IRAQI STATISTICIANS JOURNAL

https://isj.edu.iq/index.php/isj

ISSN: 3007-1658 (Online)

Hypothesis Testing for Non-Normal Multiple Compact Regression Model

Ons Edin Musa [1] and Noor Saleem Mohammed Ali[2]

[1]College of Physical Education and Sports Science, Mustansiriyah University, Baghdad, Iraq.
ons.mary@uomustansiriyah.edu.iq
[2]College of Administration and Economics, Baghdad University
noor.ali1101@coadec.uobaghdad.edu,iq.

ARTICLE INFO

Article history:

Received	07 March 2025,
Revised	07 March 2025,
Accepted	30 March 2025,
Available online	31 March 2025

Keywords:
Multiple compact regression model,
Bayesian Approach,
Bayes Factor,
Generalized multivariate transmuted Bessel distribution,
Kernel functions,
Jaundice percentage in the blood component data.

ABSTRACT

Generalized multivariate transmuted Bessel distribution belongs to the family of probability distributions with a symmetric heavy tail. It is considered a mixed continuous probability distribution. It is the result of mixing the multivariate Gaussian mixture distribution with the generalized inverse normal distribution. On this basis, the paper will study a multiple compact regression model when the random error follows a generalized multivariate transmuted Bessel distribution.

Assuming that the shape parameters are known, the parameters of the multiple compact regression model will be estimated using the maximum likelihood method and Bayesian approach depending on non-informative prior information. In addition, the Bayes factor was used as a criterion to test the hypotheses. A Gaussian distribution rule selects the bandwidth parameter and the kernel function based on the Gauss kernel function and quartic kernel function. It estimates the model parameters are under quadratic loss function. The researchers concluded that the posterior probability distribution of $\underline{\theta}$ is a multivariate t distribution. Applying the findings to real data related to the jaundice percentage in the blood component as a response variable, red blood cell volume and red blood cell sedimentation as parametric influencing variables, and white and red cells as nonparametric influencing variables, the researchers concluded that when the shape parameters increase, the values of the mean square error criteria of $\underline{\theta}$ And the variance parameter decreases.

1. Introduction:

The parameters of a multiple general linear regression model are often estimated when the error term is the multivariate normal distribution. Still, there are a number of cases where the observations of the error term are uncorrelated or may belong to probability distributions with symmetric heavy tails, i.e. heavier than the normal situation. In such a case, mixed distributions are more appropriate. One of these distributions is the generalized multivariate transmuted Bessel distribution, which is considered more general than models that include both distributions multivariate normal and multivariate T as special cases. (Barndorff-Nielsen, (1978)), This distribution has practical applications in a variety of areas, including the presentation of financial stock market data, quality control and filtering data, and random signal analysis.

Corresponding author E-mail address: ons.mary@uomustansiriyah.edu.iq
https://doi.org/10.62933/pjseay72

(Thabane and Haq, 2003) Studied the properties of the generalized multivariate transmuted Bessel distribution with its exceptional cases and confirmed that mixed distributions such as the mixed multivariate normal distribution and the multivariate T distribution are special cases of it and that it is a symmetric mixed distribution resulting from the multivariate normal distribution with the generalized inverse Gaussian distribution as well as its applications in Bayesian analyses of the general normal linear regression model assuming the generalized inverse Gaussian distribution as a prior distribution for the scale parameter. In the same year, **(Thabane & Drekic)** tested simple linear hypotheses about the population mean where the random error follows a multivariate generalized transmuted Bessel distribution in addition to a test for two groups with the same means and confirmed that this distribution is more general than the multivariate T distribution, the multivariate normal distribution. They also derived the probability distribution of (Hotelling-T2) and (Scheffe-T2) for the two tests mentioned above. Tested **(Choi et al., 2009)** a Bayesian statistical hypothesis for a compact multiple normal regression model and assumed that the parametric part of the model is a multidimensional linear function while the nonparametric part is an infinite series of trigonometric functions. They concluded that when the sample size increases, the Bayes factor under the null hypothesis of the linear function is consistent, i.e. it approaches infinity. In contrast, it approaches zero under the alternative hypothesis of the compact linear function.

The first section of the research included a general introduction, the second section described the multiple generalized transmuted Bessel compact regression model and the third section mentioned the smoothing parameter and kernel functions used in the research. In the fourth section, the model parameters were estimated using the maximum likelihood method and the Bayes method when the initial information was not available in the fifth section. The sixth section included the use of the Bayes factor as a criterion for testing hypotheses, and the seventh section applied real data related to the number of people infected with jaundice and its percentage in the blood component for the year 2021. The last section mentioned the most important theoretical and applied conclusions.

2. Multiple Generalized Transmuted Bessel Compact Regression Model:

Multiple Compact linear regression model is one of the multiple semi-parametric regression models, symbolized by the symbol (MCLM). It is a special case of the ensemble models. It is one of the models that depends on linear, parametric variables and other nonlinear, nonparametric variables. Usually, the variables of the compact model are continuous, and these linear and nonlinear variables affect the dependent variable.

(You et al.,2013) (Przystalski, 2014)

The following formula can represent multiple compact linear regression models:

$$Y_I = X_I'\beta + M(T_I) + E_I \quad I = 1,2,\dots,N \qquad \dots(1)$$

Where $X_I'\beta$ It is considered the parametric part of the model, $M(T_I)$ It is considered the nonparametric part and an unknown smoothing function. The model in Equation (1) can be rewritten in matrix form as follows:

$$\underline{Y} = X\underline{\beta} + \aleph\,\underline{\delta} + \underline{E} \qquad \dots(2)$$

Where $\underline{Y}$ Represents the vector of observations of the dependent variable of degree *(N×1)*, and *N* represents the number of observations. *X* represents the matrix of observations of the explanatory variables of degree *(N×P)*, and *P* represents the number of explanatory variables. $\underline{\beta}$ Represents the vector parameters of the parametric part of degree *(P×1)*. $\aleph$ is a full-rank matrix indicating the weights of the kernel function of degree *(N×K*), and *K* represents the number of nonparametric explanatory variables. $\underline{\delta}$ represents the vector parameters of the nonparametric part of degree *(K×1)* and $\underline{E}$ Represents the vector random errors of degree *(N×1)*. The model can be rewritten in Equation (2) in the following form: (Al Mouel et al., 2017) (Salih & Aboudi (2021))

$$\underline{Y} = W\underline{\theta} + \underline{E} \qquad ...(3)$$

$$Where: \quad W = [X \quad \aleph] \quad , \quad \underline{\theta} = [\underline{\beta} \quad \underline{\delta}]'$$

$$\aleph = \mathrm{diag}\left(\mathrm{ke_h}(\mathrm{t} - \mathrm{T_i})\right) \quad , \quad \mathrm{ke_h}(.) = \frac{1}{\mathrm{h}}\,\mathrm{ke}\left(\frac{\cdot}{\mathrm{h}}\right)$$

$\mathrm{ke_h}(.)$: Kernel functions are defined as positive, continuous, and symmetrical, and their integral is equal to the integer. (Langrene & Warin (2019))

h: The smoothing parameter is positive and more important than the kernel function and can be selected according to the researcher's experience.
(Langrene & Warin (2019))

Assuming that the ($\underline{E}$) follows a generalized multivariate transmuted Bessel distribution, the probability density function can be found using the concept of Bayes' theorem by combining the mixed multivariate normal distribution and the generalized inverse Gauss distribution as follows:

$$\underline{E}\big|Z \sim N\left(\underline{0}, Z\sigma^2 I_N\right) \quad , \quad Z \sim GIG(\lambda, \psi, \nu)$$

Probability density function of the random error vector conditional on the variable Z takes the following form:

$$f\left(\underline{E}\big|Z\right) = \frac{1}{(2\pi\, Z\, \sigma^2)^{\frac{N}{2}}}\; e^{-\frac{1}{2\sigma^2 Z}\,(\underline{E})'(\underline{E})}$$

Equation (4) represents the mixed multivariate normal distribution; the probability density function of the (Z) is as follows: (Salih & Aboudi (2022))
(Silva & Lopes (2006))

$$P(Z) = \frac{\left(\frac{\lambda}{\psi}\right)^{\frac{\upsilon}{2}}}{2\,K_\upsilon(\sqrt{\lambda\psi})}\, Z^{\upsilon-1}\, exp\left[-\frac{1}{2}\left(\left(\frac{\psi}{Z}\right) + \lambda Z\right)\right] \quad , Z > 0 \qquad ...(5)$$

Where:

$\lambda\,, \psi$: Scale parameters.

υ: Shape parameter.

$K_\upsilon(.)$: Third-order Bessel function of order υ is defined as: (Silva & Lopes (2006))

$$K_n(x) = 0.5\int_0^\infty t^{n-1}\exp\left(-0.5\, x(t + t^{-1})\right) dt \qquad x > 0 \qquad ...(6)$$

According to the concept of mixed distributions, the probability distribution of the unconditional random error vector is as follows:

$$f\left(\underline{E}\right) = \int_0^\infty f\left(\underline{E}\big|Z\right) P(Z) \quad dZ$$

$$f(E) = \frac{\left(\frac{\lambda}{\psi}\right)^{\frac{N}{4}} K_{\frac{2v-N}{2}}\left(\sqrt{\lambda\psi\left(1+\frac{\underline{E}'\,\underline{E}}{\psi\sigma^2}\right)}\right)}{(2\pi\,\sigma^2)^{\frac{N}{2}} \quad K_v\left(\sqrt{\lambda\psi}\right)} * \left(1+\frac{\underline{E}'\,\underline{E}}{\psi\sigma^2}\right)^{\frac{2v-N}{4}} \qquad \dots (7)$$

Whereas (λ, ψ, ν) represent shape parameters.

Equation (7) represents the generalized multivariate transmuted Bessel distribution of the random error vector, which is described as follows:

$\underline{E} \sim GMTB\left(\underline{0}, \sigma^2, I_N, \lambda, \psi, v\right)$

Since the vector observations of the dependent variable $\underline{Y}$ Equation (3) is a linear combination in terms of the vector random errors that follows a generalized multivariate transmuted Bessel distribution, the probability distribution of $\underline{Y}$ Follows a generalized multivariate transmuted Bessel distribution as follows:

$$f\left(\underline{Y}\middle|Z\right) = \frac{1}{(2\pi\ \sigma^2 Z)^{\frac{N}{2}}}\ e^{-\frac{1}{2\sigma^2 Z}\left(\underline{Y}-W\underline{\theta}\right)'\left(\underline{Y}-W\underline{\theta}\right)} \qquad \dots (8)$$

Based on the concept of mixed distributions, the unconditional probability distribution of $\underline{Y}$ is as follows: (Thabane & Drekic (2003)) (Saieed & Salih (2013)) (Saieed & Salih (2014))

$$f\left(\underline{Y}\right) = \frac{\left(\frac{\lambda}{\psi}\right)^{\frac{N}{4}} \quad K_{\frac{2v-N}{2}}\left(\sqrt{\lambda\psi\left(1+\frac{\left(\underline{Y}-W\underline{\theta}\right)'\left(\underline{Y}-W\underline{\theta}\right)}{\psi\sigma^2}\right)}\right)}{(2\pi\,\sigma^2\,)^{\frac{N}{2}} \quad K_v\left(\sqrt{\lambda\psi}\right)}$$

$$* \left(1+\frac{\left(\underline{Y}-W\underline{\theta}\right)'\left(\underline{Y}-W\underline{\theta}\right)}{\psi\sigma^2}\right)^{\frac{2v-N}{4}} \qquad \dots (9)$$

This distribution can be expressed descriptively as follows:

$\underline{Y} \sim GMTB\left(W\underline{\theta}\ ,\ \sigma^2, I_N\ , \lambda\ , \psi\ , v\ \right)$

3. Smoothing Parameter and Kernel Functions:

Kernel functions are used to estimate both regression functions spectral and probability density functions. That the method of selecting the smoothing parameter (h) is an essential part of estimating the nonparametric regression curve, and that choosing the smoothing parameter is more important than choosing the kernel function, and its properties are a non-random, symmetrical and positive boundary parameter. Table (1) shows the kernel functions used in the paper, as well as the selection of the smoothing parameter based on the thumb rule method. (Langrene & Warin (2019))

Table 1: Some Kernel Functions and Method of Selecting the Smoothing Parameter.

Kernel	K(x)		$h = \hat{\sigma}\, CV(k)\, n^{-\frac{1}{5}}$
Quartic	$(15/16)(1-x^2)^2$	$I(\lvert x\rvert \le 1)$	$\boldsymbol{CV(k) = 2.78}$
Gauss	$(2\pi)^{-0.5}\exp(-x^2/2)$	$I(\lvert x\rvert < \infty)$	$\boldsymbol{CV(k) = 1.06}$

4. Maximum Likelihood Estimators for Parameters of the Multiple Generalized Transmuted Bessel Compact Regression Model:

If there are n observations of the response variable $\underline{Y}$ Explanatory variables for the parametric and nonparametric parts, then the probability function of the vector observations of the response variable conditional on Z, has a multivariate normal distribution and is described in Equation (8). As for the probability function of the unconditional random vector Y, using the concept of mixed distributions, it is written as follows.(Saieed & Salih (2013))

$$L(\underline{\theta},\sigma^2) = \int_0^\infty f\left(\underline{Y}|\underline{\theta},\sigma^2,Z\right) P(Z)\, dZ \qquad \ldots(10)$$

Due to the difficulty of finding the maximum likelihood estimator of $\underline{\theta}$ From Equation (10), the concept of mixed distributions was used as follows:

$$L(\underline{\theta},\sigma^2|Z) = (2\pi\,\sigma^2 Z)^{-\frac{N}{2}} \exp\left(-\frac{\left(\underline{Y}-W\underline{\theta}\right)'\left(\underline{Y}-W\underline{\theta}\right)}{2\sigma^2 Z}\right) \qquad \ldots(11)$$

By taking the natural logarithm of both sides of Equation (11) and taking the first partial derivative relative to the vector$\underline{\theta}$, we obtain:

$$\frac{\partial\, ln\, L(\underline{\theta},\sigma^2|Z)}{\partial\underline{\theta}} = \frac{\left(W'\underline{Y} - W'W\underline{\theta}\right)}{\sigma^2 Z} \qquad \ldots(12)$$

$$\frac{\partial\, ln\, L(\underline{\theta},\sigma^2)}{\partial\underline{\theta}} = \frac{\left(W'\underline{Y} - W'W\underline{\theta}\right) K_{v-1}\left(\sqrt{\lambda\psi}\right)}{\sigma^2\, K_v\left(\sqrt{\lambda\psi}\right)\left(\frac{\lambda}{\psi}\right)^{\frac{-1}{2}}} \qquad \ldots(13)$$

When Equation (13) is equated to a zero vector, we get the following:

$$\hat{\underline{\theta}} = \left(W'W\right)^{-1} W'\underline{Y} \qquad \ldots(14)$$

To estimate the parameterσ^2, we take the natural logarithm of both sides of Equation (11) and take the first partial derivative relative to σ^2. We get:

$$\frac{\partial\, ln\, L(\underline{\theta},\sigma^2|Z)}{\partial\sigma^2} = \frac{-N}{2\sigma^2} + \frac{\left(\underline{Y}-W\underline{\theta}\right)'\left(\underline{Y}-W\underline{\theta}\right)}{2(\sigma^2)^2 Z} \qquad \ldots(15)$$

When equation (15) is equated to a zero vector, we get the following

$$\hat{\sigma}^2 = \frac{\left(\underline{Y}-W\hat{\underline{\theta}}\right)'\left(\underline{Y}-W\hat{\underline{\theta}}\right)\ \left(\frac{\lambda}{\psi}\right)^{\frac{1}{2}}\ K_{v-1}\left(\sqrt{\lambda\psi}\right)}{N\ \ K_v\left(\sqrt{\lambda\psi}\right)} \qquad \ldots(16)$$

5. Bayes Estimator for Parameters of the Multiple Generalized Transmuted Bessel Compact Regression Model:

Under this type of information, the initial probability is non-standard in most cases, but in 1961, Jeffreys found a standard formula for the initial probability with little information. Under the compact regression model previously defined in Equation (3), the initial distribution of $\underline{\theta}$ and σ^2 is proportional to the square root of the determinant of the information matrix. It is defined as follows: (Jefferys,(1961)).

$$I\left(\underline{\beta}\right) = -E_{\underline{Y}}\left(\frac{\partial^2 \ln f\left(\underline{Y}\,\middle|\underline{\beta}\right)}{\partial\underline{\beta}\,\partial\underline{\beta}'}\right) \quad \dots (17)$$

$$Where\ \underline{\beta} = \left(\underline{\theta}',\ \sigma^2\right)'$$

The information matrix is square, symmetrical and positive, and on the basis of this information, the parameters of the model defined in (3) will be estimated.

We take the natural logarithm of both sides of the probability density function. $\underline{Y}$ Conditional on the random variable Z, which was previously defined in Equation (11), and take the second partial derivative relative to $\underline{\theta}$ and σ^2, which is equal to a zero vector. Then, the joint initial distribution of $\underline{\theta}$ and σ^2 is as follows:

$$P\left(\underline{\theta},\sigma^2\right) \propto \left(\sigma^2\right)^{-\left(\frac{P\times K}{2}+1\right)}$$

By combining the joint initial distribution of $\underline{\theta}$ and σ^2 defined by equation (18) with the probability function of $\underline{Y}$ Conditional on the random variable Z, we obtain the kernel of the joint posterior distribution of the vector $\underline{\theta}$ and σ^2 conditional on the variable Z according to Bayes' theorem as follows:

$$P\left(\underline{\theta},\sigma^2\middle|\underline{Y}\,,Z\right) \propto P\left(\underline{\theta},\sigma^2\right) f\left(\underline{Y}\middle|\underline{\theta},\sigma^2,Z\right) \quad \dots (19$$

After substituting the components of Equation (19) and performing algebraic operations, we obtain the joint posterior distribution of $\underline{\theta},\sigma^2$ conditional on the random variable Z as follows:

$$P\left(\underline{\theta},\sigma^2\middle|\underline{Y}\,,Z\right) = \frac{\left(\frac{\left(\underline{Y}-W\hat{\underline{\theta}}\right)'\left(\underline{Y}-W\hat{\underline{\theta}}\right)}{2Z}\right)^{\frac{N}{2}} \left|W'W\right|^{\frac{1}{2}}}{\Gamma\left(\frac{N}{2}\right)\ \left(2\pi\sigma^2 Z\right)^{\frac{P\times K}{2}}} \left(\sigma^2\right)^{-\left(\frac{N}{2}+1\right)} e^{-\frac{1}{2\sigma^2 Z}\left(\left(\underline{\theta}-\hat{\underline{\theta}}\right)' W'W\left(\underline{\theta}-\hat{\underline{\theta}}\right)\right)}\ e^{-\frac{1}{2\sigma^2 Z}\left(\left(\underline{Y}-W\hat{\underline{\theta}}\right)'\left(\underline{Y}-W\hat{\underline{\theta}}\right)\right)} \quad \dots (20)$$

Equation (20) represents the product of the multivariate normal distribution with parameters $\left(\hat{\underline{\theta}}\,,\sigma^2 Z\left(W'W\right)^{-1}\right)$ and the multivariate inverse gamma distribution with parameters $\left(\frac{N}{2}\,,\frac{\left(\underline{Y}-W\hat{\underline{\theta}}\right)'\left(\underline{Y}-W\hat{\underline{\theta}}\right)}{2Z}\right)$.

The joint posterior distribution of the parameter vector $\underline{\theta}$ and σ^2 unconditional on Z is:

$$P(\underline{\theta},\sigma^2|\underline{Y}) = \int_0^\infty P(\underline{\theta},\sigma^2|\underline{Y},Z)\, P(Z)\, dZ \qquad \dots(21)$$

$$P(\underline{\theta},\sigma^2|\underline{Y}) = \frac{\left(\frac{(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})}{2}\right)^{\frac{N}{2}} \left|W'W\right|^{\frac{1}{2}} \left(\frac{\lambda}{\psi}\right)^{\frac{N+P\times K}{4}}}{\Gamma\left(\frac{N}{2}\right)\ (2\pi\sigma^2)^{\frac{P\times K}{2}}\ K_v\left(\sqrt{\lambda\psi}\right)}$$

$$\text{x}\left[1+\frac{(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})+(\underline{\theta}-\underline{\hat{\theta}})'W'W(\underline{\theta}-\underline{\hat{\theta}})}{\psi\sigma^2}\right]^{\frac{2v-N-(P\times K)}{4}}$$

$$\text{x}\, K_{\frac{2v-N-(P\times K)}{2}}\left(\sqrt{\lambda\psi\left(1+\frac{(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})+(\underline{\theta}-\underline{\hat{\theta}})'W'W(\underline{\theta}-\underline{\hat{\theta}})}{\psi\sigma^2}\right)}\right) \dots(22)$$

Marginal posterior distribution of the parameter vector $\underline{\theta}$ It is found as follows

$$P(\underline{\theta}|\underline{Y}) = \int_0^\infty \int_0^\infty P(\underline{\theta},\sigma^2|\underline{Y},Z)\ \ P(Z)\, d\sigma^2\ \ dZ$$

$$P(\underline{\theta}|\underline{Y}) = \int_0^\infty \frac{\Gamma\left(\frac{N+(P\times K)}{2}\right)\left|W'W\right|^{\frac{1}{2}}}{\Gamma\left(\frac{N}{2}\right)\ (2\pi\, Z)^{\frac{P\times K}{2}}} \frac{\left(\frac{(N-(P\times K))s^2}{2Z}\right)^{\frac{N}{2}}}{\left(\frac{(n-(P\times K))s^2+(\underline{\theta}-\underline{\hat{\theta}})'W'W(\underline{\theta}-\underline{\hat{\theta}})}{2Z}\right)^{\frac{N+(P\times K)}{2}}}$$

$$\times\ P(Z)\, dZ \qquad \dots(23)$$

$$P(\underline{\theta}|\underline{Y}) = \frac{\Gamma\left(\frac{N+(P\times K)}{2}\right)\left|W'W\right|^{\frac{1}{2}}}{\Gamma\left(\frac{N}{2}\right)\ \left((N-(P\times K))s^2\,\pi\right)^{\frac{P\times K}{2}}}\left[1+\frac{(\underline{\theta}-\underline{\hat{\theta}})'W'W(\underline{\theta}-\underline{\hat{\theta}})}{(N-(P\times K))s^2}\right]^{-\frac{N+(P\times K)}{2}} \qquad \dots(24)$$

Equation (24) represents a multivariate t-distribution with dimension $(P\times K)$and parameters.$\left(\underline{\hat{\theta}}\,, s^2\left(W'W\right)^{-1}, (N-(P\times K))\right)$

Where

$$(N-(P\times K))s^2 = (\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}}) \qquad \dots(25)$$

Bayes estimator of the parameter vector $\underline{\theta}$ Under the quadratic loss function is:

$$\underline{\hat{\theta}_B} = \underline{\hat{\theta}} \qquad \dots(26)$$

It is the same as the maximum likelihood estimator of the vector $\underline{\theta}$ Defined in Equation (14).

$$P\left(\sigma^2 \middle| \underline{Y}\right) = \int_0^{\infty}\int_{-\infty}^{\infty} P\left(\underline{\theta}, \sigma^2 \middle| \underline{Y}, Z\right)\; P(Z)\; d\underline{\theta}\; dZ$$

$$= \int_0^{\infty} \frac{\left(\frac{(N-(P\times K))s^2}{2Z}\right)^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} (\sigma^2)^{-\left(\frac{N}{2}+1\right)}\, exp\left(-\frac{(N-(P\times K))s^2}{2\,\sigma^2 Z}\right) P(Z)\, dZ \;\ldots (27)$$

$$P\left(\sigma^2 \middle| \underline{Y}\right) = \frac{\left(\frac{(N-(P\times K))s^2}{2}\right)^{\frac{N}{2}} \left(\frac{\lambda}{\psi}\right)^{\frac{N}{4}}}{\Gamma\left(\frac{N}{2}\right) K_v\left(\sqrt{\lambda\psi}\right) (\sigma^2)^{\frac{N}{2}+1}} \left[1+\frac{(N-(P\times K))s^2}{\psi\sigma^2}\right]^{\frac{2v-N}{4}}$$

$$\times\; K_{\frac{2v-N}{2}}\left(\sqrt{\lambda\psi\left(1+\frac{(N-(P\times K))s^2}{\psi\sigma^2}\right)}\right) \qquad \ldots (28)$$

The marginal posterior distribution of the parameter σ^2 is found as follows:

Since the marginal posterior distribution of the parameter σ^2 is not a common probability distribution; it is a proper distribution. Therefore, the Bayesian estimator of σ^2 will be found using the properties of mathematical expectation as follows:

$$\hat{\sigma}^2_B = E_Z E_{\sigma^2}\left(\sigma^2 \middle| \underline{Y}, Z\right) = \frac{\sqrt{\frac{\lambda}{\psi}}\; K_{v-1}\left(\sqrt{\lambda\psi}\right) (N-(P\times K))s^2}{(N-2)\; K_v\left(\sqrt{\lambda\psi}\right)} \qquad \ldots (29)$$

6. Bayesian Hypotheses Testing:

Bayes factor is one of the criteria used in testing Bayesian hypotheses. It is defined as the probability between two statistical hypotheses that results from dividing the posterior probabilities to the initial ones for the null hypothesis H_0 divided by the result of dividing the posterior probabilities to the initial ones for the alternative hypothesis H_1. This factor is expressed mathematically as follows: (Saieed & Salih (2014))

$$BF = \frac{P\left(\underline{Y}\middle|H_0\right)}{P\left(\underline{Y}\middle|H_1\right)} \qquad \ldots (30)$$

Jeffery's (1961) presented a table showing whether or not H_0 is preferred in different cases. (Jeffery's, (1961)).

Table 2: Shows BF values for H_0 preference.

Negative	BF<1
Barely worth mentioning	1≤ BF<3
Positive	3≤ BF<12
Strong	12≤ BF<150
Very strong	BF>150

Thus, the Bayes factor (BF) can be considered a statistical indicator of accepting or rejecting H_0 and represents the statistical laboratory in the Bayesian approach. Therefore, it is necessary to find the value of the Bayes factor for the hypothesis to be tested and compare that value with the values set by Jeffery's shown in the Table above, which determine whether the preference is to accept or reject H_0.

In order to test a simple hypothesis about the parameters vector $\underline{\theta}$ Of compact regression against a compound hypothesis, those two hypotheses are defined as follows:

$$H_0 : \underline{\theta} = \underline{\theta}_0, \quad \sigma^2 > 0$$
$$H_1 : \underline{\theta} \neq \underline{\theta}_0, \quad \sigma^2 > 0 \qquad \dots(31)$$

Under the above statistical hypothesis and using Equation (30), the Bayes factor becomes as follows:

$$BF = \frac{\int_0^\infty \int_0^\infty f\left(\underline{Y}\middle|\underline{\theta}_0\,, \sigma^2, Z\right) P(Z)\, P(\sigma^2)\, d\sigma^2\, dZ}{\int_0^\infty \int_0^\infty \int_{-\infty}^\infty f\left(\underline{Y}\middle|\underline{\theta}\,, \sigma^2, Z\right) P(Z)\, P(\sigma^2)\, P\left(\underline{\theta}\right) d\underline{\theta}\ d\sigma^2\, dZ} \qquad \dots(32)$$

The numerator in the Bayes factor formula above represents the probability function of the random vector. $\underline{Y}$ conditional on the variable Z defined in Equation (8) under the hypothesis H_0 multiplied by the initial distribution of σ^2 defined below:

$$P(\sigma^2) \propto \frac{1}{\sigma^2} \qquad \dots(33)$$

The generalized inverse Gauss distribution defined in Equation (5), after substituting the components of the numerator of Equation (32) and integrating relative to σ^2, we obtain:

$$P\left(\underline{Y}\middle|H_0\right) = \int_0^\infty \frac{1}{(2\pi\, Z)^{\frac{N}{2}}} \frac{\Gamma\left(\frac{N}{2}\right)}{\left[\frac{\left(\underline{Y} - W\underline{\theta}_0\right)'\left(\underline{Y} - W\underline{\theta}_0\right)}{2Z}\right]^{\frac{N}{2}}}\ P(Z)\quad dZ \qquad \dots(34)$$

$$P\left(\underline{Y}\middle|H_0\right) = \frac{\Gamma\left(\frac{N}{2}\right)\left[\left(\underline{Y} - W\underline{\theta}_0\right)'\left(\underline{Y} - W\underline{\theta}_0\right)\right]^{-\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}\left(\frac{1}{2}\right)^{\frac{N}{2}}} \qquad \dots(35)$$

In the same way, the denominator of the Bayes factor under hypothesis H_1 was found as follows:

$$P\left(\underline{Y}\middle|H_1\right) = \int_0^\infty \int_0^\infty \int_{\underline{\theta}} (2\pi)^{-\frac{N}{2}} (\sigma^2 Z)^{-\frac{N}{2}}\, e^{-\frac{\left(\underline{Y} - W\underline{\theta}\right)'\left(\underline{Y} - W\underline{\theta}\right)}{2\sigma^2 Z}}$$
$$\times\ \frac{1}{\sigma^2}\, P(Z)\ \ d\underline{\theta}\, d\sigma^2\, dZ \qquad \dots(36)$$

$$P\left(\underline{Y}\middle|H_1\right) = \int_0^\infty \int_0^\infty (2\pi)^{-\frac{N}{2}} (\sigma^2 Z)^{-\frac{N}{2}}\, e^{-\frac{\left(\underline{Y} - W\underline{\hat{\theta}}\right)'\left(\underline{Y} - W\underline{\hat{\theta}}\right)}{2\sigma^2\, Z}}\ \frac{(2\pi)^{\frac{P\times K}{2}} (\sigma^2 Z)^{\frac{P\times K}{2}}}{\left|W'W\right|^{\frac{1}{2}}}$$
$$\times\ \frac{1}{\sigma^2}\ P(Z)\quad d\sigma^2\, dZ \qquad \dots(37)$$

$$P(\underline{Y}|H_0)=\int_0^\infty \frac{\left|W'W\right|^{-\frac{1}{2}}}{(2\pi\, Z)^{\frac{N-(P\times K)}{2}}}\ \frac{\Gamma\left(\frac{N-(P\times K)}{2}\right)}{\left[\frac{(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})}{2\,Z}\right]^{\frac{N-(P\times K)}{2}}}\ P(Z)\ \ dZ \qquad \dots(38)$$

$$P(\underline{Y}|H_1)=\frac{(2\pi)^{\frac{-N+(P\times K)}{2}}\ \left|W'W\right|^{-\frac{1}{2}}\Gamma\left(\frac{N-(P\times K)}{2}\right)}{\left(\frac{(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})}{2}\right)^{\left(\frac{N-(P\times K)}{2}\right)}} \qquad \dots(39)$$

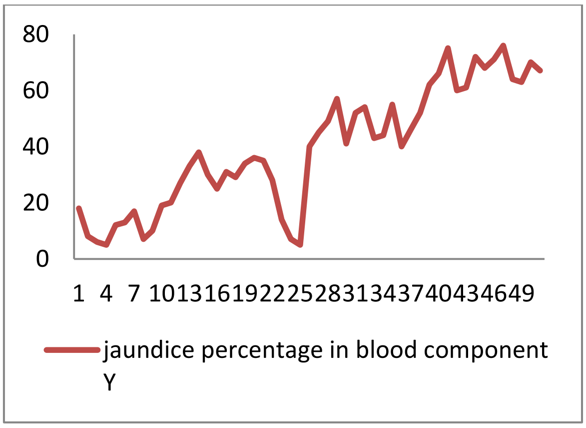
When substituting both equations (35) and (39) into Equation (30), we obtain the Bayes factor for testing hypothesis (31) as follows:

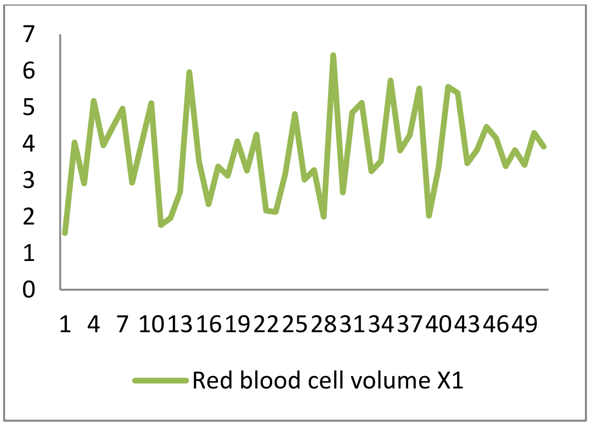
$$BF=\frac{\Gamma\left(\frac{N}{2}\right)\left[(\underline{Y}-W\underline{\theta}_0)'(\underline{Y}-W\underline{\theta}_0)\right]^{-\frac{N}{2}}\left[(\underline{Y}-W\underline{\hat{\theta}})'(\underline{Y}-W\underline{\hat{\theta}})\right]^{\frac{N-(P\times K)}{2}}}{\Gamma\left(\frac{N-(P\times K)}{2}\right)\ (\pi)^{\frac{P\times K}{2}}\ \ \left|W'W\right|^{-\frac{1}{2}}} \qquad \dots(40)$$

Since $\underline{\hat{\theta}}$ It was previously defined in Equation (14).

7. Application:

In this section, what was reached in the sixth section was applied to real data related to the number of people with jaundice and blood components in Kirkuk Governorate for the year 2021 and 51 people. The research data represents people with jaundice and its percentage in the blood component as a response variable, red blood cell volume and red blood cell sedimentation as explanatory variables, and white cells and red cells as nonparametric explanatory variables. The following Figure shows the behaviour of people with jaundice and its percentage in the blood component, in addition to the influential variables, both parametric and nonparametric. (Abdel Wahed, (2021))





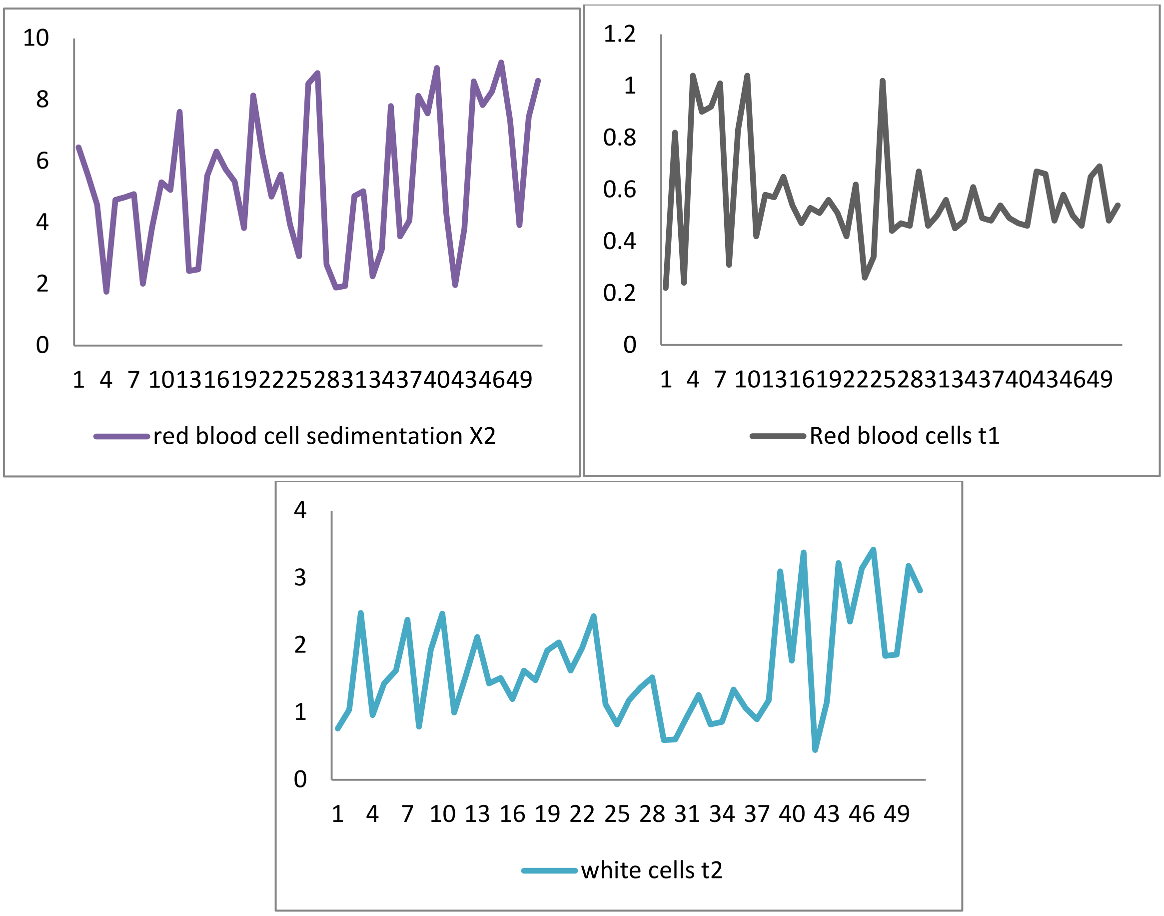


Figure (1): Behavior of patients with jaundice and its percentage in the blood component, in addition to the behaviour of the influential parametric and nonparametric variables.

Before the analysis process, it is necessary to know whether the data follows the generalized multivariate transmuted Bessel distribution. The researchers (Salih & Aboudi (2021)) suggested a method for conducting a goodness of fit test, which is based on mixed distributions represented by the chi-square test for several random samples. Different samples were chosen to conduct the matching in the research, as shown in the Table below:

Table 3. Chi-Square Test Values for Matching Data.

Samples (λ, ψ, v)	Chi2-calculate	Chi2-tab. $(= \mathbf{0.01})$
(1,3,2)	8.2142	9.2103
(5,9,4)	7.0239	9.2103
(10,12,8)	7.2248	9.2103
(15,14,10)	6.9987	9.2103

We note from Table (3) that the calculated values of the chi-square test for random samples are smaller than the tabular value under a significance level of (0.01), which indicates the acceptance of the null hypothesis, which states that the data follow the generalized transmuted Bessel distribution.

7.1 Bayes Estimators for Parameters of a Multiple Generalized Transmuted Bessel Compact Regression Model:

In this section, the Bayesian estimators of the parameter vector ($\underline{\theta}$) and the variance parameter (σ^2) of the multiple generalized transmuted Bessel compact regression model will be found when the previous information is not available, and the model estimators will be compared based on the mean square error criterion. Table (4) shows the values of the mean square error criterion for the parameter vector $\underline{\theta}$ Estimator under different kernel functions.

Table (4): Values of the mean square error criterion for the $\underline{\theta}$ estimator

Suggested models	$\underline{\theta}$ Bayes estimator			Rank
	Kernel Function (λ,ψ,v)	Gauss	Quartic	
First	$(1,3,2)$	11.2364	11.5478	4
Second	$(5,9,4)$	9.0325	9.5214	3
Third	$(10,12,8)$	8.3278	8.4214	2
Forth	$(15,14,10)$	7.9852	7.3021	1

We note from Table (4) that the best Bayesian estimator for $\underline{\theta}$ It was for the fourth proposed model and the Gauss kernel function due to it having the lowest value for the criterion. In addition, we note from the Table above that the more the values of the shape parameters *(λ, ψ, v)* increased, the values of the mean square error criterion decreased, and the estimated vector parameters ($\underline{\theta}$) is:

$$\underline{\hat{\theta}}_B = [7.8899 \quad 9.0155 \quad 2.2845 \quad -46.6980 \quad 7.1273]$$

The following Figure shows the behaviour of real and estimated data.

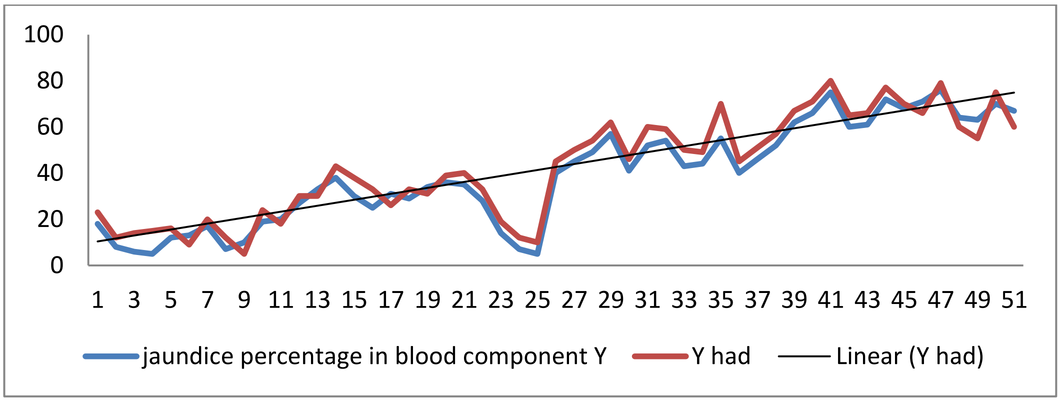


Figure (2): Behavior of real and estimated data

We notice from Figure (2) that the behaviour of the estimated data is the same as the behaviour of the real data, which indicates the suitability of the data to the estimated model.

Table (5) shows the values of the mean square error criterion for the variance parameter. σ^2.

Table (5): Values of the mean square error criterion for the $\underline{\theta}$ estimator

Suggested models	σ^2 Kernel Function (λ, ψ, v)	Bayes estimator Gauss	Quartic	Rank
First	$(1, 3, 2)$	9.2141	9.4521	4
Second	$(5, 9, 4)$	9.1100	9.3659	3
Third	$(10, 12, 8)$	8.0324	8.1697	2
Forth	$(15, 14, 10)$	7.9852	7.9992	1

We note from Table (5) that the best Bayesian estimator for σ^2 was for the fourth proposed model and the Gauss kernel function due to it having the lowest value for the criterion. In addition, we note from the Table above that the more the values of the shape parameters *(λ, ψ, v)* increased, the values of the mean square error criterion decreased, and the estimated variance parameter σ^2 was$(\hat{\sigma}^2 = 4.321)$.

7.2 Bayesian Hypotheses Testing of a Multiple Generalized Transmuted Bessel Compact Regression Model:

To test the statistical hypothesis defined in Equation (31), the Bayes factor criterion, which is described in Equation (40), will be calculated, and then the computed criterion values will be compared with the values shown in Table (2). Table (6) shows the values of the Bayes factor criterion based on previously unavailable information, and the assumed parameter vector is:

$$\underline{\theta}_0 = [6.9902 \quad 8.9854 \quad 2.0855 \quad -44.3210 \quad 8.0394]$$

Table (6): Bayes Factor (BF) criterion values for the proposed models

Proposed Models	BF. Gauss kernel	Quartic Kernel	Decision on accepting the null hypothesis
First	55.3251	51.2145	Strong
Second	88.0241	90.0212	Strong
Third	95.3368	100.3252	Strong
Forth	135.214	141.6987	Strong

We note from the result of Table (6) that the null hypothesis H_0 will be accepted, which means that the sample was drawn from a generalized transmuted Bessel population.

8. Conclusion:

In this section, the most important findings of the researchers will be mentioned in the theoretical and applied study:

1. Maximum likelihood estimator of $\underline{\theta}$ For a multiple generalized transmuted compact, the Bessel regression model is the same as if the model error were normally distributed.
2. Posterior probability distribution of $\underline{\theta}$ is a multivariate t-distribution defined in Equation (24).
3. Bayesian estimator of $\underline{\theta}$ When prior information is very little, it is the same as the maximum likelihood estimator.
4. Posterior probability distribution of σ^2 is an unknown probability distribution but one of the proper probability distributions.
5. Through the applied study, we notice that the more the values of the shape parameters increase, the mean square error criterion decreases of the parameter vector estimator $\underline{\theta}$, variance parameter and for Gauss kernel function.
6. Based on the Bayes factor criterion, it was found that the proposed models were drawn from a generalized transmuted Bessel population.

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