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Data Modelling and Analysis Using Odd Lomax Generalized Exponential Distribution: an Empirical Study and Simulation

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ABSTRACT

This study is based on the formulation of a new probabilistic model called the generalized singular Lomax exponential distribution using the singular Lomax generator and the exponential distribution. The proposed distribution is very flexible in modelling age data with both decreasing and increasing (non-monotonic) shapes. We define the probability density function (pdf) and cumulative distribution function (CDF) for the proposed distribution. Some mathematical properties of the proposed distribution such as the quantum function and moments are derived and for the incomplete moments the Renyi entropy of the proposed distribution is also obtained. Furthermore, the paper discusses model parameters in three different techniques, with Monte Carlo simulations to determine the efficiency of estimating the HOLGE distribution, with a comparison with three benchmarks to determine the best estimation method. A practical application is also carried out on two types of data that include the survival times of 72 guinea pigs infected with virulent tuberculosis bacilli, where the efficiency of the HOLGE distribution analysis is determined by comparing it to six other distributions using 4 information criteria and 4 statistical measures, which demonstrated the efficiency and flexibility of the HOLGE distribution.

1. Introduction

Statistical distributions play a fundamental role in data analysis and statistical modelling, describing how the possible values of a random variable are distributed. Distributions are classified into two main types: discrete distributions, which deal with discrete values such as integers, and continuous distributions, which deal with values that take a continuous range of true values. In statistics, distributions are used to understand the nature of data, estimate parameters, and test hypotheses. Each distribution is characterized by a probability density function (PDF) or probability mass function (PMF), as well as parameters that describe it such as mean, variance, and kurtosis. Continuous distributions belong to different families, each of which has specific mathematical properties that suit specific applications. Accordingly, many continuous statistical families have been introduced based on this principle. Many researchers in statistics introduced new families distribution like the MKi-G [1], GME family [2], MT-X [3], MOTL-G family [4], , logarithmic family [5], ITL-H family [6], SHE-G family [7], NOGEE−G family [8], WEE-X family [9], GOM-G family [10], and the OL-G family on which the proposed distribution is

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based has the CDF, and pdf function respectively as follows [11]:

$$F_{OLG}(x,\zeta,\beta,\xi) = 1 - \left(1 - \frac{\Psi(x,\xi).log(1-(\Psi(x,\xi))}{\beta}\right)^{-\zeta} \quad (1)$$

$$f_{OLG}(x,\zeta,\beta,\xi) = \frac{\zeta}{\beta}\psi(x,\xi)\left(1 - \frac{\Psi(x,\xi).log(1-(\Psi(x,\xi))}{\beta}\right)^{-(\zeta+1)}\left[\frac{\Psi(x,\xi)}{1-\Psi(x,\xi)} - \log(1-\Psi(x,\xi))\right] \quad (2)$$

$$,x,\zeta,\beta,\xi > 0$$

This study is trying to include a new distribution product and explain its features. A family formation combines individual distribution of Lomax T-X. The OLG family suggested that it is better than a distributor distribution. This distribution of the product is important because it explains how to decrease the data increase. The second goal is to show the general distribution as standard. The family also considered distribution statistics. In addition, estimation MLE, LSE, and WLSE to estimate the model parameters. A simulation study is performed to evaluate the distribution performance.

The problem if the study is that the current probability distributions face difficulties in representing the life expectancy data that have non-monotonic (non-unidirectional) behaviour, and there is no specific statistical model that has enough flexibility to represent the different patterns in the actual data, in addition to the lack of studies that provide a comparative analysis between the methods of estimating the parameters of the distributions, which makes the choice of the optimal method unclear, and finally the need for a new distribution that has enough flexibility to represent the real data, and provides superior performance compared to the traditional distributions.

This paper aims to develop and present a new probability distribution called the Odd Lomax Generalized exponential (HOLGE) distribution. The new distribution is based on combination of Lomax generator with the General Exponential distribution, making it more flexible in modeling lifespan data that take increasing or decreasing forms. The research focuses on:

- Deriving basic functions such as the pdf and CDF.
- Analyzing mathematical properties such as the Quantile function, moments, and Renny entropy.
- Evaluation the performance of estimation using three methods (MLE, LSE, and WLSE).
- Verifying the efficiency of the distribution using Monte Carlo simulation to compare the estimation performance.
- Applying the distribution to real data, including the several time of 72 guinea pigs infected with tuberculosis, and comparing the efficiency of HOLGE with six other distributions using four information criteria and four statistical measures.

2. odd Lomax Generalized exponential (HOLGE) distribution

If we take the Generalized exponential distribution as a base line model , the CDF and pdf is $\Psi(x) = (1-\exp(-\lambda x))^{\theta}$and $\psi(x) = \theta\lambda\exp(-\lambda x)\,(1-\exp(-\lambda x))^{\theta-1}$ [8] respectively and substitute in Eq.(1) , and Eq.(2) to get the CDF and pdf for odd Lomax Generalized exponential distribution (HOLGE), respectively by:

$$F(x,\zeta,\beta,\lambda,\theta) = 1 - \left(1 - \frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\zeta} \quad (3)$$

$$f(x,\zeta,\beta,\lambda,\theta) = \frac{\zeta\theta\lambda}{\beta}exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1} \quad (4)$$

$$\times\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-(\zeta+1)}$$
$$\times\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}}-\log(1-(1-exp(-\lambda x))^{\theta})\right]$$

The Survival function can be getting from following equation [12]:

$$S(x)=1-F(x,\zeta,\beta,\lambda,\theta)$$

$$S(x)=\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\zeta} \quad (5)$$

On the other hand, Eq. yields the hazard function of the HOLEG distribution:

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\zeta\theta\lambda exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1}\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}}-log\left(1-(1-exp(-\lambda x))^{\theta}\right)\right]}{\beta\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log(1-(1-exp(-\lambda x))^{\theta})}{\beta}\right)} \quad (6)$$

Figure 1, 2, and 3 represents CDF, pdf, and Survival of HOLGE distribution, respectively

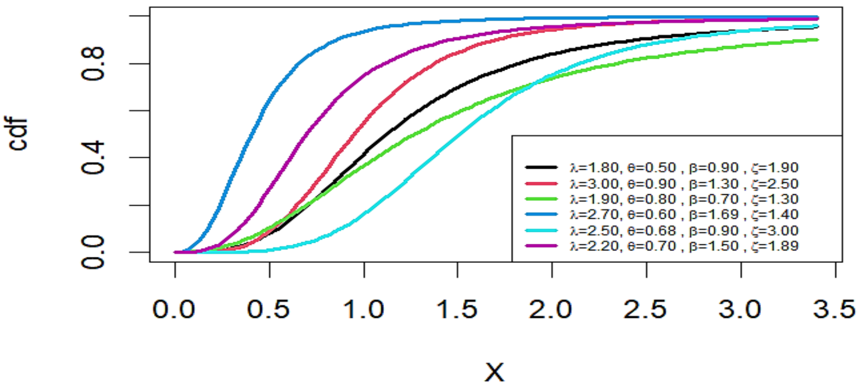

Figure 1. plot CDF function for HOLGE distribution with different value of parameters

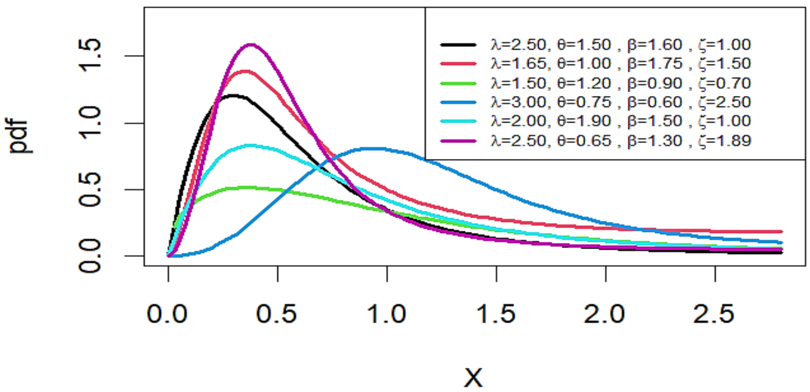

Figure 2. plot pdf function for HOLGE distribution with different value of parameters

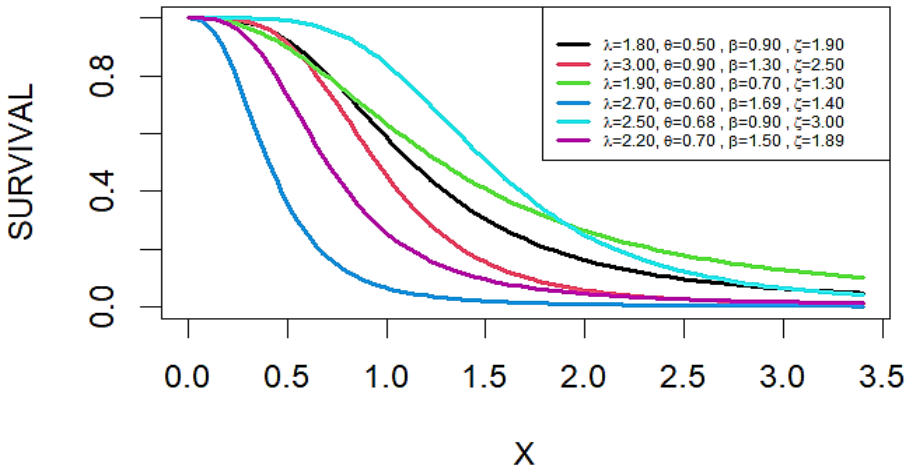

Figure 3. plot Survival function for HOLGE distribution with different value of parameters

Figure 1 shows how the cumulative function gradually increases until it reaches 1 as X increases, reflecting the nature of the cumulative distribution that determines the probability that random variables are less than or equal to a certain value. It also shows that changing the parameters affects the slope of the curve and the speed of its rise, confirming the flexibility of the distribution in representing different types of data. Figure 2 shows how the shape of the distribution changes with different parameters, as it can be symmetrical, skewed to the left, or skewed to the right. Some curves show non-monotonic behavior, indicating the ability of the distribution to represent data with nonlinear patterns. Figure 3 shows the curves and shows how the survival probabilities decrease as X increases, reflecting how the data is distributed in phenomena that depend on the time of failure or survival. Changing the parameters affects the speed of the function's decline, indicating the ability of the distribution to represent different types of data.

3. Mathematical Properties of HOLGE distribution

3.1. CDF and pdf expansion of HOLGE distribution

We can expand the CDF of HOLGE distribution using Eq.(3) as follows :Using binomial series expansion ,and logarithm expansion [13], [14] we get:

$$F_{OLG}(x,\zeta,\beta,\lambda,\theta) = 1 - \Omega_{i,j,k,l,p}(exp(-\lambda x)^{p\theta} \quad (7)$$

where $\Phi_{i,j,k,l,p} = \sum_{i=j=k=l=p=0}^{\infty} \frac{\Gamma(\zeta+k)\beta^{-k}(-1)^{j+k+l+p}+pd_{k,i}}{K!\Gamma(\zeta)} \binom{2k+1}{j}\binom{j}{r}\binom{r}{p}$

using logarithmic and binomial series expansion, we obtain:

Using Eq.(4) we can use the following to enlarge the pdf of HOLGE distribution: by

$$f_{OLGE}(x,\zeta,\beta,\lambda,\theta) = \mathrm{N}_{k,r,z,n,p,w,q} exp(-(q+1)\lambda x) - \mathrm{M}_{k,s,h,q,w,v}\, exp(-(w+1)\lambda x) \quad (8)$$

where

$$N_{k,r,z,n,p,q} = \sum_{k=r=z=n=p=q=0}^{\infty} \frac{\zeta\Gamma(\zeta+1+k)\beta^{-(k+1)}(-1)^{r+z+n+p+q} d_{k,r}}{k!\,\Gamma(\zeta+1)} \binom{r+2k+1}{z}\binom{z}{n}$$

$$M_{k,l,s,n+w+v} = \sum_{k=s=h=q=w=0}^{\infty} \frac{\zeta(-1)^{s+h+q+w} d_{k+1,i}\Gamma(\zeta+1+k)\beta^{-(k+1)}}{k!\,\Gamma(\zeta+1)} \theta\lambda \binom{2k+l+1}{s}\binom{s}{n}\binom{n\theta+\theta-1}{w}$$

3.2. Quantile function of HOLGE distribution

From Eq.(3) we can get the Quantile function of HOLGE distribution by form:

$$Q(u) = -\frac{1}{\lambda} log\left[1 - \left[\frac{\beta - \frac{\beta}{(1-u)^{\frac{1}{\xi}}}}{\beta - \frac{\beta}{(1-u)^{\frac{1}{\xi}}} + W_{-1}\beta - \frac{\beta}{(1-u)^{\frac{1}{\xi}}} e^{-\left(\beta - \frac{\beta}{(1-u)^{\frac{1}{\xi}}}\right)}}\right]^{\frac{1}{\theta}}\right] \quad (9)$$

Table1: describes the quantiles for specific HOLGE distribution parameter values.

u	$(\zeta, \beta, \lambda, \theta)$				
	(0.8,0.9,0.4,0.7)	**(0.6,0.4,0.4,0.5)**	**(0.5,0.6,0.5,0.6)**	**(0.7,0.5,0.8,0.7)**	**(0.7,0.9,0.7,0.3)**
0.1	**0.4677730**	**0.3881650**	**0.2642874**	**0.3357971**	**0.01896465**
0.2	**0.8447083**	**0.9181508**	**0.5303704**	**0.6385789**	**0.06534235**
0.3	**1.2657410**	**1.7236572**	**0.8630717**	**1.0229124**	**0.14547628**
0.4	**1.7842550**	**3.0818034**	**1.3194431**	**1.5773329**	**0.27640158**
0.5	**2.4799354**	**5.7051502**	**2.0090051**	**2.4899302**	**0.49315246**
0.6	**3.5147624**	**11.6594927**	**3.1950238**	**4.2634135**	**0.87357898**
0.7	**5.3020886**	**27.1964642**	**5.6775205**	**8.4090352**	**1.62246967**
0.8	**9.2516909**	**80.6063635**	**12.6140683**	**20.5541608**	**3.44759120**
0.9	**22.9410706**	**90.8283298**	**44.3942459**	**46.7873741**	**10.19877909**

The table shows the quantile values of the HOLGE distribution at different parameters. Quantiles represent the points that divide the data into specific parts, such as 10th, 20th, 50th percentile, etc. the values in the table illustrate how the HOLGE distribution can be used to determine values that fall at different percentiles of the distribution, which helps in risk analysis and decision making. The variation in values reflects the flexibility of the distribution in representing data with varying characteristics.

3.3. Moment

In Eq.(7). Let x be a random variable with pdf. The HOLGE distribution's n^{th} moment is thus provided by [15], [16]:

$$\mu_n = E(x^n) = \int_0^{\infty} x^n f(x) dx$$

Substituting Eq.(7) in above equation to get:

$$\mu_n = N_{k,r,z,n,p,w,q} I_1 dx - M_{k,s,h,q,w,v} I_2 dx \quad (10)$$

where

$$I_1 = \int_0^{\infty} x^n exp(-(q+1)\lambda x)\, dx$$

$$I_2 = \int_0^{\infty} x^n exp(-(w+1)\lambda x)$$

For I_1 , $let\ y = (q+1)\lambda x$

$$x = \frac{y}{(q+1)\lambda} \Rightarrow dx = \frac{1}{(q+1)\lambda} dy$$

Then

$$I_1 = \int_0^{\infty} \left(\frac{y}{(q+1)\lambda}\right)^n exp(-y) \frac{1}{(q+1)\lambda} dy$$

By simplify I_1 to get a final form as follows:

$$I_1 = \frac{1}{((q+1)\lambda)^{n+1}} \int_0^{\infty} y^n exp(-y)\, dy$$

$$I_1 = \frac{\Gamma(n+1)}{\left((q+1)\lambda\right)^{n+1}}$$

By same way for I_2 to get a final form:

$$I_2 = \frac{\Gamma(n+1)}{\left((w+1)\lambda\right)^{n+1}}$$

To get:

$$\mu_n = \Gamma(n+1)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{n+1}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{n+1}}\right] \tag{11}$$

The following formula yields the variance of the HOLGE distribution: $(\sigma^2 = \mu_2 - \mu_1^2)$.

Kurtosis (KU), and skewness (SK) are determined by [17], [18]:

$$SK_{OLGE} = \frac{\Gamma(4)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{4}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{4}}\right]}{\left(\Gamma(3)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{3}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{3}}\right]\right)^{\frac{3}{2}}} \tag{12}$$

$$KU_{OLGE} = \frac{\Gamma(5)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{5}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{5}}\right]}{\left(\Gamma(3)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{3}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{3}}\right]\right)^{2}} - 3 \tag{13}$$

Table.2 some values of moments for HOLGE

ξ	β	λ	θ	m_1	m_2	m_3	m_4	$Var(X)$	SK_{OLGE}	KU_{OLGE}
0.2	**0.4**	**0.3**	**0.1**	**0.051107**	**0.026077**	**0.017507**	**0.013178**	**0.023465**	**4.157576**	**19.37978**
			0.2	**0.086512**	**0.044386**	**0.029694**	**0.022284**	**0.036902**	**3.175371**	**11.31117**
		0.5	**0.3**	**0.116536**	**0.060538**	**0.040423**	**0.030258**	**0.046957**	**2.713841**	**8.256435**
			0.4	**0.138493**	**0.074959**	**0.050593**	**0.038018**	**0.055778**	**2.465204**	**6.766102**
	0.6	**0.7**	**0.5**	**0.190653**	**0.10028**	**0.066228**	**0.049051**	**0.063932**	**2.08555**	**4.877706**
			0.6	**0.211793**	**0.116526**	**0.078155**	**0.058252**	**0.07167**	**1.964817**	**4.290108**
		0.9	**0.7**	**0.229774**	**0.126288**	**0.084191**	**0.062415**	**0.073492**	**1.875966**	**3.913518**
			0.8	**0.244146**	**0.139414**	**0.094406**	**0.070491**	**0.079806**	**1.813595**	**3.62682**

3.4. Moment Generating Function

Eq.(11) moments generating function (mgf) for HOLGE distribution has a from [19], [20]:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\Gamma(n+1)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{n+1}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{n+1}}\right]\right] \tag{14}$$

3.5. Incomplete Moments

The n^{th}incomplete moments of HOLGE distribution has a form [21], [22]:

$$M_n(y) = \int_{-\infty}^{y} x^n f(x)dx$$

By same way in integral of moments we get a final form for n^{th}incomplete moments of HOLGE distribution:

$$M_n(y) = \frac{\Phi_{k,i,r,j,z,w,q}\Gamma\left((n+1),\lambda y(q+1)\right)}{\left((q+1)\lambda\right)^{n+1}} - \frac{\psi_{k,l,s,n,w,v}\Gamma\left((n+1),\lambda y(w+1)\right)}{\left((w+1)\lambda\right)^{n+1}} \quad (15)$$

3.6. Characteristic function

The Characteristic function of HOLGE distribution from Eq.(11) getting a form [23], [24]:

$$Q_x(t)_{OLGE} = \sum_{v=0}^{\infty}\frac{(it)^v}{v!}\left[\Gamma(n+1)\left[\frac{N_{k,r,z,n,p,w,q}}{\left((q+1)\lambda\right)^{n+1}} - \frac{M_{k,s,h,q,w,v}}{\left((w+1)\lambda\right)^{n+1}}\right]\right] \quad (16)$$

3.7. Rényi entropy

The R´enyi entropy for the HOLGE distribution from Eq.(8) [25], [26], then we get:

$$I_R(c) = \frac{1}{1-c}\log\int_0^{\infty} f(x_N)^c dx$$

$$I_R(c) = \frac{1}{1-c}\log\int_0^{\infty}\left(\mathrm{N}_{k,r,z,n,p,w,q}\exp(-(q+1)\lambda x) - \mathrm{M}_{k,s,h,q,w,v}\exp(-(w+1)\lambda x)\right)^c dx$$

Then we get a final form:

$$I_R(c) = \frac{1}{1-c}\log\left[\sum_{n=0}^{c}(-1)^n\binom{c}{n}\left(N_{k,r,z,n,p,w,q}\right)^{c-n}\left(M_{k,s,h,q,w,v}\right)^n\frac{1}{(qc-qn+c-nw)\lambda}\right] \quad (17)$$

3.8. Order statistics

For a random sample of size n down from a distribution function $F(x)$, the j^{th} order statistic's pdf and its corresponding pdf $f(x)$ are provided by [27], [28]:

$$f_{j:n}(x) = \sum_{r=0}^{n-j}k(-1)^r\binom{n-j}{r}[F(x)]^{j+r-1}f(x)$$

In this case , $f(x)$ is the HOLGE distribution's pdf and $F(x)$ is its CDF. Nonetheless, the $n = th$ order statistics' pdf for a random sample of size n selected from the HOLGE distribution is as follows:

$$f_{j:n}(x) = \sum_{r=0}^{n-j}k(-1)^r\binom{n-j}{r}\left[1-\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\zeta}\right]^{j+r-1} \times \left[\frac{\zeta\theta\lambda}{\beta}exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1}\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-(\zeta+1)}\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}} - log(1-(1-exp(-\lambda x))^{\theta})\right]\right] \quad (18)$$

So, the $f_{j:n}(x)$of minimum order statistics is obtained by substituting $\mathrm{j} = 1$ in Eq.(18) to have:

$$f_{1:n}(x) = \sum_{r=0}^{n-1}k(-1)^r\binom{n-1}{r}\left[1-\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\zeta}\right]^{r} \times \quad (19)$$

$$\left[\frac{\zeta\theta\lambda}{\beta}exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1}\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-(\zeta+1)}\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}}-log(1-(1-exp(-\lambda x))^{\theta})\right]\right]$$

The maximum order statistic for $f_{j:n}(x)$) given when $j=n$ in Eq.(18) as

$$f_{n:n}(x)=\sum_{r=0}^{n-n}k(-1)^r\binom{n-j}{r}\left[1-\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\zeta}\right]^{n+r-1}\times \quad (20)$$

$$\left[\frac{\zeta\theta\lambda}{\beta}exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1}\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-(\zeta+1)}\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}}-log(1-(1-exp(-\lambda x))^{\theta})\right]\right]$$

4. Estimation

4.1 Maximum Likelihood Estimation (MLE)

Estimation of the parameters of the HOLGE distributions can be obtained by the method of maximum likelihood. Let $X\sim HOLGE(\zeta,\beta,\lambda,\theta)$ and $\Delta=(\zeta,\beta,\lambda,\theta)^T$ be the parameter vector .The log-likelihood for Δ can be written as [29], [30]:

$$L(\Theta,x)=\prod_{i=1}^{n}f_{OLG}(x,\xi,\beta,\lambda,\theta)$$

$$L(\Theta,x)=\prod_{i=1}^{n}\frac{\xi\theta\lambda}{\beta}exp(-\lambda x)\,(1-exp(-\lambda x))^{\theta-1}\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-(\xi+1)}\left[\frac{(1-exp(-\lambda x))^{\theta}}{1-(1-exp(-\lambda x))^{\theta}}-log\left(1-(1-exp(-\lambda x))^{\theta}\right)\right]$$

we compute the log- likelihood:

$$l=l(\Delta)=nlog\xi+nlog\theta+nlog\lambda-nlog\beta-\lambda\sum_{i=1}^{n}x_i+(\theta-1)\sum_{i=1}^{n}log(1-exp(-\lambda x_i))\;-(\xi+1)\sum_{i=1}^{n}log\left(1-\frac{(1-exp(-\lambda x_i))^{\theta}.log\left(1-(1-exp(-\lambda x_i))^{\theta}\right)}{\beta}\right)+\sum_{i=1}^{n}log\left[\frac{(1-exp(-\lambda x_i))^{\theta}}{1-(1-exp(-\lambda x_i))^{\theta}}-log\left(1-(1-exp(-\lambda x_i))^{\theta}\right)\right] \quad (21)$$

4.2 Ordinary Least Squares Estimation (OLSE)

The formula can used to estimate a parameters using the Least square estimation (LSE) method [31], [32]:

$$\varphi(\theta_N)=\sum_{i=1}^{m}\left[\left[1-\left(1-\frac{(1-exp(-\lambda x))^{\theta}.log\left(1-(1-exp(-\lambda x))^{\theta}\right)}{\beta}\right)^{-\xi}\right]-\frac{1}{n+1}\right]^2 \quad (22)$$

4.3 Weighted Least Squares Estimators (WLSE)

The formula can used to estimate a parameters using the Weighted Least square estimation (WLSE) method [33]:

$$W(\theta_N) = \sum_{i=1}^{m} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\left[1 - \left(1 - \frac{(1-exp(-\lambda x))^{\theta}.log(1-(1-exp(-\lambda x))^{\theta})}{\beta} \right)^{-\xi} \right] - \frac{i}{n+1} \right]^2 \quad (23)$$

5. Simulation Study

A Monte Carlo simulation is carried out for the three methods described in the fourth section to demonstrate the accuracy of the estimation of the HOLGE distribution. In this simulation where the sizes of generated samples at n=50, 100, 150, and 200, to 1000 the values of the mean square error (MSE) and its root (RMSE) and the bias in the estimated parameters are calculated, and Table 3 displays the simulation values as follows:

Table 3 : Monte Carlo simulations conducted for the HOLGE

$r = 4$, $u = 6$, $b = 7$, $c = 5$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
50	**Mean**	$\hat{\xi}$	**1.358402**	**2.083932**	**1.464815**
		$\hat{\beta}$	**1.780466**	**2.282907**	**1.591727**
		$\hat{\lambda}$	**8.075031**	**5.855668**	**5.155367**
		$\hat{\theta}$	**10.985579**	**5.1154834**	**4.3191266**
	MSE	$\hat{\xi}$	**3.376931**	**2.455489**	**1.995376**
		$\hat{\beta}$	**4.481358**	**2.940426**	**2.355893**
		$\hat{\lambda}$	**23.749701**	**8.503083**	**10.349577**
		$\hat{\theta}$	**915.636321**	**11.4997911**	**7.3245288**
	RMSE	$\hat{\xi}$	**5.811137**	**1.567000**	**1.412578**
		$\hat{\beta}$	**6.694295**	**1.714767**	**1.534892**
		$\hat{\lambda}$	**4.873367**	**2.916005**	**3.217076**
		$\hat{\theta}$	**30.259483**	**3.3911342**	**2.7063867**
	Bias	$\hat{\xi}$	**1.358362**	**2.083892**	**1.464775**
		$\hat{\beta}$	**1.780406**	**2.282847**	**1.591667**
		$\hat{\lambda}$	**1.075031**	**1.144332**	**1.844633**
		$\hat{\theta}$	**5.985579**	**0.1154834**	**0.6808734**
100	**Mean**	$\hat{\xi}$	**1.716257**	**5.242738**	**4.9993144**
		$\hat{\beta}$	**1.949000**	**10.451472**	**10.486508**
		$\hat{\lambda}$	**8.009034**	**6.1436476**	**6.148459**
		$\hat{\theta}$	**8.495557**	**5.2588719**	**4.95088272**
	MSE	$\hat{\xi}$	**1.892481**	**329.217788**	**25.5158522**
		$\hat{\beta}$	**2.503260**	**465.784384**	**126.632229**
		$\hat{\lambda}$	**12.803613**	**7.8203061**	**8.220962**
		$\hat{\theta}$	**371.912069**	**14.3685222**	**8.94340209**
	RMSE	$\hat{\xi}$	**1.375675**	**18.144360**	**5.0513218**
		$\hat{\beta}$	**1.582169**	**21.582038**	**11.253099**
		$\hat{\lambda}$	**3.578214**	**2.7964810**	**2.867222**
		$\hat{\theta}$	**19.285022**	**3.7905834**	**2.99055214**
	Bias	$\hat{\xi}$	**1.715857**	**1.242738**	**0.9993144**
		$\hat{\beta}$	**1.948400**	**4.451472**	**4.486508**
		$\hat{\lambda}$	**1.009034**	**0.8563524**	**0.851541**
		$\hat{\theta}$	**3.495557**	**0.2588719**	**0.04911728**
150	**Mean**	$\hat{\xi}$	**1.078367**	**3.152043**	**8.10150**
		$\hat{\beta}$	**1.029940**	**4.861298**	**14.627486**

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		$\hat{\lambda}$	**7.6829482**	**6.5768103**	**6.6734885**
		$\hat{\theta}$	**6.160415**	**5.3395078**	**5.2408726**
	MSE	$\hat{\xi}$	**3.905954**	**1.030239**	**519.22862**
		$\hat{\beta}$	**3.583477**	**2.449648**	**1039.142202**
		$\hat{\lambda}$	**5.5225036**	**5.7023435**	**5.3470273**
		$\hat{\theta}$	**11.743136**	**9.8056725**	**7.1757685**
	RMSE	$\hat{\xi}$	**6.249763**	**3.209734**	**22.78659**
		$\hat{\beta}$	**5.986215**	**4.949392**	**32.235729**
		$\hat{\lambda}$	**2.3500008**	**2.3879580**	**2.3123640**
		$\hat{\theta}$	**3.426826**	**3.1314010**	**2.6787625**
	Bias	$\hat{\xi}$	**1.077967**	**3.148043**	**4.10150**
		$\hat{\beta}$	**1.029340**	**4.855298**	**8.627486**
		$\hat{\lambda}$	**0.6829482**	**0.4231897**	**0.3265115**
		$\hat{\theta}$	**1.160415**	**0.3395078**	**0.2408726**
200	**Mean**	$\hat{\xi}$	**5.917397**	**1.233490**	**4.1412378**
		$\hat{\beta}$	**5.260151**	**1.990197**	**6.9318768**
		$\hat{\lambda}$	**7.6668824**	**6.4373878**	**6.8083237**
		$\hat{\theta}$	**6.091310**	**4.826280**	**5.1407879**
	MSE	$\hat{\xi}$	**8.453297**	**1.535163**	**5.2926037**
		$\hat{\beta}$	**6.676290**	**3.996380**	**15.5014819**
		$\hat{\lambda}$	**4.2042904**	**3.1236686**	**2.2782903**
		$\hat{\theta}$	**6.474655**	**3.054968**	**2.6653278**
	RMSE	$\hat{\xi}$	**9.194181**	**1.239017**	**2.3005659**
		$\hat{\beta}$	**8.170857**	**1.999095**	**3.9371921**
		$\hat{\lambda}$	**2.0504366**	**1.7673903**	**1.5094006**
		$\hat{\theta}$	**2.544534**	**1.747847**	**1.6325832**
	Bias	$\hat{\xi}$	**5.913397**	**1.233090**	**0.1412378**
		$\hat{\beta}$	**5.254151**	**1.989597**	**0.9318768**
		$\hat{\lambda}$	**0.6668824**	**0.5626122**	**0.1916763**
		$\hat{\theta}$	**1.091310**	**0.173720**	**0.1407879**

The table compares MLE, LSE, and WLSE in terms of mean estimated values, MSE, RMSE, and bias. MLE showed the lowest MSE and least bias, indicating that it is the most accurate when estimating parameters. LSE and WLSE perform well with small samples but their performance deteriorates with large samples showing greater dispersion in estimates compared to MLE. Simulations showed that MLE is best when dealing with large sample sizes, while LSE and WLSE may be suitable for small sample but with lower accuracy.

6. Application

We present a real-world example to demonstrate the efficacy of the HOLGE distribution in fitting data. on two datasets in order to demonstrate the benefits of HOLGE and how well it fits the data. The Kolmogorov-Smirnov statistic (KS), the Anderson-Darling statistic (A), and the Cramer-vonMises statistic are the eight measures used in this comparison. statistic(W), the information criteria HQIC,BIC, AIC ,and CAIC, and the p-value corresponding to the KS -test . These are common metrics for goodness. of fit. comparison of the results obtained between the proposed distribution and six other distribution represented by:

- Beta exponential generalized exponential distribution (BeGE)
- Kumaraswamy Exponential Generalized exponential distribution (KuGE)

- Exponential generalized exponential distribution (EGGE)
- Log-gamma exponential generalized exponential distribution (LGoGE)
- Topp-Leone exponential extreme generalized exponential distribution (LEEGE)
- Generalized exponential distribution (GE)

6.1 The first data set I

The survival durations of 72 guinea pigs infected with virulent tubercle bacilli are included in the dataset utilized in this investigation. Days are used to quantify the survival periods. Bjerkedal made the first observations and reported this dataset. [11].

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58 , 5.55.

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	72	1.77	1.04	1.5	1.64	0.72	0.1	5.55	5.45	1.3	1.83	0.12

Table 4. outcomes of the distribution's criteria for Data1

Dist.	AIC	CAIC	BIC	HQIC
HOLGE	195.8465	196.4435	204.9532	199.4719
BeGE	196.7042	197.3012	205.8108	200.3296
KuGE	196.654	197.251	205.7606	200.2794
EGGE	196.6748	197.2718	205.7815	200.3002
LGamGE	196.85	197.4471	205.9567	200.4754
TEEGE	196.2712	196.8683	205.3779	199.8966
GE	248.3725	248.5464	252.9258	250.1852

Table 4 presents the results of various information criteria used to compare distribution , including AIC ,CAIC ,BIC and HQIC. The table shows that the HOLGE distribution has the lowest value across compared to other distribution , indicating its superior fit to the data . In contrast other distribution such as BeGE, KuGE ,EGGE, and LGamGE have higher values , suggesting lower efficiency in representing the data. Additionally the GE distribution recorded the highest value , clearly reflection its poor ability to fit the data compared to the other distribution. Based on these results , the HOLGE distribution can be considered the most efficient and accurate distribution for representing the data , according to the information criteria used in this analysis .

Table 5 . the statistical metrics validity for Data1

Dist.	W	A	K-S	p-value
HOLGE	0.07026161	0.4254626	0.08223026	0.7149827
BeGE	0.08640486	0.5304392	0.09246496	0.569404
KuGE	0.08504277	0.5229369	0.09249856	0.5689334
EGGE	0.08251194	0.5117872	0.09249545	0.568977
LGamGE	23.04606	143.1498	0.9995012	6.683328×10^{-63}

TEEGE	0.07706932	0.4692153	0.08710097	0.6455775
GE	0.08773156	0.6100735	0.1276226	0.1914481

HOLGE had the best fit to the data, as evidenced by its lowest W and A values and highest p-value in Table 5. When compared to HOLGE. Other distribution performed worse.

Table 6. MLE-Estimated parameter values for Data1

Dist.	$\hat{\xi}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
HOLGE	0.02760921	2.38004035	0.10931692	1.27299359
BeGE	1.6323110	1.7882730	0.7161627	1.9048520
KuGE	1.7502280	1.4976217	0.8350891	1.7956546
EGGE	1.6473992	1.726634	0.7333563	1.8653533
LGamGE	1.7852686	1.5304064	0.982951	1.8164578
TEEGE	8.7727799	1.2524582	٠.2707369	2.1877516
GE	---	---	2.009281	14052824

The MLE-based parameter estimates for various distributions are shown in Table 6. HOLGE is the best match for the data due to its high accuracy and stability. BeGE, KuGE, and LGamGE, on the other hand, exhibit greater variation while GE exhibits extreme instability, rendering it useless for data representation. Therefore, HOLGE is the preferred choice..

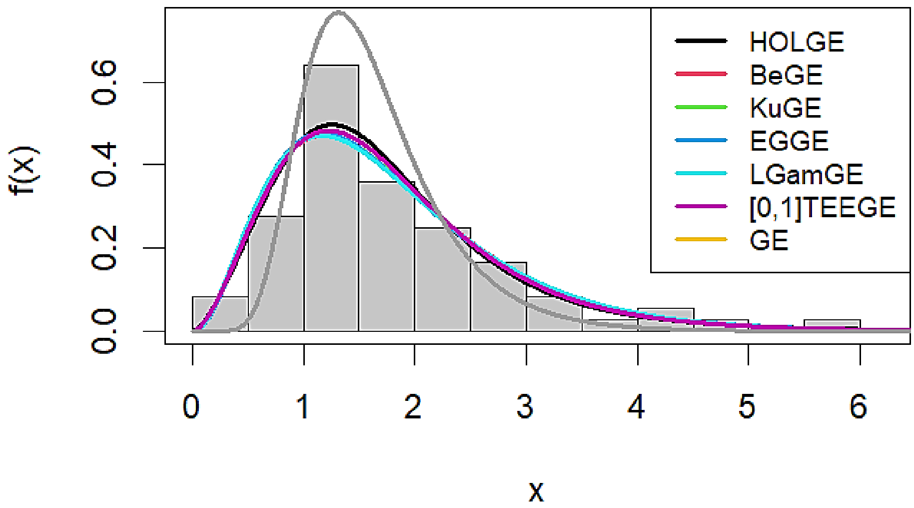

Figure 4: utilising a histogram data set1 to fit pdfs HOLGE

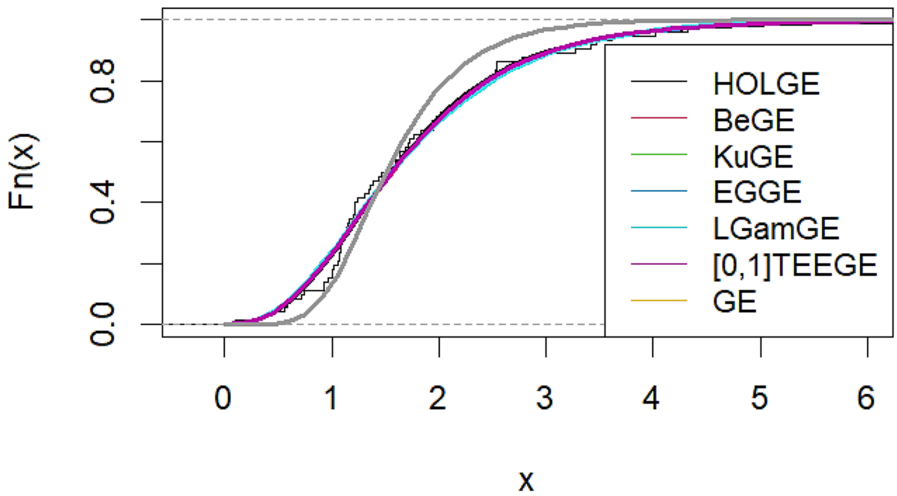

Figure 5: Dataset1 based Empirical fitted CDFs HOLGE

The HOLGE distribution is compared to other distributions using histogram data in Figure 4. HOLGE shows the best fit, while KuGE and GE exhibit weaker alignment. The empirical cumulative distribution function (Empirical CDF) is compared to various probability distributions in Figure 5. HOLGE closely follows the empirical curve, while LGamGE and GE show significant deviations. HOLGE provides the best fit to the data, demonstrating the highest alignment with both the histogram and empirical cumulative distribution function compared to other distributions.

6.2 The Second Dataset II

The data represent deaths of COVID-19 for past 153 and as shown below, at least two people have died every day over the past 153 days [34].

2, 2, 2, 2, 2, 2, 3, 2, 3, 3, 4, 2, 5, 5, 3, 2, 4, 4, 8, 4, 4, 3, 2, 3, 7, 6, 6, 11, 9, 3, 8, 7, 11, 8, 12, 12, 14, 7, 11, 12, 6, 14, 9, 9, 11, 6, 6, 5, 5, 14, 9, 15, 11, 8, 4, 7, 11, 10, 16, 2, 7, 17, 6, 8, 10, 4, 10, 7, 11, 11, 8, 7, 19, 9, 15, 12, 10, 14, 22, 9, 18, 12, 19, 21, 12, 12, 18, 8, 26, 21, 17, 13, 5, 15, 14, 11, 17, 16, 17, 23, 24, 20, 30, 18, 18, 17, 21, 18, 22, 26, 15, 13, 13, 6, 9, 17, 12, 17, 22, 7, 16, 16, 24, 28, 23, 23,19, 25, 29, 21, 9, 13, 16, 10, 17, 20, 23, 14, 12, 11, 15, 9, 18, 14, 13, 6, 16, 12, 11, 7, 3, 5, 5

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	153	11.61	6.76	١١	11.16	7.41	٢	٣٠	٢٨	0.5	-0.40	0.55

Table 7. outcomes of the distribution's criteria for Data2

Dist.	AIC	CAIC	BIC	HQIC
HOLGE	1004.357	1004.628	1016.479	1009.282
BeGE	1009.797	1010.067	1021.919	1014.721
KuGE	1028.09	1028.36	1040.212	1033.014
EGGE	1009.845	1010.115	1021.967	1014.769
LGamGE	1008.188	1008.458	1020.31	1013.112
TEEGE	1011.249	1011.519	1023.37	1016.173
GE	1037.012	1037.092	1043.072	1039.474

The evaluation criteria (AIC, CAIC, BIC, and HQIC) for various distributions are compared in Table 7, with lower values indicating a better fit to the data. The best distribution, HOLGE, has the lowest values across all evaluation criteria. BeGE, EGGE, and LGamGE are acceptable but less efficient than HOLGE. Compared to the other distributions, HOLGE is the most accurate and stable for representing the data. GE is the least compatible.

Table 8 . the statistical metrics validity for Data2

Dist.	W	A	K-S	p-value
HOLGE	0.09829107	0.7748557	0.06517917	0.5340536
BeGE	0.1966739	1.400477	0.09653148	0.1155036
KuGE	0.2696922	1.881191	0.1468048	0.002734918
EGGE	0.1983625	1.411178	0.09649334	0.1157639
LGamGE	52.58612	306.2228	0.9734491	$2.345016 \times 10^{-126}$
TEEGE	0.2206137	1.550103	0.10169	0.08448083

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GE	0.1597465	1.165366	0.1442787	0.003425011

The fact that HOLGE distribution in table 8 had the lowest values for W, A and K-S and greatest p-value indicates that it is a better match to the data than other distributions and that there is no evidence to refute the hypothesis of the distribution.

Table9 . MLE-Estimated parameter values for Data2

Dist.	$\hat{\xi}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
HOLGE	0.58447058	17.16651685	0.01353619	0.94144189
BeGE	1.8665075	0.9332271	0.1578078	1.4617922
KuGE	2.26047333	0.08807555	1.12554738	2.70831122
EGGE	0.8926672	1.4242118	0.1648765	1.9456299
LGamGE	1.7730088	0.9323336	0.1948429	1.5286722
TEEGE	0.4405843	1.6312504	0.194134	1.7287642
GE	0.09946959	1.28419899	---	---

Parameter estimation (MLE) for various distributions is compared in Table 9. HOLGE demonstrates stability and accuracy, while BeGE, LGamGE, and KuGE show variations indicating instability. GE has estimation issues. The best option for modeling is HOLGE.

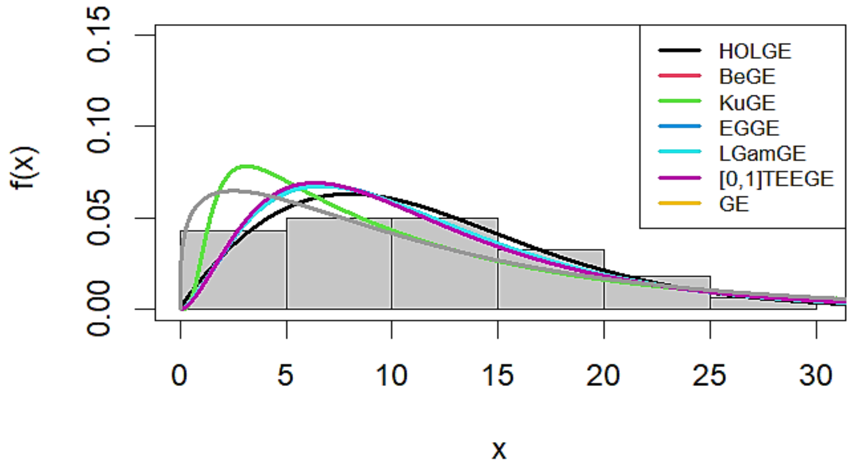

Figure ٦: utilising a histogram dataset2 to fit pdfs HOLGE

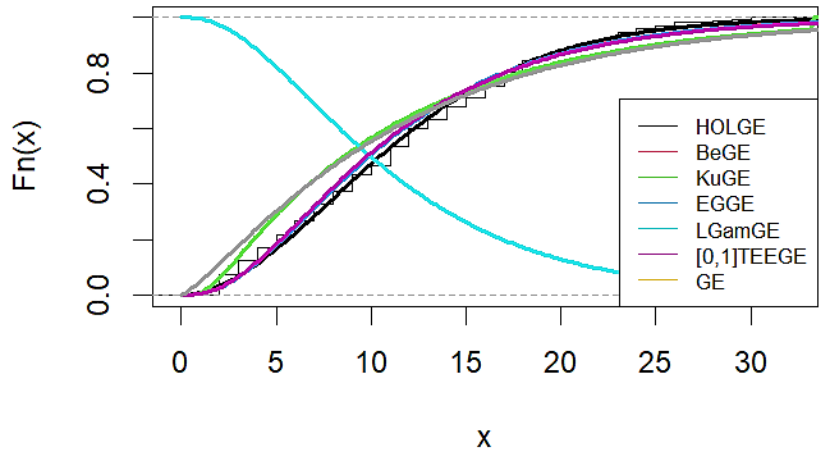

Figure ٧: Dataset2 based Empirical fitted CDFs HOLGE

Figure 6: HOLGE is the best fit to the histogram data, while LGamGE shows significant deviation.Figure 7: HOLGE closely follows the empirical CDF,

whereas LGamGE and GE exhibit weak alignment. HOLGE provides the best fit for the second dataset, making it the optimal choice compared to other distributions.

7. Conclusion

the HOLGE distribution was introduced and developed, which is characterized by its ability to model life expectancy data with increasing and decreasing (non-monotonic) patterns, making it more flexible than traditional distributions such as the exponential distribution and the Weibull distribution. the basic mathematical properties were derived, as PDF, CDF, and survival function were derived, in addition to other mathematical properties such as the quantile function, moments, Rényi entropy, and order statistics function, which allow the distribution to be used in applied data analysis.

Monte Carlo simulations showed that the MLE method is the most accurate in estimating the parameters of the HOLGE distribution, as it had the lowest MSE and lowest bias compared to LSE and WLSE methods, especially when dealing with large samples. The LSE and WLSE methods may be acceptable in small samples but are less accurate as the sample size increases. The simulation results showed that the HOLGE distribution provides high estimation accuracy, and its efficiency was verified using real data, making it a reliable distribution for use in applied and statistical studies. When HOLGE distribution applied to the two real data sets, the distribution showed the best fit with the data compared to 6 other distributions according to four information criteria and four statistical tests. The results confirmed that HOLGE distribution is applied to provide a more accurate representation of real data than traditional distributions such as Beta Exponential, Kumaraswamy Exponential, and Generalized Exponential. The results showed that the new distribution can be applied in wide fields, such as medical data analysis, engineering systems reliability, and economic modeling, as it can adapt to data with complex and nonlinear behavior. When comparing HOLGE with other distributions using statistical quality measures, the distribution achieved the lowest information criteria values (AIC, BIC, HQIC, CAIC), indicating that it provides a more accurate representation of the data than the other distributions. The Generalized Exponential (GE) distribution showed the worst performance, as its criteria values were higher, indicating that it is not suitable for representing the data under study.

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