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Slight 2-Prime Submodules and Slight 2-Prime Modules

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ABSTRACT

In this study the notion of slight 2-Prime submodules of unitary modules are introduced. Many properties related with this type of submodules are obtained. Beside this, we define a new class of modules namely slight 2-Prime modules, which is generalization of 2-Prime modules. Basic results of these modules are given.

1. Introduction

In current research, we use the supposition R is a commutative ring with identity and that each R- module $\mathcal{M}$ is a unitary right R- module . The notions $\mathbb{N} < \mathcal{M}(\mathbb{N} \leq \mathcal{M})$ stands for ℕ is a submodule of $\mathcal{M}$. (ℕ is a porper submodule of $\mathcal{M}$ (Clearly every ideal I of R is a submodule of the R- module R). $I < R$ is named prime ideal if $a.b \in I$, then $a \in I \; or \; b \in I$[7] . For $\mathbb{N} < \mathcal{M}$, ℕ is called a prime submodule if whenever $a \in R, x \in \mathcal{M}$, with $xa \in \mathbb{N}$, then $x \in \mathbb{N}$ or $a \in (\mathbb{N}\dot{:}_R M)$, where $(\mathbb{N}\dot{:}_R M) = \{r \in R: \mathcal{M}r \subseteq \mathbb{N}\}$. [11]. Recently W. Messiridi & a.t.l in [10] introduced the concept 2-Prime ideal as a generalization of prime ideal, where if $I < R$, I is said to be 2-prime if $a.b \in I \; (a, b \in R), then \; a^2 \in h \; or \; b^2 \in h.$

Fatima and Alaa in [6] generalized this notion for submodules, as follows : $\mathbb{N} < \mathcal{M}$ is named a 2-Prime submodule if $ma \in \mathbb{N}$ with $(a \in R, m \in \mathcal{M})$, implies $m \in \mathbb{N}$ or $a^2 \in (\mathbb{N}\dot{:}_R\mathcal{M})$.

By [5, Proposition 2.3] every 2-Prime submodule ℕ of $\mathcal{M}$ implies $(\mathbb{N}\dot{:}_R\mathcal{M})$ is 2-Prime ideal, but the converse may be not valid, see [5, Remark 2.4]. This motivate us to present a new concept namely slight2-Prime submodule, where $\mathbb{N} < \mathcal{M}$ is called a slight 2-Prime submodule(shortly S-2PS), if $\mathbb{N}\dot{:}_R\mathcal{M})$ is a 2-Prime ideal of R.

In S.2 of this paper many properties of this class of submodules are given. In S.3, we define a type of modules namely slight 2-Prime module as generalization of 2-prime module which is given in [6 الاء], where a module M is a 2- prime module if the zero submodule is a 2-Prime ideal. We say that M is a slight 2-Prime module(abbreviated S-2PM) if $< 0 > \leq M$ is a S-2PS. Many fundamental results related with this concept are introduced, some of them are analogues to that of 2-prime modules. Note that we shall use these abbreviations (2-PI, 2-PS, S-2-PS, 2-PM, S-2-PM) for 2-Prime ideal, 2-Prime submdule, slight 2-Prime submodule, 2- Prime module, slight 2-Prime module.

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1. Slight 2-Prime Submodules.

1.1 Definition

Let $\mathbb{N} < \mathcal{M}$. $\mathbb{N}$ is called slight 2-Prime submodule (bravely S-2-PS) if $(\mathbb{N}\dot{:}_R\mathcal{M})$ is a (2-PI).

1.2_ Remark

By [5, Proposition 2.3] every(2-PS) $\mathbb{N}\dot{:}_R$module $\mathcal{M}$ implies $(\mathbb{N}\dot{:}_R\mathcal{M})$ is a (2PI), hence $\mathbb{N}$ is (S-2-PS) of $\mathcal{M}$.

The next example explains that the converse may be not true :

The submodule $\mathbb{N} =< \frac{1}{p} + \mathbb{Z} >$ of the $\mathbb{Z}$ -module $\mathbb{Z}_{p^\infty}$ is (S2PSM), since $\left(N\dot{:}_{\mathbb{Z}}\mathbb{Z}_{p^\infty}\right) = (0)$ which is a prime ideal (So that is (2PI). On the hand $\mathbb{N}$ is not (2-PSM) because $P\left(\frac{1}{p^2} + \mathbb{Z}\right) \in \mathbb{N}$ and $\left(\frac{1}{p^2} + \mathbb{Z}\right) \notin N$ and $P^2 \in \left(\mathbb{N}\dot{:}_{\mathbb{Z}}\mathbb{Z}_{p^\infty}\right) = 0.$

According to common knowledge, an R-module M is considered multiplication when all submodule $\mathbb{N}$ of $\mathcal{M}$ $(\mathbb{N} \leq \mathcal{M})$, has an ideal of R where $\mathbb{N} = \mathcal{M}I.$
Likewise, if for every $\mathbb{N} \leq \mathcal{M}\ and \mathbb{N} = \mathcal{M}(\mathbb{N}\dot{:}_R\mathcal{M}),$ M is a multiplication module [2].

1.3_ Proposition

Let $\mathbb{N} < \mathcal{M}$, in which $\mathcal{M}$is a multiplication R-module . Then $\mathbb{N}$ is (2-PSM) if and only if $\mathbb{N}$ is a (S-2-PS).

Proof: Clearly by [5, Corollary 3.10]

1.4 Corollary

For $\mathbb{N} < \mathcal{M}$, where $\mathcal{M}$ is a multiplication R-module over a Boolean ring R $(ie\ n^2 = n, \forall n \in \mathbb{N}$. The following concepts are equivalent:

a) S-2-PS.
b) 2-PS.
c) Prime submodule.
d) Primary submodule.

The aforementioned statements are all equivalent.

Proof : (a)↔(b): follows the Proposition 1.3

(b)↔(c): Let $ax \in \mathcal{M}$ then either $x \in \mathbb{N}$ or $a^2 \in (\mathbb{N}\dot{:}_R\mathcal{M}) \rightarrow$.

As R is Boolean ring, either $x \in \mathbb{N}$ or $a \in (\mathbb{N}\dot{:}_R\mathcal{M})$. Thus $\mathbb{N}$ is considered a prime submodule.

(c)↔(d) and (c)↔(b) (are clear),

(d)↔(c): Let $xa \in \mathbb{N}$, where $a \in R, x \in \mathcal{M}$. Either $x \in \mathbb{N}\ or\ n^k \in (\mathbb{N}:\mathcal{M})$ for some $k \in Z_+$, Since $\mathbb{N}$ is defined as a primary submodule,. It is following that either $x \in \mathbb{N}\ or\ n \in (\mathbb{N}:\mathcal{M})$, when R is Boolean ring. Thus $\mathbb{N}$ is a (PSM).

1.5 Proposition

Assume $\mathcal{M}_1$and $\mathcal{M}_2$ be R-modules, $f: \mathcal{M}_1 \to \mathcal{M}_2$ be a homomorphism, and $\mathbb{N} \leq \mathcal{M}_1$ with $Kerf \subseteq \mathbb{N}$. If $\mathbb{N}$ is S-2-PS of $\mathcal{M}_1$, then $f(\mathbb{N})$ is an S-2-PSof $f(\mathcal{M}_1)$

Proof:

Since $N \subsetneqq \mathcal{M}$ and $Kerf \subseteq \mathbb{N}$, then $f(\mathbb{N}) \neq f(\mathcal{M}_1)$. As $\mathbb{N}$ is S-2-PS of M_1, $(\mathbb{N} \dot{:}_R \mathcal{M}_1)$ is a(2PI) of R. obviously $(\mathbb{N} \dot{:}_R \mathcal{M}_1) = (f(\mathbb{N}) \dot{:}_R f(\mathcal{M}_1))$, hence $(f(\mathbb{N}) \dot{:}_R f(\mathcal{M}_1))$ is a (2PI) of R.

Thus $f(\mathbb{N})$ is S-2-PS of $f(\mathcal{M}_1)$.

1.6 Lemma

Let $f: \mathcal{M}_1 \to \mathcal{M}_2$ be an R-homomorphism, $W \leq f(\mathcal{M}_1)$, then $(W \dot{:}_R f(\mathcal{M}_1) = (f^{-1}(W) \dot{:}_R \mathcal{M}_1)$

Proof:

Let $a \in (WR: f(\mathcal{M}_1)$. Then $af(\mathcal{M}_1) \subseteq W$, so that $f^{-1}f(a\mathcal{M}_1) \subseteq f^{-1}(W)$, but $M_1 a \subseteq f^{-1}f(\mathcal{M}_1 a)$, hence $aM_1 \subseteq f^{-1}(W)$. Thus $a \in (f^{-1}(W)R: \mathcal{M}_1)$ and so $(WR: f(\mathcal{M}_1) \subseteq (f^{-1}(W)R: \mathcal{M}_1)$. The reverse inclusion is similarly.

Note that if f is an epimorphism, then $(W \dot{:}_R \mathcal{M}_2) = (f^{-1}(W)R: \mathcal{M}_1)$

Proposition 1.7

If $f: \mathcal{M}_1 \to \mathcal{M}_2$ be an R- epimorphism, W is (S-2-PS) in $\mathcal{M}_2$. Then $f^{-1}(W)$ is (S-2-PS) in $\mathcal{M}_1$.

Proof : Since W is a (S-2-PS) of M_2 then $W \neq \mathcal{M}_2$ and

$(W:_R \mathcal{M}_2)$ is a (2-PI) of R. Hence $f^{-1}(W) \neq \mathcal{M}_1$ as f is an epimorphism. Beside this by lemma 2.6, $(W:_R \mathcal{M}_2) = (f^{-1}W :_R \mathcal{M}_1)$. Thus $(f^{-1}(W) :_R \mathcal{M}_1)$ is a (2PI) of R and $f^{-1}(W)$ is a (2-PS).

Remark 1.8

The condition f is an epimorphism is a necessary condition in proposition 2.7, for example:

Consider $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$ as $\mathbb{Z}$ -modules, $f: \mathbb{Z}_8 \to \mathbb{Z}_{16}$ defined by $f(\underline{0}) = f(\overline{4}) = \overline{0}, f(\overline{1}) = f(\overline{5}) = (\overline{4}), f(\overline{2}) = f(\overline{6}) = \overline{8}, f(\overline{3}) = f(\overline{7}) = \overline{12}$. Let $W = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}\} < \mathbb{Z}_{16}$. Then $(W:_Z : \mathbb{Z}_{16}) = 4\mathbb{Z}$ is a (2PSM) of $\mathbb{Z}_{16}$. But $f^{-1}(W) = \mathbb{Z}_8$ which is not (S-2-PS) of $\mathbb{Z}_8$.

Proposition 1.9

Let $\mathbb{N}$ be a (S-2-PS) of an R-module $\mathcal{M}$. Then $(\mathbb{N} \dot{:}_M I)$ is an S-2-PS for each idempotent ideal I of R $(i.e\ I = I^2)$.

Proof: To provide $(\mathbb{N} \dot{:}_M I)$ is a (S2PSM) of $\mathcal{M}$, $((\mathbb{N} \dot{:}_{\mathcal{M}} I) \dot{:}_R \mathcal{M})$ is a (2-PI) of R, that must prove . Let $a.b \in ((\mathbb{N} \dot{:}_M I) \dot{:}_R \mathcal{M})$, where $a, b \in R$, that is $\mathcal{M}ab \subseteq (N \dot{:}_R I)$. Hence $\mathcal{M}abI \subseteq \mathbb{N}$ and so $abI \subseteq (\mathbb{N} \dot{:}_R \mathcal{M})$ which is (2-PI) of R, hence either $a^2(\mathbb{N} \dot{:}_R \mathcal{M})\ or\ b^2 I^2 \subseteq (\mathbb{N}: \mathcal{M})$.

If $a^2(\mathbb{N}: \mathcal{M})$, then $\mathcal{M}a^2 \subseteq \mathbb{N}$ and $\mathcal{M}a^2 I \subseteq \mathbb{N}I \subseteq \mathbb{N}$ that $\mathcal{M}a^2 \subseteq (\mathbb{N} \dot{:}_M I)$. Therfore $a^2 \in ((\mathbb{N} \dot{:}_M I) \dot{:}_R \mathcal{M})$.

If $b^2I^2 \subseteq (\mathbb{N}\dot{:}_{\mathcal{M}} I)$, Then $b^2 I \subseteq (\mathbb{N}\dot{:}_R \mathcal{M})$, (since $I^2 = I$). This implies $\mathcal{M}b^2 I \subseteq \mathbb{N}$ and so $\mathcal{M}b^2 \subseteq (\mathbb{N}\dot{:}_M I)$. Thus $b^2 \in ((\mathbb{N}\dot{:}_{\mathcal{M}} I)\dot{:}_R \mathcal{M})$

Remark 1.10

The condition I is an idempotent ideal can't be dropped from Proposition 2.9, as an illustrative example :

Consider $\mathbb{Z}_{16}$ as $\mathbb{Z}$ -module, $\mathbb{N} = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}\}$. N is an (S-2-PS) of M. Let $I = 8\mathbb{Z}$. Clearly I is not idempotent and $\left(\mathbb{N}\dot{:}_{Z_{16}} I\right) = \mathbb{Z}_{16}$ which is not (S-2-PS) of $\mathcal{M} = \mathbb{Z}_{16}$.

Recalling a module $\mathcal{M}$ over R is named cancellation if for each I , J $\leq R, \mathcal{M}I = MJ$, [9]

Proposition 1.11

Let$\mathcal{M}$ be a cancellation R-module, let $I < R$. Then $\mathcal{M}I$ is a (S-2-PS) of $\mathcal{M}$ only when I is a (2-PI) of R.

<u>Proof</u> : Clearly $(\mathcal{M}I\dot{:}_R \mathcal{M}) = I$. Hence $\mathcal{M}$I is a (S-2PS) only when I is a (2-PI) of R.

Corollary 1.12

Let M be a multiplication R-module that has been faithfully and finitely generated. The below statement are equivalent:

1- $\mathcal{M}$I is a (S-2-PS)

2- I is a (2-PI) of R

3- $\mathcal{M}$ I is a (2-PS)

Proof:

(1) ↔ (2) S, Since $\mathcal{M}$ is a multiplication R- module, $R - module, M\ is\ a\ cancellation\ module$ by [2, Theorem 3.1]. hold by Proposition 2.11 (2)↔(3) It pursue by Proposition 2.3.

If every submodule of module M is a finite intersection of its primary submodules, then M is named Laskerian module.[4]

Proposition 1.13

Assume M be a Laskerian R-module with finite generators and ϖ is a (S-2-PS) of $\mathcal{M}$. The rad ϖ is a (S-2-PS) of $\mathcal{M}$, where rad ϖ is all the prime submodules intersections containing ϖ.

<u>Proof</u>: Since $\mathcal{M}$ is finitely generated Laskerian R-module, then $\sqrt{(\varpi\dot{:}_R \mathcal{M})} = (rad\varpi\dot{:}_R \mathcal{M})$ by [8, Theorem 5]. But ϖ is a S-2-PS of $\mathcal{M}$, that is$(\varpi\dot{:}_R \mathcal{M})$ is a (2-PI) of R, which implies that $\sqrt{(\varpi\dot{:}_R \mathcal{M})}$ is a prime ideal [10] and so (2PI).

Thus $(rad\varpi\dot{:}_R \mathcal{M}$ is a (2PI) of R and so rad ϖ is a (S-2-PS) of $\mathcal{M}$.

A module $\mathcal{M}$ of a ring R is known as Comultiplication if for every $\varpi < \mathcal{M}$, there exists $I \leq R$ so that $\varpi = annI_{\mathcal{M}}$. Equivalently for each $\varpi \leq \mathcal{M}$, $\varpi = (0\vdots_M ann\, \varpi)$ [1].

Proposition 1.14

Assume $\mathbb{N} < \mathcal{M}$,where $\mathcal{M}$ a Comultiplication R-module .Then $\mathbb{N}$ will be (2-PI) of R is idempotent ($ie\ I^2 = I$) and (0) is a (S-2-PS). Then every $\mathbb{N} < \mathcal{M}$ is a (S-2-PS) of M and ($\mathbb{N}$: $\mathcal{M}$) is a prime ideal of R.

Proof: As $\mathbb{N} < \mathcal{M}$ and $\mathcal{M}$ is a comultiplication R -module

Then $\mathbb{N} = (0\vdots_M I)$ for some ideal I of R_2, I≠R.

As (0) is a (S-2-PS) of $\mathcal{M}$ and I is an idempotent ideal, so that $\mathbb{N} = (0\vdots_M I)$ is a (S-2-PS) of $\mathcal{M}$ by Proposition 2.9.

Hence $(\mathbb{N}\vdots_R \mathcal{M})$ is a(2-PI). As all ideal of R is idempotent, so that$(\mathbb{N}\vdots_R \mathcal{M})$ is a Prime ideal.

Proposition 1.15

Assume $\mathcal{M}$ is considered as an R-module, let$\{K_i\}_{eI}$ is considered a chain of (2-PSM) of $\mathcal{M}$. Then $\bigcap_{i\in I} K_i$ is a (S2PSM) of $\mathcal{M}$.

Proof:

It is clear that$(\bigcap_{i\in I} K_i :_R \mathcal{M}) \neq R, (\bigcap_{i\in I} K_i :_R \mathcal{M}) = \bigcap_{i\in I}(K_i :_R \mathcal{M})$

Let $a, b \in R$such that $a.b \in \bigcap_{i\in I} \left(K_i\vdots_R \mathcal{M}\right)$. Assume that there exist $m, n \in I$ such that $a^2 \notin \left(K_m\vdots_R \mathcal{M}\right)$and $b^2 \notin \left(K_n\vdots_R M\right)$. Since $\{K_i\}_{i\in I}$ is a chain, so it could be assumed $K_m \subseteq K_n$. Then $\left(K_m\vdots_R \mathcal{M}\right) \subseteq \left(K_n\vdots_R \mathcal{M}\right)$. On the other hand $a.b \in \left(K_m\vdots_R \mathcal{M}\right)$, So either $a^2 \in \left(K_m\vdots_R \mathcal{M}\right)\ or b^2 \in \left(K_m\vdots_R \mathcal{M}\right)$. However each case implies contradiction. Thus either $a^2 \in \bigcap_{i\in I} (K_i\vdots_R \mathcal{M})$ or $b^2 \in \bigcap_{i\in I} (K_i\vdots_R \mathcal{M})$.

Now we define the following:

Definition 1.16

Assume $\mathbb{N}$ is a (S-2-PS) of the module $\mathcal{M}$, let $\mathbb{C} \leq \mathcal{M}$. $\mathbb{N}$ is called a minimal (S-2-PS) of K if there is no (S-2-PS) $\mho$ of $\mathcal{M}$ such that $\mathbb{C} \subset \mho \subset \mathbb{N}$. $\mathbb{N}$ is said to a minimal (S-2-PS) of $\mathcal{M}$ if $\mathbb{N}$ is a minimal (S-2-PS) of (0).

Example 1.17

Assume $\mathcal{M}$ be the $\mathbb{Z}$ -module $\mathbb{Z}$, $\mathbb{N} = 4\mathbb{Z}$, $K = 8\mathbb{Z}$. Then $\mathbb{N}$ is a minimal (S-2-PS) of K. But $\mathbb{N}$ is not a minimal (S-2-PS) of $\mathbb{Z}$, Since $(0) \subseteq 8\mathbb{Z} \subseteq 4\mathbb{Z}$ and $8\mathbb{Z}$ is an (S-2-PS)of $\mathbb{Z}$

Proposition 1.18

Every S-2-PS of a module $\mathcal{M}$ contains a minimal (S-2-PS) of $\mathcal{M}$.

Proof: Assume $\mathbb{N}$ be a (S-2-PS) of a (S-2-PS) of $\mathcal{M}$ and $F = \{$K:K is a (S-2-PS) of $\mathcal{M}$ and $K \subseteq \mathbb{N}\}$. $F \neq \phi$ since $N \in F$. Let $\{K_i\}_{i\in I}$ be a chain in F, then by Proposition 2.14, $\cap_{i\in I} K_i$ is a (S2PSM) and $\cap_{i\in I} K_i \subseteq \mathbb{N}$. Suppose there exists a ((S-2-PS) T of M such that $(0) \subseteq T \cap_{i\in I} K_i \subseteq \mathbb{N}$. Then $T \in F$ and $T = \cap_{i\in I} K_i$. Thus $\cap_{i\in I} K_i$ is a minimal (S-2-PS) and $\cap_{i\in I} K_i \subseteq \mathbb{N}$

Proposition 1.19

Assume $\mathbb{N}$ be a (S-2-PS) of a module $\mathcal{M}$, S is a multiplicative closed subset of R. Then $S^{-1}N$ is a (S-2-PS) of S^{-1}R module $S^{-1}\mathcal{M}$. Provided $\mathcal{M}$ is finitely generated.

Proof:

Since $\mathbb{N}$ is a (S-2-PS of $\mathcal{M}$, then $(\mathbb{N} :_R \mathcal{M})$ is (2-PI) of R. Hence by [10, Proposition 1.3.2.], $S^{-1}(\mathbb{N} :_R \mathcal{M})$ is a (2-PI) of R. But $S^{-1}(\mathbb{N} :_R \mathcal{M}) = (S^{-1}\mathbb{N} :_{S^{-1}R} S^{-1}\mathcal{M})$ because $\mathcal{M}$ is finitely generated, see [7, Proposition 3:14, P43]. Thus $(S^{-1}\mathbb{N} :_{S^{-1}R} S^{-1}\mathcal{M})$ is a (2-PI) of R and so is $S^{-1}\mathbb{N}$ is a (S-2PS) of S^{-1}R-module $S^{-1}\mathcal{M}$.

Now, we focus on the direct sum of two (S-2-PS) for the corresponding modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Theorem 1.20

Let $\mathbb{N}_1 < \mathcal{M}_1$ and $\mathbb{N}_2 < \mathcal{M}_2$ respectively. If $\mathbb{N}_1 \oplus \mathbb{N}_2$ is an (S-2-PS) of $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then $\mathbb{N}_1$ and $\mathbb{N}_2$ are (S-2-PS) of $\mathcal{M}_1$and $\mathcal{M}_2$ (respectively). The converse hold if R is a chained ring.

Proof:

If $\rho_1: \mathcal{M}_1 \oplus \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ be the natural projection. Then $p_1(\mathbb{N}_1 \oplus \mathbb{N}_2) = \mathbb{N}_1$ and $Kerf\ \rho_1 = (0) \oplus \mathbb{N}_2 \subseteq \mathbb{N}_1 \oplus \mathbb{N}_2$. Hence Proposition 2.5, $\mathbb{N}_1$ is an (S-2-PS) of $\mathcal{M}_1$. Similary $\mathbb{N}_2$ is a (S-2-PS) of M_2.

Conversely: Since $(\mathbb{N}_1 \oplus N_2 :_R \mathcal{M}_1 \oplus \mathcal{M}_2) = (\mathbb{N}_1 :_R \mathcal{M}_1) \cap (\mathbb{N}_2 :_R \mathcal{M}_2)$ and R is a chained ring, then either $(\mathbb{N}_1 :_R \mathcal{M}_1) \subseteq (\mathbb{N}_2 :_R \mathcal{M}_2)$ or $((\mathbb{N}_2 :_R \mathcal{M}_2) \subseteq (\mathbb{N}_1 :_R \mathcal{M}_1)$. Thus either $\left(\mathbb{N}_1 \oplus \mathbb{N}_2 \vdots_R \mathcal{M}_1 \oplus \mathcal{M}_2\right) = \left(\mathbb{N}_1 \vdots_R \mathcal{M}_1\right)$ which is a (2PI) of R (since a S-2-Pr-), or $\left(\mathbb{N}_1 \oplus \mathbb{N}_2 \vdots_R \mathcal{M}_1 \oplus M_2\right) = \left(\mathbb{N}_2 \vdots_R \mathcal{M}_2\right)$ which is a (2PI) of R (since N_2 is an (S-2-PS) of $\mathcal{M}_2$).

Remark 1.21

The condition R is a chained ring can't be dropped from Proposition 1.19, for example:

Consider $\mathbb{Z}_{16} \oplus \mathbb{Z}$ as $\mathbb{Z}$ -module let $\mathbb{N}_1 = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}\}$, $\mathbb{N}_2 = 3\mathbb{Z}$. Each $\mathbb{N}_1$ and $\mathbb{N}_2$ are (S-2-PS) submodules of $\mathbb{Z}_{16}$ and $\mathbb{Z}$ (respectively). But $\left(\mathbb{N}_1 \oplus \mathbb{N}_2 \vdots_Z \mathbb{Z}_{16} \oplus \mathbb{Z}\right) = 12\mathbb{Z}$ which is not a 2-Prime ideal of $\mathbb{Z}_{16} \oplus \mathbb{Z}$. Thus $\mathbb{N}_1 \oplus \mathbb{N}_2$ is not a (S-2-PS) of $\mathbb{Z}_{16} \oplus \mathbb{Z}$.

Proposition 1.22

Let $\mathcal{M}_1 and\ \mathcal{M}_2$ be modules , let $\mathbb{N}_1 < \mathcal{M}_1$ and $\mathbb{N}_1 < \mathcal{M}_2$ respectively. Then:

1) If $\mathbb{N}_1$ is (S-2-PS)of $\mathcal{M}_2$, it leads to $\mathbb{N}_1 \oplus \mathcal{M}_2$ is (S2PSM) of $\mathcal{M}_1 \oplus M_2$
2) If $\mathbb{N}_2$ is a (S-2-PS) of M_2, it leads to $M_1 \oplus \mathbb{N}_2$ is a (S2PSM) of $\mathcal{M}_1 \oplus \mathcal{M}_2$

Proof: It is easy.

Recall that if $\mathcal{M}_i$ is an R_i-module, $i = 1,2$, and R be the ring $R_1 \times R_2$, so that $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is module where $(m_1, m_2)(r_1, r_2) = (m_1 r_1, m_2 r_2), \forall (m_1, m_2) \in \mathcal{M}, (r_1, r_2) \in R$.

Theorem 1.23

Let $R = R_1 x R_2$, $\mathcal{M} = \mathcal{M}_1 x \mathcal{M}_2$ be R-module. When ℕ and W are proper $\mathcal{M}_1$ *and* $\mathcal{M}_2$ submodules, respectively. So that

1) N is a (S-2-PS) of $\mathcal{M}_1$, if and only if $\mathbb{N} \times \mathcal{M}_2$ is a (S2PSM) of $\mathcal{M}$.
2) W is a (S-2-PS) of M_2, if and only if $\mathcal{M}_1 \times W$ is a (S2PSM) of $\mathcal{M}$.

Proof: First

$\left(N \times \mathcal{M}_2 :_{R_1 \times R_2} \mathcal{M}_1 \times \mathcal{M}_2\right) = \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times \left(\mathcal{M}_2 :_{R_1} \mathcal{M}_2\right) = \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$

Let $(a, b), (c, d) \in R_1 \times R$ such that $(a, b).(c, d) \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$.

Hence $(ac, bd) \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$ and so that $ac \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$ and $bd \in R_2$. As N is (S2PSM) of $\mathcal{M}_1$, $\left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$ is a (2PI) of R_1 It follows that either $a^2 \in \left(\mathbb{N} :_{R_1} M_1\right)$ *or* $c^2 \in \left(\mathbb{N} :_{R_1} M_1\right)$. Then clearly $(a^2, b^2) \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$ *or* $(c^2, d^2) \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$, *ie* $(a, b)^2 \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$. Thus $\left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2$ is a (2PI) of M_1.

To prove ℕ is (S-2-PS) of $\mathcal{M}_1$.

Let $a, b \in R_1$ such that $a.b \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$, hence for each $c, d \in R_2$, $(ab, cd) \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) \times R_2 = \left(\mathbb{N} \times \mathcal{M}_2 :_{R_1 \times R_2} \mathcal{M}_1 \times \mathcal{M}_2\right)$. That is $(a, c).(b, d) \in \left(\mathbb{N} \times \mathcal{M}_2 :_{R_1 \times R_2} \mathcal{M}_1 \times M_2\right)$ which is a (2PI) of $R_1 \times R_2$. Hence either $(a, c)^2 \in \left(\mathcal{M}_1 \times \mathcal{M}_2 :_{R_1 \times R_2} \mathcal{M}_1 \times \mathcal{M}_2\right)$ or $(b, d)^2 \in \left(\mathbb{N} \times \mathcal{M}_2 :_{R_1 \times R_2} \mathcal{M}_1 \times \mathcal{M}_2\right)$. It follows that either $\left(a^2 \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) and\ c^2 \in R_2\right)$ or $\left(b^2 \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right) and\ d^2 \in R_2\right)$. Thus either $a^2 \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$ or $b^2 \in \left(\mathbb{N} :_{R_1} \mathcal{M}_1\right)$. So that ℕ is a (2PSM) of $\mathcal{M}_1$.

Theorem 1.24

Let $R = R_1 \times R_2$, $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ be R-module. When $\mathbb{N} = \mathbb{N}_1 \times \mathbb{N}_2$ is an (S-2-PS) of $\mathcal{M}$ Then either

1) ℕ is a (S-2-PS) of $\mathcal{M}_1$ and $\mathbb{N}_2 = \mathcal{M}_2$, or
2) $\mathbb{N}_1 = \mathcal{M}_1$ and $\mathbb{N}_2$ is a (S-2-PS) of $\mathcal{M}_2$, or
3) $\mathbb{N}_1$ and $\mathbb{N}_2$ are (S-2-PS) of both $\mathcal{M}_1$ and $\mathcal{M}_2$.

Proof: Since $N_1 \times N_2 <$of $\mathcal{M}_1 \times \mathcal{M}_2$. Then there are 3 states:

1) $\mathbb{N}_1 \leq \mathcal{M}_1$ so that $\mathbb{N}_2 = \mathcal{M}_2$
2) $\mathbb{N}_1 = \mathcal{M}_1$ so that $\mathbb{N}_2 \lneqq \mathcal{M}_2$
3) $\mathbb{N}_1 \lneqq \mathcal{M}_1$ so that $\mathbb{N}_2 \lneqq \mathcal{M}_2$

State (1): implies that $\mathbb{N} = \mathbb{N}_1 \times \mathcal{M}_2$, and byTheorem 1.22 $\mathbb{N}_1$ is S2Pr- submodule of $\mathcal{M}_1$

of M_1

State (2): implies that $\mathbb{N} = \mathcal{M}_1 \times \mathbb{N}_2$ and Theorem 2.22 yields that , $\mathbb{N}_2$ is an (S-2-PS) of $\mathcal{M}_2$

State (3): $\mathbb{N}_1 \subsetneqq \mathcal{M}_1$ and $N_1 \subsetneqq \mathcal{M}_1$ imply $(\mathbb{N}_1 :_{R_1} \mathcal{M}_1) \subsetneqq R_1$ and $(\mathbb{N}_2 :_{R_2} \mathcal{M}_2) \subsetneqq R_2$. To prove $\mathbb{N}_1$ and $\mathbb{N}_2$ are (S-2-PS) of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

we must show that : $(\mathbb{N}_1 :_{R_1} \mathcal{M}_1)$ and $(\mathbb{N}_2 :_{R_2} \mathcal{M}_2)$ are (2-PI) of R_1 and R_2, respectively.

Let $a, b \in R$ and $c, d \in R_2$ such that $a.b \in (\mathbb{N}_1 :_{R_1} \mathcal{M}_1)$ and $.d \in (N_2 :_{R_2} \mathcal{M}_2)$; that is $(ab, cd) \in \left((\mathbb{N}_1 :_{R_1} \mathcal{M}_1\right) \times \left(\mathbb{N}_2 :_{R_2} \mathcal{M}_2\right) = (\ (\mathbb{N}_1 \times \mathbb{N}_2 :_{R_1 \times R_2}\ \ \mathcal{M}_1 \times \mathcal{M}_2)$.

Hence $(a, c).(b, d) \in ((\mathbb{N}_1 \times \mathbb{N}_2 :_{R_1 \times R_2}\ \ \mathcal{M}_1 \times \mathcal{M}_2)$.

But $\mathbb{N} = (\mathbb{N}_1 \times \mathbb{N}_2$ is a (S-2-PS) of $\mathcal{M}_1 \times \mathcal{M}_2)$.

, so that $((\mathbb{N}_1 \times \mathbb{N}_2 \vdots_{R_1 \times R_2} \mathcal{M}_1 \times \mathcal{M}_2)$.

is a 2Pr- ideal of $R = R_1 \times R$. It follows that either

$(a, c)^2 \in ((\mathbb{N}_1 \times N_2 :_{R_1 \times R_2}\ \ \mathcal{M}_1 \times \mathcal{M}_2)$.

or $(b, d)^2 \in \left((\mathbb{N}_1 \times \mathbb{N}_2 :_{R_1 \times R_2}\ \ \mathcal{M}_1 \times\ \mathcal{M}_2).\right)$ This implies either

I) $(a^2, c^2) \in \left((\mathbb{N}_1 :_{R_1} \mathcal{M}_1\right) \times \left(N_2 :_{R_2} \mathcal{M}_2\right)$, and so $a^2 \in \left((\mathbb{N}_1 :_{R_1} \mathcal{M}_1\right)$ and $c^2 \in \left(N_2 :_{R_2} M_2\right)$. or
II) $(b^2, d^2) \in \left((\mathbb{N}_1 \times \mathbb{N}_2 :_{R_1 \times R_2}\ \ \mathcal{M}_1 \times \mathcal{M}_2\right)$.This implies $b^2 \in \left((\mathbb{N}_1 :_{R_1} \mathcal{M}_1\right)$ and $d^2 \in \left(N_2 :_{R_2} \mathcal{M}_2\right)$

Thus each case ((I) or (II)) implies $\left((\mathbb{N}_1 :_{R_1} \mathcal{M}_1\right)$ and $\left(N_2 :_{R_2} \mathcal{M}_2\right)$ are (2-PI) of R_1 and R_2 (respectively).

Therefore $\mathbb{N}_1$ and $\mathbb{N}_2$ are (S-2-PS) of $\mathcal{M}_1$ and $\mathcal{M}_2$ (respectively).

2. Slight2-Prime Modules

The current section introduced a new class of modules namely slight2-Prime module (S-2-PM) as a generalization of 2-Prime modules(2PM). The requisite properties of this type of modules are presented.

Definition 2.1

A module $\mathcal{M}$ over R is named is slight2-Prime module(briefly S-2-PM) if (0) is (S-2-PS). In other words $\mathcal{M}$ is (S-2-PM) if $(0 \vdots_R M) = ann_R \mathcal{M}$ is a (29PI) of R.

Example and Remarks 2.2

1) All 2-prime module(2-PM) is (S-2-PM) , however, it is not conversely.

Proof:

Let $\mathcal{M}$ be a (2-PM). Then (0) is a (2-PS), hence $(0\dot{:}_R\mathcal{M}) = ann\mathcal{M}$ is a (2-PI) of R. Thus $\mathcal{M}$ is (S-2PM).

Assume the $\mathbb{Z}$-module $\mathbb{Q}$. It is (S-2PM) since $ann_Z\mathbb{Q} = (0)$ which a prime ideal of $\mathbb{Z}$, hence (2-PI). But $\mathbb{Q}$ is not (2-PM) .

2) The $\mathbb{Z}$ -module $\mathbb{Z}_4$ is (S-2-PM) since It is (2-PM), the $\mathbb{Z}$ -module $\mathbb{Z}_6$ is not (S-2-PM) since $ann_Z\mathbb{Z}_{16} = 6\mathbb{Z}$ which is not (2-PI) in $\mathbb{Z}$, where$2.3 \in 6\mathbb{Z}$, but $2^2 \in 6\mathbb{Z}$ and $3^2 \notin 6\mathbb{Z}$.
3) Not every nonzero submodule of (S-2-PM), for instance : Assume $\mathcal{M}$ be the $\mathbb{Z}$ -module $\mathbb{Z}\oplus\mathbb{Z}_6$ as $\mathbb{Z}$ -module

subsequently $ann_Z\mathcal{M} = (0)$ which is (2-PI).

If $N = (0)\oplus\mathbb{Z}_6$, then $ann_{\mathbb{Z}}\mathbb{N} = 6\mathbb{Z}$ which is not 2-Pl of $\mathbb{Z}$, hence $\mathbb{N}$ is not (S-2-PM).

Notice that $\mathbb{N}$is a direct summand of $\mathcal{M}$, hence a direct summand of (S-2-PM) is not necessarily (S-2-PM).

4- The homomorphic image of (S-2-PM) is not necessarily (S-2-PM), for example: Let $\rho: \mathbb{Z}\oplus\mathbb{Z}_6 \longrightarrow \mathbb{Z}_6$ be the natural epimorphism, $\rho(\mathbb{Z}\oplus\mathbb{Z}_6) = \mathbb{Z}_6$ which is (S-2-PM). (see part (3)).

The concepts (2-PM) and (S-2-PM) are equivalent under the category of multiplication modules.

Proposition 2.3

For a multiplication module $\mathcal{M}$. thereafter, $\mathcal{M}$ is (S-2-PM) if and only if $\mathcal{M}$ is (2-PM).

Proof : (⇐) It is easy

(⇒)If $\mathcal{M}$ is (S-2-PM), then (0) is (S-2-PS), ie $((0)\dot{:}_R\mathcal{M})$ is (2-PI), hence (0) is (2-PS) of $\mathcal{M}$ by Proposition 1.3. Thus $\mathcal{M}$ is (2-PM).

Corollary 2.4

Assume $\mathcal{M}$ is considered as a cyclic R-module. So that $\mathcal{M}$ is a (S-2-PM) if and only if $\mathcal{M}$ is (2-PM).

Proposition 2.5

Let $\mathcal{M}$ be faithful module. It leads to the statements below being equivalent:

1) $\mathcal{M}$ is a (S-2-PM).
2) R is 2-Prime ring
3) R is S-Prime ring.

Proof

(1)⇒(2): Since $\mathcal{M}$ is an (S-2-PM), $ann\mathcal{M}$ is a (2-PI). But $ann\mathcal{M} = 0$, since $\mathcal{M}$ is faithful, so (0) is a prime ideal of R. Thus R is a 2Prime ring.

(2)⇒(1) is similarly.

(2)⇔(3) it follows by Corollary 2.4.

Corollary 2.6

Assume $\mathcal{M}$ is considered as a faithful module, in which R is an integral domain. So that $\mathcal{M}$ be (S-2-PM).

Proof:

Since R is an integral domain, R is a prime ring, hence R is a 2-Prime ring. But $\mathcal{M}$ is faithful, so that $\mathcal{M}$ is a (S-2-PM) by Proposition 2.5.

Proposition 2.7

Let R be a chained ring, $\mathcal{M}_1$and $\mathcal{M}_2$ be (S-2-PM). Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a (S-2-PM).

Proof:

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are (S-2-PM), $ann_R\mathcal{M}_1$ and $ann_R\mathcal{M}_2$ are (2-PI). Also $ann_R(\mathcal{M}_1 \oplus \mathcal{M}_2) = ann_R\mathcal{M}_1 \cap ann_R\mathcal{M}_2$. But R is a chained ring, so that $ann_R(\mathcal{M}_1 \oplus \mathcal{M}_2) = ann_R\mathcal{M}_1$ or $ann_R(\mathcal{M}_1 \oplus \mathcal{M}M_2) = ann_R\mathcal{M}_2$. Thus $\mathcal{M}_1 \oplus \mathcal{M}_2$ is (S-2-PM).

Next we have the following:

Proposition 2.8

Let $\mathcal{M}$ be a finitely generated (S-2-PM) module, S is a multiplicative subset of R. Then $S^{-1}\mathcal{M}$ is an (S-2-PM) S^{-1}R-module.

Proof:

Since $\mathcal{M}$ is a (S2PM), then (0) as a (S2PM) of $\mathcal{M}$. Then by Proposition 1.18, $S^{-1}(0)$ is an(S-2-PM) of $S^{-1}\mathcal{M}$ Thus $S^{-1}\mathcal{M}$ is an (S2PM) S^{-1}R-module.

References

[1] H. Ansari and Farshadifar "The dual notion of multiplication modules, Taiwanese J.Math. 11(4)(2007),1189-1201.

[2] Z.A.EL. Bast and P.F. Smith, Multiplication Modules, Comm. In Algebra 16(1988),755-779.

[3] J.Dauns, Prime Modules and one sided ideals in Ring Theory and Algebra, Proceeding of Third Oklaham Confrence, B.R. McDonald (editor), Dekker, New York,(1980) ,301-433.

[4] W.Heinzer and D.Lantz), The Laskerian Property in Commutative Rings, J.Algebra ,72 (1981),101-114.

[5] F.D. Jasim and A.A. Elewi, 2-Prime submodule of modules, Iraqi Journal of Science, 63(8) (2022),3605-3611,

[6]. F.D. Jasim, 2-Quasi-Prime Submodules and 2-Quasi-Prime Modules, Ms.C. Thesis, university of Baghdad,(2022).

[7] M.D. Larsen & P.J. Mccarlthy , Multiplicative Theory of Ideals, Academic, Press, New York. ,(1981)

[8] C.P. Lu)," M-radical of Submodules of Modules", Math. Japan,34(2)(1989),211-219.

[9] A.S. Mijbass, On Cancellation Modules, Ms.C Thesis, University of Baghdad,(1992).
[10] W. Messirdi and C.Beddani) ,2-Prime ideals and their multiplications, Journal of Algebra, Taibah Univ. Dep. of Math. Madinah Kingdom of Saudi-Arabia, No.6806/1436,(2015)
[11] S.A. Saymach, on Prime submodules, Univ. Noc. Tucmare Ser.A.29 (1979),121-136.