
$\Phi_0 \leftarrow \{\Sigma_{V_0}\}$
$s \leftarrow 0$
while $w_0(\Phi_s) > d$ **do**
  take $\psi_s \in \Phi_s$ **with** $w_0(\psi_s) = w_0(\Phi_s)$
  $\Phi_{s+1} \leftarrow (\Phi_s \setminus \{\psi_s\}) \cup \{\mathfrak{D}^{\omega}\psi_s : \omega \in \{0,1\}^{d+1} \setminus \{0\}\}$
  $s \leftarrow s + 1$
end while
$\mathfrak{s}_{\mathfrak{f}} \leftarrow s$

We will show this algorithm:

Lemma 5. For every finite function f: G → [−1, 1] we have:

1. $\mathfrak{s}_{\mathfrak{f}} < (2d+2)^k$
2. $\|f\|_{\square(\Phi_1)} \geq \|f\|_{U^{d+1}}^{2^{d+1}}$
3. $\|f\|_{\square(\Phi_{\mathfrak{s}_{\mathfrak{f}}})} \leq 2^{\binom{k}{d}} \|\Gamma_f^{(k)}\|_{\square_d^k}$
4. $\Phi_s \cup \{\Sigma_{V_0}\}$ is in d-normal form for all s ≥ 1.
5. $\|f\|_{\square(\Phi_{s+1})} \geq \|f\|_{\square(\Phi_s)}^{2^{d+2}}$ $for\ 1 \leq s < \mathfrak{s}_{\mathfrak{f}}$

With the help of this lemma, statement (ii) of Theorem 4 is easily obtained:

Proof of Theorem 4. (ii). Suppose $\Gamma_A^{(k)}$ that ε-pseudorandom is of order d. Using case 2 of **Lemma 5,** we obtain $\|A-\delta\|_{\square(\Phi_{s_f})} \leq 2^{\binom{k}{d}}\varepsilon$ for Application of item 4

Recursively from s = sf - 1 to s = 1, we conclude that $\|A-\delta\|_{\square(\Phi_1)}^{2^{(d+2)s_f}} \leq 2^{\binom{k}{d}}\varepsilon$. Using item 1 together with the bound $s_f < (2d+2)^k$ from item 5, we deduce that.

$$\|A-\delta\|_{U^{d+1}}^{2^{(d+2)(2d+2)^k}} \leq \left(\|A-\delta\|_{U^{d+1}}^{2^{d+1}}\right)^{2^{(d+2)s_f}} \leq \|A-\delta\|_{\square(\Phi_1)}^{2^{(d+2)s_f}} \leq 2^{\binom{k}{d}}\varepsilon,$$

And so A is 2εck,d-uniform of degree d for $c_{k,d} = 2^{-(d+2)(2d+2)^k}$.
Then it suffices to prove Lemma 5, which we do in the next step.
Proof of Lemma 5. Throughout this proof, we will denote the support of the linear system Φs by Vs, that is: $V_s = \bigcup_{\phi\in\Phi_s} \operatorname{supp}(\phi)$. Note that $V_0 \subset V_1 \subset \ldots \subset V_s$.
Case 1. Using identity (3) (which motivated our definition of dual forms), we have that:

$$\begin{aligned}\|f\|_{U^{d+1}}^{2^{d+1}} &= \mathbb{E}_{\mathbf{x}\in G^{V_0}}\left[f(\Sigma_{V_0}(\mathbf{x}))\mathcal{D}_{d+1}f(\Sigma_{V_0}(\mathbf{x}))\right]\\ &= \mathbb{E}_{\mathbf{x}\in G^{V_0\cup I_d}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{\omega\in\{0,1\}^{d+1}\setminus\{0\}} f\left(\mathfrak{D}^{\omega}\Sigma_{V_0}(\mathbf{x})\right)\right]\\ &= \mathbb{E}_{\mathbf{x}\in G^{V_1}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{\phi\in\Phi_1} f(\phi(\mathbf{x}))\right].\end{aligned}$$

Since $\|f\|_\infty \leq 1$, this last term is a maximum $\|f\|_{\square(\Phi_1)}$, as you want.
Case 2. By the definition of sf, we have that w0(φ) ≤ d for all φ ∈ Φsf. Choose optimal functions: G ⟶ [−1, 1] uφ, φ ∈ Φsf, so that:

$$\|f\|_{\square(\Phi_{s_f})} = \mathbb{E}_{\mathbf{x}\in G^{V_{s_f}}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{\phi\in\Phi_{s_f}} u_\phi(\phi(\mathbf{x}))\right].$$

By the averaging principle, we can fix the variables indexed by Vsf\ V0 to a fixed value $\mathbf{y}_{V_{s_f}\setminus V_0} \in G^{V_{s_f}\setminus V_0}$

$$\|f\|_{\square(\Phi_{s_f})} \leq \mathbb{E}_{\mathbf{x}_{V_0}\in G^{V_0}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{\phi\in\Phi_{s_f}} u_\phi(\phi(\mathbf{x}_{V_0},\mathbf{y}_{V_{s_f}\setminus V_0}))\right].$$

Note that each function in the above product depends on at most d variables xV0, so we can write it as hB(xB) for some set B⊂ V0 of size at most d and some function: GB ⟶ [−1, 1] write hB. It results that:

$$\|f\|_{\square(\Phi_{s_f})} \leq \max_{h_B:G^B\to[-1,1],\forall B\in\binom{V_0}{d}} \mathbb{E}_{\mathbf{x}_{V_0}\in G^{V_0}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{B\in\binom{V_0}{d}} h_B(\mathbf{x}_B)\right].$$

Decompose each function hB in the above expression into its positive part h′B: = max {hB, 0} and its negative part h″B: = max {−hB, 0}, so hB = h ′ B − h ′ Write B. By opening the resulting product into $2^{\binom{k}{d}}$ expressions and using the triangle inequality, we conclude that this expression is bounded:

$$2^{\binom{k}{d}} \max_{g_B:G^B\to[0,1],\forall B\in\binom{V_0}{d}} \left|\mathbb{E}_{\mathbf{x}_{V_0}\in G^{V_0}}\left[f\left(\sum_{v\in V_0} x_v\right)\prod_{B\in\binom{V_0}{d}} g_B(\mathbf{x}_B)\right]\right|.$$

This is exactly $2^{\binom{k}{d}}\|\Gamma_f^{(k)}\|_{\square_d^k}$, what completes the proof.

Case 3. Prove 1 ≤ s ≤ sf and let φ ∈ Φs be of any form. By construction, we have that $\phi = \mathfrak{D}^{\omega}\psi_t$ for some ω ∈ {0, 1} and some 0 ≤ t < s (where ψt ∈ Φt is the heavy form chosen by the algorithm at step t). Copy Id = [d + 1] used in step t with {(t + 1, 1), . . ., (t + 1, d + 1)} show, and write:

$$\operatorname{supp}(\psi_t)\cap V_0 = \{(0,j_1),(0,j_2),\ldots,(0,j_{w_0(\psi_t)})\}.$$

Then it is $\mathfrak{D}^{\omega}\psi_t$ easy to check that it is the only form in Φs∪{ΣV0} that is supported:

$$\{(0,j_i):\ 1\leq i\leq d+1,\ \omega_i=0\}\cup\{(t+1,i):\ 1\leq i\leq d+1,\ \omega_i=1\}.$$

Finally, ΣV0 is the only form in Φs ∪ {ΣV0} that uses all variables (0, d + 1), ..., (1, 0). And so this system is in d-normal form.

Case 4. Choose the optimal functions uφ: uφ: G ⟶ [−1, 1], φ ∈ Φs, so that:

$$\|f\|_{\square(\Phi_s)} = \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[f(\Sigma_{V_0}(\mathbf{x}))\prod_{\phi\in\Phi_s} u_\phi(\phi(\mathbf{x}))\right].$$

We shift our focus to the function uψ, where ψ = ψs ∈ Φs is the linear form of the maximum weight selected by the algorithm at step s. We can rewrite the above expectation as $\mathbb{E}_{\mathbf{x}\in G^{V_s}}[u_\psi(\psi(\mathbf{x}))g(\psi(\mathbf{x}))]$, where:

$$g(z) := \mathbb{E}_{\mathbf{y}\in G^{V_s}:\psi(\mathbf{y})=z}\left[f(\Sigma_{V_0}(\mathbf{y}))\prod_{\phi\in\Phi_s\setminus\{\psi\}} u_\phi(\phi(\mathbf{y}))\right].$$

Note that $\|g\|_\infty \leq 1$, since all functions in its definition have a limit of 1. Since $\|u_\psi\|_\infty \leq 1$, we have by Cauchy-Schwartz:

$$\begin{aligned}\|f\|_{\square(\Phi_s)} &= \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[u_\psi(\psi(\mathbf{x}))g(\psi(\mathbf{x}))\right] \\ &\leq \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[g(\psi(\mathbf{x}))^2\right]^{1/2} \\ &= \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[g(\psi(\mathbf{x}))f(\Sigma_{V_0}(\mathbf{x}))\prod_{\phi\in\Phi_s\setminus\{\psi\}}u_\phi(\phi(\mathbf{x}))\right]^{1/2}.\end{aligned}$$

Now we use the fact (from Item 3.) that Φs ∪ {ΣV0} is in d-normal form. Since all the expected expressions above have bound 1, we conclude by Lemma 4 that this last expression is maximal $\|g\|_{U^{d+1}}^{1/2}$ and therefore:

$$\|f\|_{\square(\Phi_s)} \leq \|g\|_{U^{d+1}}^{1/2}$$

Then we closed $\|g\|_{U^{d+1}}$. N .notice that:

$$\begin{aligned}\|g\|_{U^{d+1}}^{2^{d+1}} &= \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[g(\psi(\mathbf{x}))\mathcal{D}_{d+1}g(\psi(\mathbf{x}))\right] \\ &= \mathbb{E}_{\mathbf{x}\in G^{V_s}}\left[f(\Sigma_{V_0}(\mathbf{x}))\prod_{\phi\in\Phi_s\setminus\{\psi\}}u_\phi(\phi(\mathbf{x}))\cdot\mathcal{D}_{d+1}g(\psi(\mathbf{x}))\right].\end{aligned}$$

By constructing $\mathfrak{D}^\omega\psi$ linear forms, we have for each $\mathbf{x}_{V_s}\in G^{V_s}$ constant:

$$\mathcal{D}_{d+1}g(\psi(\mathbf{x}_{V_s})) = \mathbb{E}_{\mathbf{x}_{V_{s+1}\setminus V_s}\in G^{V_{s+1}\setminus V_s}}\prod_{\omega\in\{0,1\}^{d+1}\setminus\{0\}}g(\mathfrak{D}^\omega\psi(\mathbf{x}_{V_{s+1}}));$$

It results that:

$$\|g\|_{U^{d+1}}^{2^{d+1}} = \mathbb{E}_{\mathbf{x}\in G^{V_{s+1}}}\left[f(\Sigma_{V_0}(\mathbf{x}))\prod_{\phi\in\Phi_s\setminus\{\psi\}}u_\phi(\phi(\mathbf{x}))\prod_{\omega\in\{0,1\}^{d+1}\setminus\{0\}}g(\mathfrak{D}^\omega\psi(\mathbf{x}))\right].$$

Since $\Phi_{s+1} = (\Phi_s\setminus\{\psi\})\cup\{\mathfrak{D}^\omega\psi : \omega\in\{0,1\}^{d+1}\setminus\{0\}\}$, this last term is the maximum $\|f\|_{\square(\Phi_{s+1})}$. Then we conclude with inequality (4):

Case 5. Denote the final value of sf in the SystemCut(n,d) algorithm by sf(n) (that is, when V0| = n|). We will show by induction that sf(n) < (2d + 2) n for every n ≥ 1.First, we note that sf(n) is equal to the number of times we enter the loop in the pattern rhythm. In particular, sf (n) = 0 if n ≤ d, as in this case W_0 (ΣV_0) = n ≤ d and we do not enter the loop. This takes care of the base case for induction.

After we first enter the loop, we replace the shape ΣV_0 of weight n with shapes $\mathfrak{D}^\omega\Sigma_{V_0}, \omega\in\{0,1\}^{d+1}\setminus\{0\}$ that have weights.

$$w_0(\mathfrak{D}^{\omega}\Sigma_{V_0}) = (d+1-|\omega|) + (n-d-1) = n-|\omega|.$$

It results that:

$$\mathfrak{s}_{\mathfrak{f}}(n) = 1 + \sum_{\omega\in\{0,1\}^{d+1}\setminus\{0\}} \mathfrak{s}_{\mathfrak{f}}(n-|\omega|) = 1 + \sum_{i=1}^{d+1}\binom{d+1}{i}\mathfrak{s}_{\mathfrak{f}}(n-i).$$

By induction hypothesis, we have:

$$\sum_{i=1}^{d+1}\binom{d+1}{i}\mathfrak{s}_{\mathfrak{f}}(n-i) < \sum_{i=1}^{d+1}\binom{d+1}{i}(2d+2)^{n-i}$$

$$= (2d+2)^n \sum_{i=1}^{d+1}\binom{d+1}{i}\left(\frac{1}{2d+2}\right)^i$$

$$= (2d+2)^n\left(\left(1+\frac{1}{2d+2}\right)^{d+1} - 1\right).$$

Using 1 + x ≤ ex for all x ≥ 0, we conclude that:

$$\mathfrak{s}_{\mathfrak{f}}(n) < 1 + (2d+2)^n\left(e^{1/2}-1\right) < (2d+2)^n,$$

Which concludes the proof by induction.

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الرسوم البيانية الفائقة العشوائية الزائفة

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مستخلص البحث:
ندرس التفاعل بين مفاهيم شبه العشوائية للرسوم البيانية الفائقة .على وجه الخصوص، نظهر ارتباطا قويا بين مقاييس نوع التناقض لشبه العشوائية في إعداد الرسم البياني الفائق .استغلال هذا الصدد، ونحن نقدم قائمة طويلة من الخصائص شبه العشوائية المتباينة فيما يتعلق، وتبين أن هذه الخصائص كلها متكافئة (بمعنى تشونغ، غراهام وويلسون) مع حدود متعددة الحدود على ترابطها.
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