AL-Qadisiya Journal for pure Science Vol. 13 No. 4 year 2008
The first scientific congress for science col. Which's hold in March Year 2008

# On $\Theta-$convergent of Nets in Bitopological spaces

By
Ihsan Jabbar Kadhim Al-Fatlawee
University Of AL-Qadisiya
College of Science
Department of Mathematics

**Abstract**. We study the concept of $\theta-$convergence of nets in bitopological spaces and we find the relation among this concept and some new concepts as $\theta-$cluster points of a set, $\theta-$adherent points of a set ,and $\theta-$closed sets in bitopological spaces .And we introduce the concept of the $\theta-$Hausdorff bitopological space and we fined the relation between this new concept and the concept of $\theta-$convergence of nets in bitopological spaces.

**1. Introduction**. A ***bitopological space*** **[2]** , is a non empty set $X$ with two non identical topologies $\sigma$ and $\rho$ denoted by $(X,\sigma,\rho)$.By a ***direct set*** **[4,5]** , we mean a pair $(A,\geq)$ consisting of a non empty set $A$ and a binary relation $\geq$ defined on $A$ which directs $A$. Let $(A,\geq)$ be a directed set and let $B \subset A$. Then $B$ is said to be a ***residual subset*** of $A$ **[4,5]** , if there exists an element $a \in A$ such that $b \in A, b \geq a \Rightarrow b \in B$.If for every $a \in A$, there exists an element $b \in B$ such that $b \geq a$, then $B$ is said to be a ***cofinal subset*** **[4,5]** .

**Definition 1.1.[4,5].** *Let $(A,\geq)$ be a directed set and let $\eta$ be an arbitrary mapping of $A$ into a set $X$. Then $\eta$ is said to be* ***a net*** *in $X$ and is denoted by $(\eta, X, A, \geq)$.*

**Definition 1.2.[4,5].***Let $(\eta, X, A, \geq)$ be a net and let $Y \subset X$. Then $\eta$ is said to be* ***eventually*** *in $Y$, if there exists a residual subset $B \subset A$ such that $\eta(B) \subset Y$, and is said to be* ***frequently*** *in $Y$, if there exists a cofinal subset $D \subset A$ such that $\eta(D) \subset Y$.*

**Definition 1.3.[4,5].***Let $(\eta, X, A, \geq)$ and $(\pi, X, B, \geq *)$ be two nets . Then $\pi$ is said to be a* ***subnet*** *of $\eta$ if there exists a mapping $\varphi : B \to A$ such that*
*(i) $\pi = \eta \circ \varphi$ and*
*(ii) for each $a \in A$, there exists an element $b \in B$ such that $\varphi(x) \geq a$ for every $x \geq *b \in B$.*

**Definition 1.4.[4].***Let $(A,\geq)$ and $(B,\geq *)$ be two directed sets . Then a mapping $\psi : A \to B$ is said to be* ***isotone*** *,if $x \geq y \Rightarrow \psi(x) \geq *\psi(y)$ $(x, y \in A)$.*

**Theorem 1.5.[4]** *.Let $\psi$ be an isotone map of directed set $(B,\geq *)$ into directed set $(A,\geq)$ such that $\psi(B)$ is cofinal in $A$ .Let $(\eta, X, A, \geq)$ be a net..Then $\eta \circ \psi$ is a subnet of $\eta$.*

**2.Main results.**

**Definition 2.1.** *Let $(X,\sigma,\rho)$ be a bitopological space and let $A$ be a subset of $X$. A point $x \in X$ is said to be a $(\sigma-\rho)-\theta-$* ***cluster*** *point of $A$, if for every $\sigma-nhd$ $N$ of $x$ the intersection $(\rho-Cl(N)-\{x\})\bigcap A$ is non empty.*

**Definition 2.2.**[1] *Let $(X,\sigma,\rho)$ be a bitopological space and let $A$ be a subset of $X$. A point $x \in X$ is said to be a $(\sigma-\rho)-\theta-$* ***adherent*** *point of $A$, if for every $\sigma-nhd$ $N$ of $x$ the intersection $(\rho-Cl(N))\bigcap A$ is non empty.*

Clearly a $(\sigma-\rho)-\theta-$cluster point of a subset $A$ of a bitopological space $(X,\sigma,\rho)$ is a $(\sigma-\rho)-\theta-$adherent point of $A$.



**Definition 2.3.** *Let $(X,\sigma,\rho)$ be a bitopological space a subset $A$ of $X$ is said to be $(\sigma-\rho)-\theta-$closed, if it consist of all it is $(\sigma-\rho)-\theta-$cluster points.*

**Definition 2.4.** *A bitopological space $(X,\sigma,\rho)$ is said to be $(\sigma-\rho)-\theta-$* ***Hausdorff***, *if for every two distinct points $x$ and $y$ in $X$, there exist two $\sigma-$open sets $U$ and $V$ such that $x \in U, y \in V$ and $\rho-Cl(U)\bigcap \rho-Cl(V)=\phi$.*

**Definition 2.5.**[3] *Let $(X,\sigma,\rho)$ be a bitopological space and let $(\eta,X,A,\geq)$ be a net in $X$. We say that $\eta$ is $(\sigma-\rho)-\theta-$* ***converges(*** ***$\sigma-$weakly convergent with respect to $\rho$)*** *to a point $x \in X$, if for every $\sigma-nhd$ $N$ of $x$, $\eta$ is eventually in $\rho-Cl(N)$.*

**Theorem 2.6.** *Let $(X,\sigma,\rho)$ be a bitopological space and let $Y \subset X$. A point $x \in X$ is a $(\sigma-\rho)-\theta-$adherent point of $Y$ if and only if there exists a net in $Y$ $(\sigma-\rho)-\theta-$converges to a point $x \in X$.*

**Proof.** The "if " part .Let $(\eta,Y,A,\geq)$ be a net in $Y$ $(\sigma-\rho)-\theta-$converges to a point $x \in X$. And let $N$ be a $\sigma-nhd$ of $x$. Then there exists $a \in A$ such that for every $b \in A, b \geq a \Rightarrow \eta(b) \in \rho-Cl(N)$. Now , $a \geq a$ therefore $\eta(a) \in \rho-Cl(N)\bigcap Y$ so that $\rho-Cl(N)\bigcap Y \neq \phi$. It follows that $x$ is a $(\sigma-\rho)-\theta-$adherent point of $Y$.

The "only if " part .Let $x$ *is a* $(\sigma-\rho)-\theta-$adherent point *of* $Y$. Then for every $\sigma-nhd$ $N$ of $x$ , $\rho-Cl(N)\bigcap Y \neq \phi$. Let $\Omega$ be the collection of all $\sigma-nhds$ of $x$ , then $\Omega$ directed by the inclusion relation $\subset$. Since $\rho-Cl(N)\bigcap Y \neq \phi$ $\forall N \in \Omega$, by axiom of choice $\forall N \in \Omega$ we must choose $x(N) \in \rho-Cl(N)\bigcap Y$. Now consider the mapping $\eta:\Omega \to Y$ such that $\eta(N)=x(N), \forall N \in \Omega$. Then evidently $\eta$ is a net in $Y$ $(\sigma-\rho)-\theta-$converges to a point $x \in X$.■

**Theorem 2.7.** *Let $(X,\sigma,\rho)$ be a bitopological space and let $Y \subset X$. A point $x \in X$ is a $(\sigma-\rho)-\theta-$cluster point of $Y$ if and only if there exists a net in $Y-\{x\}$ $(\sigma-\rho)-\theta-$converges to a point $x \in X$.*

**Proof**. The " if " part . Let $(\eta, Y-\{x\}, A, \geq)$ be a net in $Y-\{x\}$ which is $(\sigma-\rho)-\theta-$ converges to a point $x \in X$.And let $N$ be a $\sigma-nhd$ of $x$.Then there exists $a \in A$ such that for every $b \in A, b \geq a \Rightarrow \eta(b) \in \rho-Cl(N)$.Now, $a \geq a$ therefore $\eta(a) \in \rho-Cl(N) \cap (Y-\{x\})$ so that $\rho-Cl(N) \cap (Y-\{x\}) \neq \phi$. It follows that $x$ is a $(\sigma-\rho)-\theta-$ cluster point of $Y$.

The " only if " part . Let $x$ *is a* $(\sigma-\rho)-\theta-$ cluster point *of* $Y$. Then for every $\sigma-nhd$ $N$ of $x$ , $\rho-Cl(N) \cap (Y-\{x\}) \neq \phi$. Let $\Omega$ be the collection of all $\sigma-nhds$ of $x$ , then $\Omega$ directed by the inclusion relation $\subset$. Since $\rho-Cl(N) \cap (Y-\{x\}) \neq \phi$ $\forall N \in \Omega$ ,by axiom of choice $\forall N \in \Omega$ we must choose $x(N) \in \rho-Cl(N) \cap (Y-\{x\})$. Now consider the mapping $\eta : \Omega \rightarrow Y-\{x\}$ such that $\eta(N) = x(N), \forall N \in \Omega$.Then evidently $\eta$ is a net in $Y-\{x\}$ $(\sigma-\rho)-\theta-$ converges to a point $x \in X$.■

**Theorem 2.8**. *Let* $(X, \sigma, \rho)$ *be a bitopological space.* A subset $Y$ *of* $X$ *is a* $(\sigma-\rho)-\theta-$*closed if and only if no net in* $Y$ $(\sigma-\rho)-\theta-$*converges to a point in* $X-Y$.

**Proof.** The "if " part. Suppose that no net in $Y$ $(\sigma-\rho)-\theta-$converges to a point in $X-Y$.Suppose if possible that $Y$ is not $(\sigma-\rho)-\theta-$closed. Then there exists a is *a* $(\sigma-\rho)-\theta-$cluster point $x$ of $Y$ such that $x \notin Y$, that is $x \in X-Y$. Then by Theorem 2.7. , there exists a net in $Y-\{x\}$ and consequently a net in $Y$ $(\sigma-\rho)-\theta-$converges to a point $x \in X-Y$. But this is a contradiction .

The "only if " part. Suppose that $Y$ is a $(\sigma-\rho)-\theta-$closed set. Suppose if possible there exists a net $\eta$ in $Y$ $(\sigma-\rho)-\theta-$converges to a point $x \in X-Y$.Since $x \notin Y$, $\eta$ is also a net in $Y-\{x\}$ $(\sigma-\rho)-\theta-$converges to a point $x \in X-Y$. Hence by Theorem 2.7. $x$ is a $(\sigma-\rho)-\theta-$cluster point of $Y$. But this is a contradiction since $x \notin Y$. Hence no net in $Y$ $(\sigma-\rho)-\theta-$converges to a point in $X-Y$.■

**Theorem 2.9.** *A bitopological space* $(X, \sigma, \rho)$ *is* $(\sigma-\rho)-\theta-$*Hausdorff if and only if every net in* $X$ *can* $(\sigma-\rho)-\theta-$*converges to at most one point* .

**Proof.** The "if " part. Suppose that every net in $X$ can $(\sigma-\rho)-\theta-$converges to at most one point and suppose if possible , that the space $(X, \sigma, \rho)$ is not $(\sigma-\rho)-\theta-$Hausdorff. Then there exist two distinct points $x$ *and* $y$ *in* $X$ such that for every $\sigma-nhd$ $N$ of $x$ and every $\sigma-nhd$ $M$ of $y$ the intersection $\rho-Cl(N) \cap \rho-Cl(M)$ is non empty .Let $\Omega$ and $\Phi$ be the collection of all the $\sigma-nhds$ of $x$ and $y$ respectively. Then $\Omega$ and $\Phi$ are directed by the inclusion relation $\subset$. Consider the Cartesian product $P = \Omega \times \Phi$. In this product , we define a relation $\geq$ as follows . Let $(A,B)$ and $(C,D)$ be two element of $P$. We define $(A,B) \geq (C,D)$ if and only if $A \subset C$ and $B \subset D$. Then $P$ is directed by $\geq$. We have $\rho-Cl(N) \cap \rho-Cl(M) \neq \phi$, $\forall (N,M) \in P$. Hence by the axiom of choice we may choose a point $x(N,M)$ in $\rho-Cl(N) \cap \rho-Cl(M)$ for all $(N,M) \in P$. Now consider the mapping

$$\eta : P \rightarrow X : \ \eta(N,M) = x(N,M) \ \ \forall \ \ (N,M) \in P .$$

Then $\eta$ is a net in $X$ $(\sigma-\rho)-\theta-$converges to both $x$ and $y$ in $X$.But this is a contradiction. Hence $(X, \sigma, \rho)$ must be $(\sigma-\rho)-\theta-$Hausdorff.

The " only if " part. Suppose that $(X,\sigma,\rho)$ be a $(\sigma-\rho)-\theta-$Hausdorff .Let $x$ and $y$ be two distinct points of $X$ .Then there exist a $\sigma-nhd$ $N$ of $x$ and $\sigma-nhd$ $M$ of $y$ such that $\rho-Cl(N)\bigcap\rho-Cl(M)=\phi$ . Since a net cannot be eventually in each of two disjoint sets it is evident that no net in $X$ can $(\sigma-\rho)-\theta-$converges to both $x$ and $y$ in $X$ .Hence a net in $X$ can $(\sigma-\rho)-\theta-$converges to at most one point .■

**Definition 2.10**. *Let $(X,\sigma,\rho)$ be a bitopological space and let $\eta$ be a net in $X$. Then a point $x\in X$ is said to be a $(\sigma-\rho)-\theta-$* ***cluster*** *point of a net $\eta$ if $\eta$ is frequently in the $\rho-$closure of every $\sigma-nhd$ of $x$.*

**Remark 2.11**. *Let $(X,\sigma,\rho)$ be a bitopological space. If a point $x\in X$ is a $(\sigma-\rho)-\theta-$ limit point of a net $\eta$ in $X$, then it is $(\sigma-\rho)-\theta-$cluster point of $\eta$.*

**Theorem 2.12**. *Let $(X,\sigma,\rho)$ be a bitopological space and let $(\eta,X,A,\geq)$ be a net in $X$. Let $\Phi$ be the collection of subset of $X$ satisfying the following two conditions*

*(i) $\eta$ is frequently in $\rho-Cl(F)$, $\forall$ $F\in\Phi$.*

*(ii) If $S,T\in\Phi$, then there exists a member $U\in\Phi$ $\ni$ $U\subset S\cap T$.*

*Then there exists a subnet of $\eta$ which is eventually in $\rho-Cl(F)$, $\forall$ $F\in\Phi$.*



**Proof.** It is evident from (ii) that $\Phi$ is directed by the inclusion relation $\subset$. Now consider the subset $B$ of the Cartesian product $A\times\Phi$ defined by

$$B=\{(a,U):a\in A,U\in\Phi,\ \ and\ \ \eta(a)\in\rho-Cl(U)\}.$$

We define a binary relation $\geq*$ in $B$ as follows :

Let $(a,S)$ and $(b,T)$ be any two members of $B$. Then $(a,S)\geq*(b,T)$ if and only if $a\geq b$ and $S\subset T$ .Then $B$ is directed by $\geq*$. Define a mapping

$$\psi:B\to A:\ \ \psi((a,S))=a\quad\forall(a,S)\in B$$

Hence $\psi$ is an isotone mapping. For there $\psi[B]$ is cofinal in $A$. It follows by Theorem 1.5. the mapping $\varphi=\eta\circ\psi:B\to X$ is a subnet of $\eta$. We now show that this subnet is eventually in the $\rho-$closure of each member of $\Phi$. Let U be an arbitrary member of $\Phi$. By (i), there exists a member $a\in A$, such that $\eta(a)\in\rho-Cl(U)$. Hence by definition of $B$, $b=(a,U)$ is a member of $B$. Now let $d=(c,V)$ be an arbitrary element of $B$ such that $d\geq*b$. Then $c\geq a$ and $V\subset U$. We have $\varphi(d)=(\eta\circ\psi)(d)=\eta(\psi(d))=\eta(\psi(c,V))=\eta(c)\in\rho-Cl(V)\subset\rho-Cl(U)$ .Thus there exists an element $b\in B$, such that for every $d\in B$, $d\geq*b$ $\Rightarrow\varphi(d)\in\rho-Cl(U)$.■

**Theorem 2.13.** *Let $(X,\sigma,\rho)$ be a bitopological space. A point $x\in X$ is a $(\sigma-\rho)-\theta-$cluster point of a net $(\eta,X,A,\geq)$ if and only if a subnet $(\pi,X,B,\geq*)$ which $(\sigma-\rho)-\theta-$converges to $x$.*

**Proof.** The "if" part. Suppose that $\eta$ has a subnet $\pi$ which $(\sigma-\rho)-\theta-$converges to $x$. To prove that $x$ is a $(\sigma-\rho)-\theta-$cluster point of a net $\eta$. Let $N$ be a $\sigma-nhd$ of $x$ and let $a\in A$. Since $\pi$ is a subnet of $\eta$, there exists a mapping $\varphi:B\to A$ such that

(i) $\pi=\eta\circ\varphi$.

(ii) For each $c \in A$, there exists an element $d \in B$ such that $\varphi(x) \geq c, \forall \quad x \geq *d \quad \in B$. Hence by (ii) , corresponding to $a \in A$, there exists an element $b \in B$ such that $\varphi(x) \geq a, \forall \quad x \geq *b$ . Since $\pi\ (\sigma-\rho)-\theta-$ converges to $x$, there exists an element $p \geq *b \in B$ such that $\pi(p) \in \rho-Cl(N)$. Now , let $\varphi(p)=q$. Then $q \in A$ and $q \geq a$. Also

$$\eta(q)=\eta(\varphi(p))=(\eta \circ \varphi)(p)=\pi(p) \in \rho-Cl(N).$$

Thus we have shown that for each element $a \in A$, there exists an element $q \geq a \in A$ such that $\eta(q) \in \rho-Cl(N)$. Hence $\eta$ is frequently in $\rho-Cl(N)$ for every $\sigma-nhd\ N$ of $x$. It follows that $x$ is a $(\sigma-\rho)-\theta-$ cluster point of $\eta$.

The " only if " part. Let a point $x \in X$ is a $(\sigma-\rho)-\theta-$ cluster point of a net $(\eta, X, A, \geq)$ and let $\Omega$ be the collection of all $\sigma-nhds$ of $x$. If $L$ and $M$ are any two members of $\Omega$, then $L \cap M$ is also a member of $\Omega$. Also since $x$ is a $(\sigma-\rho)-\theta-$ cluster point of $\eta$,then $\eta$ is frequently in the $\rho-$closure of each member of $\Omega$. Hence by Theorem 2.12. ,there exists a subnet $\pi$ of $\eta$ which is eventually in the $\rho-$closure of each member of $\Omega$. This implies that $\pi$ is $(\sigma-\rho)-\theta-$ converges to $x$.■

**Theorem 2.14.** *Let $(X,\sigma,\rho)$ be a $(\sigma-\rho)-\theta-$ Hausdorff bitopological space. Then every $(\sigma-\rho)-\theta-$convergent net has a unique $(\sigma-\rho)-\theta-$cluster point and this is the unique $(\sigma-\rho)-\theta-$limit point of that net.*

**Proof.** We know that in a $(\sigma-\rho)-\theta-$ Hausdorff bitopological space every $(\sigma-\rho)-\theta-$convergent net has a unique $(\sigma-\rho)-\theta-$limit point by Theorem2.9. Let $p$ be a unique $(\sigma-\rho)-\theta-$limit point of a $(\sigma-\rho)-\theta-$convergent net $\eta$ in $X$. Since every $(\sigma-\rho)-\theta-$limit point is also $(\sigma-\rho)-\theta-$cluster point, $p$ is a $(\sigma-\rho)-\theta-$cluster point of $\eta$. Suppose if possible that $\eta$ has another $(\sigma-\rho)-\theta-$cluster point $q$ distinct from $p$. Since $(X,\sigma,\rho)$ be a $(\sigma-\rho)-\theta-$ Hausdorff bitopological space, there exists two disjoint $\sigma-nhds$ $N$ of $p$ and $M$ of $q$,such that $\rho-Cl(N) \cap \rho-Cl(M)=\phi$. Since $p$ is a $(\sigma-\rho)-\theta-$limit point of $\eta$, then $\eta$ is eventually in $\rho-Cl(N)$ and since $\rho-Cl(M)$ is disjoint from $\rho-Cl(N)$,then $\eta$ cannot be frequently in $\rho-Cl(M)$. But this contradicts our supposition that $q$ is
$(\sigma-\rho)-\theta-$cluster point of $\eta$. Hence $\eta$ cannot has two distinct $(\sigma-\rho)-\theta-$cluster points.■

**References.**

[1] C.G.Kariofillis "On Pairwise almost compactness " Ann.Soc.Sci.Bruxelles (100)(1986), 129-137.
[2] J.C.Kelly "Bitopological Spaces " proc. London Math. Soc. 3 (13)(1963) ,71-89.
[3] J.Ewert ,Slupsk , Poland." Weak forms of Continuous, Quasi-Continuous and Cliquishness of Maps With Respect to Two Topologies " Glasnik Matematicki 21(41)(1986).179-189.
[4] J.N.Sharma,"Topology" Published by Krishna Pracushna , Mandir , and printed at Mano, (1977).
[5] S.Willard , "General Topology " ,Addison-Wesry Pub.Co.,Inc.(1970).