Iraqi Journal for Computer Science and Mathematics

Manuscript 1253

A hybrid technique for approximating the solution of fractional order partial integro-differential equations

Ahmed K. Mohsin

Fajir A. Abdulkhaleq

Osama H. Mohammed

IRAQI JOURNAL FOR COMPUTER SCIENCE AND MATHEMATICS 2025;6:139–149

ORIGINAL STUDY

A Hybrid Technique for Approximating the Solution of Fractional Order Partial Integro-Differential Equations

Ahmed K. Mohsin [a,*], **Fajir A. Abdulkhaleq** [b], **Osama H. Mohammed** [c]

[a] Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq
[b] Ministry of Education, Baghdad, Iraq
[c] Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Jadriya, Baghdad, Iraq

ABSTRACT

In this paper, we discuss the numerical solution of fractional order partial integro-differential equations. The type of fractional derivative used is a Caputo derivative. The method proposed in this paper known as transform optimal perturbation iteration method. This method combines the optimal perturbation iteration method and the Laplace transform in order to converge to the exact solution. The proposed method is highly efficient and provides the means of controlling the approximate solutions convergence. Illustrative examples prove that the suggested approach is very accurate when compared with the exact solution and the results of the existing methods.

Keywords: Fractional order, Caputo derivative, Laplace transform, Optimal perturbation iteration method

1. Introduction

The subject of fractional calculus has applications in diverse and widespread fields of engineering and science such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signals processing. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. The fractional derivative models are used for accurate modelling of those systems that require accurate modelling of damping. In these fields, various analytical and numerical methods including their applications to new problems have been proposed in recent years. Similar to how fractional exponents are an extension of exponents with integer values, fractional calculus is a branch of mathematics that emerges from the conventional definitions of calculus integral and derivative operators. Applications of fractional calculus are becoming more widespread in all associated scientific and engineering domains. Several books and associated review papers reported on some of the findings [1–17]. However, the use of this extremely potent tool in numerous study domains is still in its infancy. Currently, the dynamics of the complex real world are now included in the fractional calculus, and new ideas are starting to be used and assessed using real data. Patents were granted in some cases, which makes the fractional order quite hopeful. The fractional order community still plays a big part in encouraging applications, despite the fact that fractional calculus was first utilized more than three centuries ago and is employed in numerous scientific and engineering fields. Caputo revised the definition of the Riemann-Liouville fractional derivative in his 1967 paper [18], by substituting the fractional integral operator for the standard derivative's order. In contrast to the initial requirements required when using the Riemann-Liouville fractional

Received 18 December 2024; revised 24 February 2025; accepted 7 March 2024.
Available online 2 May 2025

* Corresponding author.
E-mail addresses: ahmed_almosawi@uomustansiriyah.edu.iq (A. K. Mohsin), ms-teacher@yahoo.com (F. A. Abdulkhaleq), osama.hameed@nahrainuniv.edu.iq (O. H. Mohammed).

https://doi.org/10.52866/2788-7421.1253

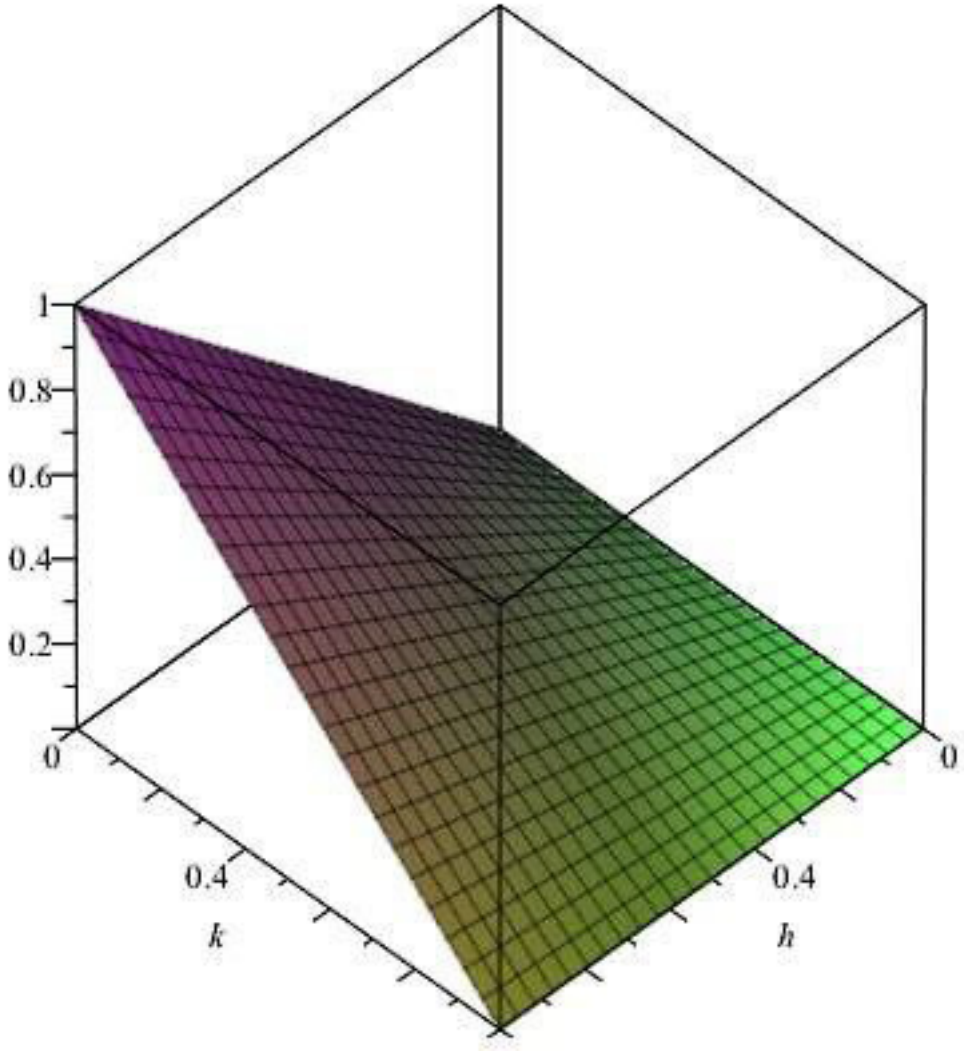

Fig. 1. The exact solution of example 1 when $\beta = 1$.

derivative, which entail fractional order conditions, the Laplace transform of this new derivative is thus dependent on integer order initial conditions.

This paper primary goal is to ascertain the numerical solution of the fallowing special class of fractional order partial integro-differential equation:

$$^{c}D_{k}^{\beta}p(h,k) = Q(h,k) + \int_{0}^{k} A(k,w)F(p(h,w))dw \quad (1.1)$$

With the initial condition

$$p(h,0) = B(h) \quad (1.2)$$

Where $Q(h,k)$ is known continues function, $A(k,w)$ is a kernel of the Eq. (1.1) and $F(p(h,k))$ is a non-linear operator in $p(h,k)$.

2. Preliminaries

Definition 2.1: The fractional order Riemann-Liouville (R-L) concept of an order $\beta > 0$ can be provided as:

$$I_{k}^{\beta}p(k) = \frac{1}{\Gamma(\beta)}\int_{0}^{k}(k-\tilde{\tau})^{\beta-1}p(\tilde{\tau})\,d\tilde{\tau},\ k>0,\ \beta\in\mathcal{R}^{+}$$

Definition 2.2: The Caputo fractional derivative with an order $\beta > 0$ can be provided as:

$$^{c}D_{k}^{\beta}p(k) = \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_{0}^{k}\frac{p^{(m)}(\tilde{\tau})}{(k-\tilde{\tau})^{\beta+1-m}}d\tilde{\tau}, & m-1<\beta<m \\ \frac{d^{m}}{dk^{m}}p(k), & \beta=m \end{cases}$$

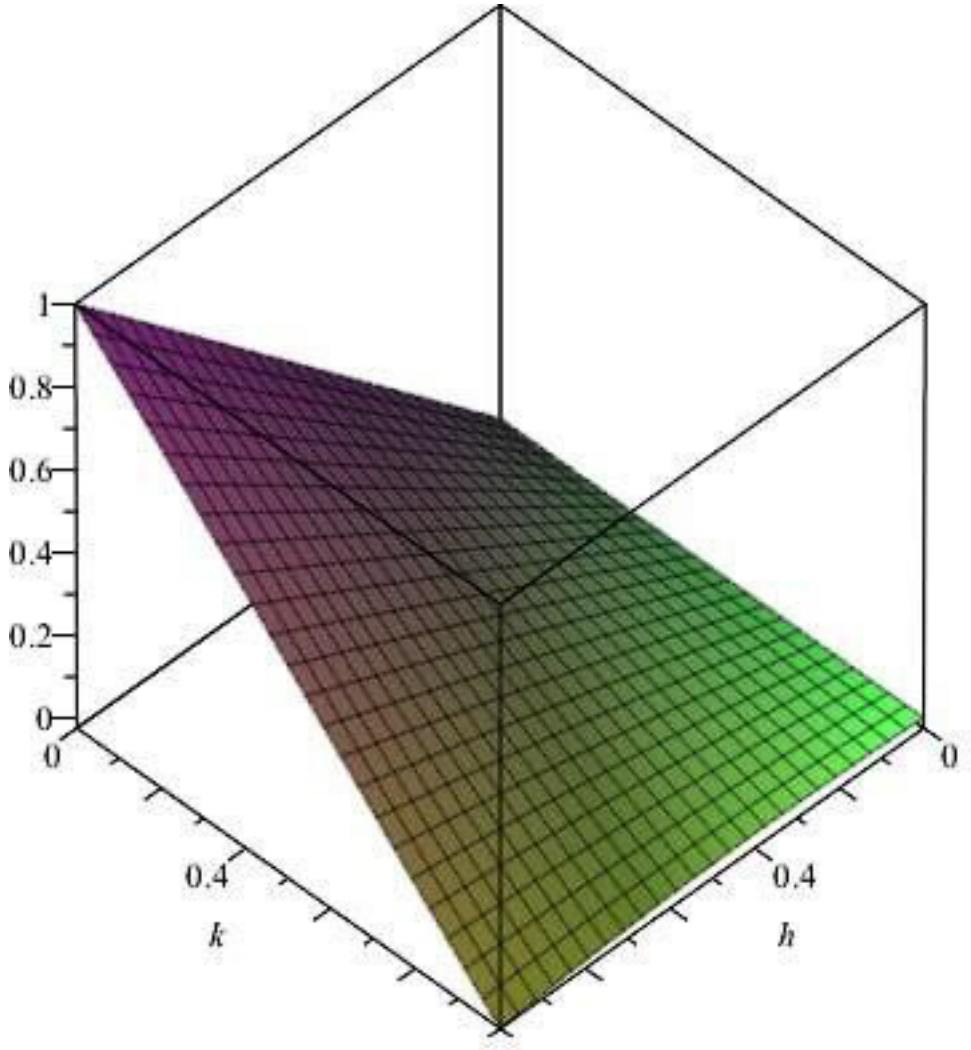

Fig. 2. The approximate solution of example 1 when $\beta = 1$.

The Caputo fractional derivative Laplace transform is provided by

$$\mathcal{L}\{{}^{c}D_{k}^{\beta}p(k)\} = s^{\beta}\mathcal{L}\{p(k)\} - \sum_{\nu=0}^{n-1} s^{\beta-\nu-1}p_i^{(\nu)}(k), \quad i \in \mathbb{N}$$

3. Theory of optimal perturbation iteration transform method (OPITM)

Consider the fractional order partial integro-differential equation given in formula:

$$ {}^{c}D_{k}^{\beta}p(h,k) = Q(h,k) + \int_{0}^{k} A(k,w)F(p(h,w))dw,$$

$$(h,k) \in (\widetilde{x},\widetilde{y}) \times (0,\widetilde{T}),\ 0 < \beta \leq 1 \tag{3.1}$$

Such that

$$p\,(h,0) = B(h) \tag{3.2}$$

Firstly, we applying the Laplace transform on both sides of Eq. (3.1) and by using Definition 2.2, yields:

$$\mathcal{L}\{p\,(h,k)\} = \frac{1}{s}p\,(h,0) + \left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{Q(h,k) + \int_{0}^{k} A(k,w)F(p(h,w))dw\right\} \tag{3.3}$$

Now, applying the inverse Laplace transform on both sides of Eq. (3.3) with use the initial condition of

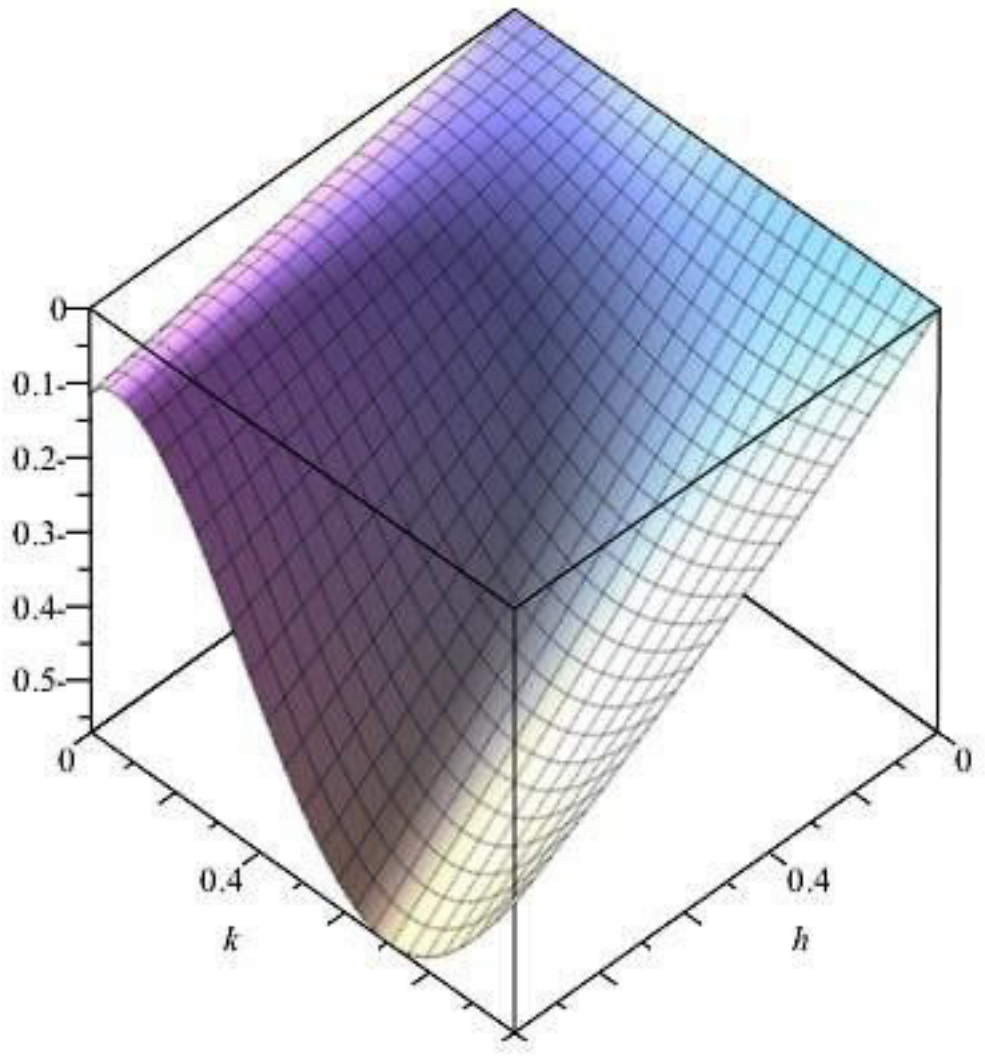

Fig. 3. The approximate solution of example 1 when $\beta = 0.5$.

Eq. (3.1), we get:

$$p(h,k) = \mathcal{L}^{-1}\left\{\frac{1}{s}B(h)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{Q(h,k) + \int_0^k A(k,w)F(p(h,w))dw\right\}\right\} \tag{3.4}$$

Secondly, we apply the perturbation iteration method in order to decompose the nonlinear term, as follows:

Step (1): The perturbation parameter can be artificially embedded into Eq. (3.4) as

$$p(h,k) = \mathcal{L}^{-1}\left\{\frac{1}{s}B(h)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{Q(h,k) + \varepsilon\int_0^k A(k,w)F(p(h,w))dw\right\}\right\} \tag{3.5}$$

Or compactly

$$U = U\left(p, \int p, \varepsilon\right) = 0 \tag{3.6}$$

Step (2): To lessen the number of calculations, one can reconsider Eq. (3.6) as

$$U\left(p, \int p, \varepsilon\right) = \xi + g(h,k) = 0 \tag{3.7}$$

Where $g(h,k)$ is known function. In addition, we can decompose Eq. (3.7) into two parts as $\xi = N + L$, where L denotes linear term and N is the nonlinear term that will be discretized.

Step (3): In order to construct the iteration scheme, we use the idea of classical perturbation theory. When the approximation solutions may be taken in the

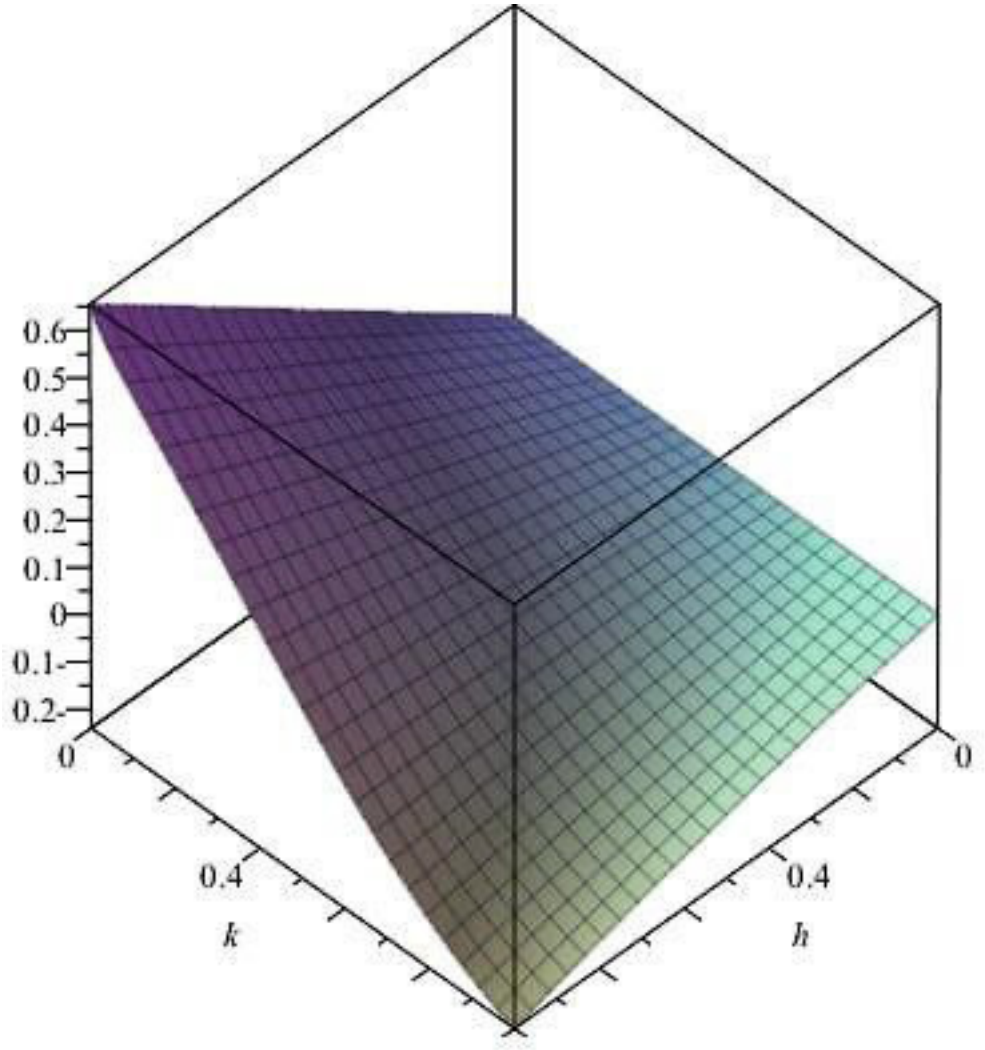

Fig. 4. The approximate solution of example 1 when $\beta = 0.75$.

perturbation expansion as [21]:

$$p_{n+1} = p_n + \varepsilon(p_c)_n \tag{3.8}$$

We must find the correction term $(p_c)_n, n \in \mathbb{N}$. Substituting Eq. (3.8) into Eq. (3.7), we get a simpler perturbation. So, by expanding the crucial part N in a Taylor series, it given by the following algorithm:

$$\begin{aligned} &N\left(p, \int p, 0\right) + N_p\left(p, \int p, 0\right)(p_c)_n\varepsilon \\ &\quad + N_{\int p}\left(p, \int p, 0\right)\left(\int (p_c)_n\right)\varepsilon + N_\varepsilon\left(p, \int p, 0\right)\varepsilon \\ &\quad + L + g(h,k) = 0 \end{aligned} \tag{3.9}$$

Now, we can call the Eq. (3.9) is OPITM, because we will add some parameters to optimize the results. Finding the correction term $(p_c)_0$ will be done using the Eq. (3.9) and with the aid of the initial function p_0, in order to establish the iteration steps.

Step (4): To improve the method results accuracy and efficiency, we express by the following formula:

$$p_{n+1} = p_n + a_n(p_c)_n \tag{3.10}$$

Where $a_0, a_1, a_2, \ldots$ are the convergence control parameters, which changes to ensure convergence of the method.

So, we get more of the approximation solutions as:

$$\begin{aligned} &p_1 = p_1(h,k;a_0) = p_0 + a_0(p_c)_0 \\ &p_2 = p_2(h,k;a_0,a_1) = p_1 + a_1(p_c)_1 \\ &\ldots \\ &p_m = p_m(h,k;a_0,a_1,\ldots,a_{m-1}) = p_{m-1} \\ &\qquad + a_{m-1}(p_c)_{m-1} \end{aligned} \tag{3.11}$$

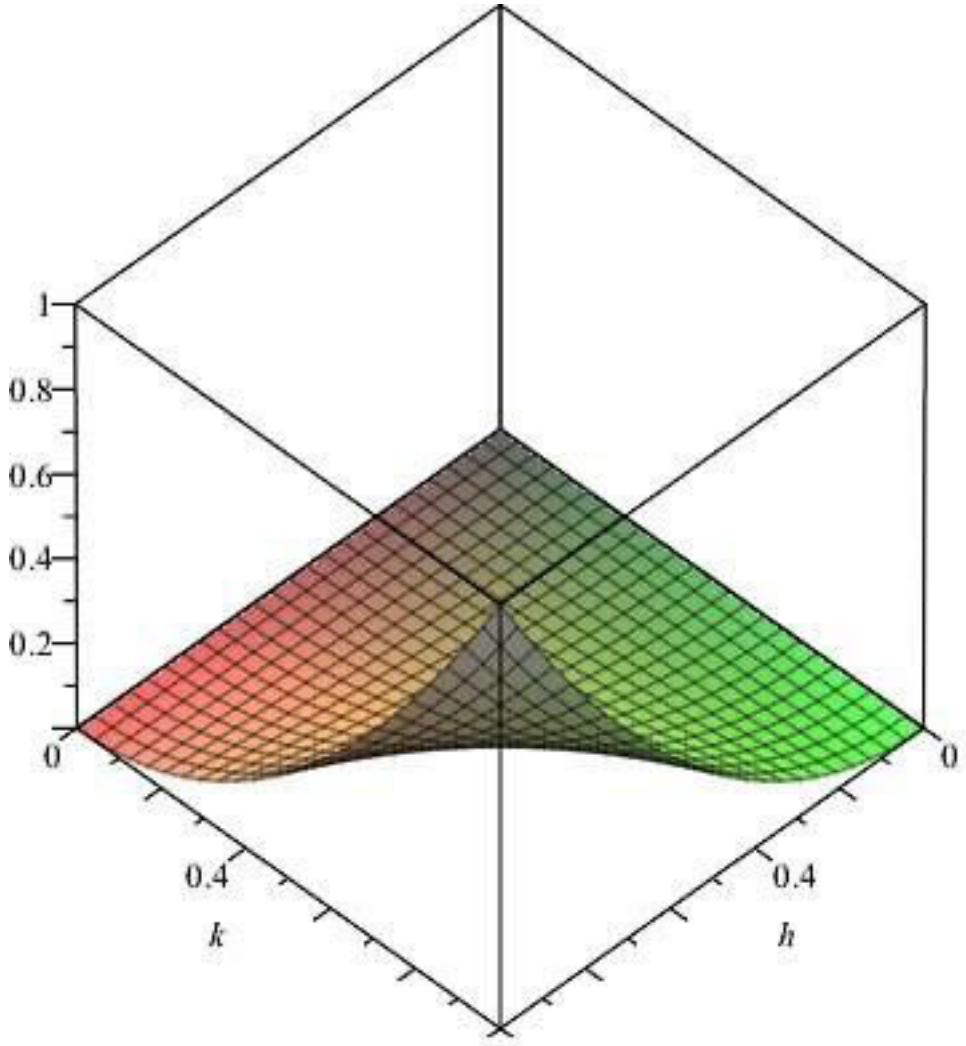

Fig. 5. The exact solution of example 2 when $\beta = 1$.

For the convergence of the method, we refer the reader to see [20].

Step (5): Substituting m^{th}-order OPITM solution p_m into Eq. (3.7), by doing so the general issue becomes the subsequent residual:

$$Re\left(h, k; a_0, a_1, \ldots, a_{m-1}\right) = \xi\left(p_m(h, k; a_0, a_1, \ldots, a_{m-1})\right) + g(h, k) \tag{3.12}$$

If $Re(h, k; a_0, a_1, \ldots, a_{m-1}) = 0$. Then, the approximation solution $p_m(h, k; a_0, a_1, \ldots, a_{m-1})$ is the exact solution. Though nonlinear differential equations don't typically have this situation, the functional can be minimized as follows:

$$J(a_0, a_1, \ldots, a_{m-1}) = \int_0^{\widetilde{T}} \int_{\widetilde{x}}^{\widetilde{y}} (Re(h, k; a_0, a_1, \ldots, a_{m-1}))^2 dhdk \tag{3.13}$$

where $\widetilde{x}, \widetilde{y}$ and $\widetilde{T}$ are extracted from the problem's domain. So, we can obtain the optimal values $a_0, a_1, a_2, \ldots$ from the following equations

$$\frac{\partial J}{\partial a_0} = \frac{\partial J}{\partial a_1} = \cdots = \frac{\partial J}{\partial a_{m-1}} = 0 \tag{3.14}$$

Also, can be obtain the optimal values $a_0, a_1, a_2, \ldots$ from

$$Re\left(h_0, k_0; a_i\right) = Re\left(h_1, k_1; a_i\right) = \cdots = Re\left(h_{m-1}, k_{m-1}; a_i\right) = 0, \quad i = 0, 1, \ldots, m-1 \tag{3.15}$$

where $(h_i, k_i) \in (\widetilde{x}, \widetilde{y}) \times (0, \widetilde{T})$.

This technique is known as the collocation method. For comprehensive details on collocation me thud and ideal unknown parameter values,

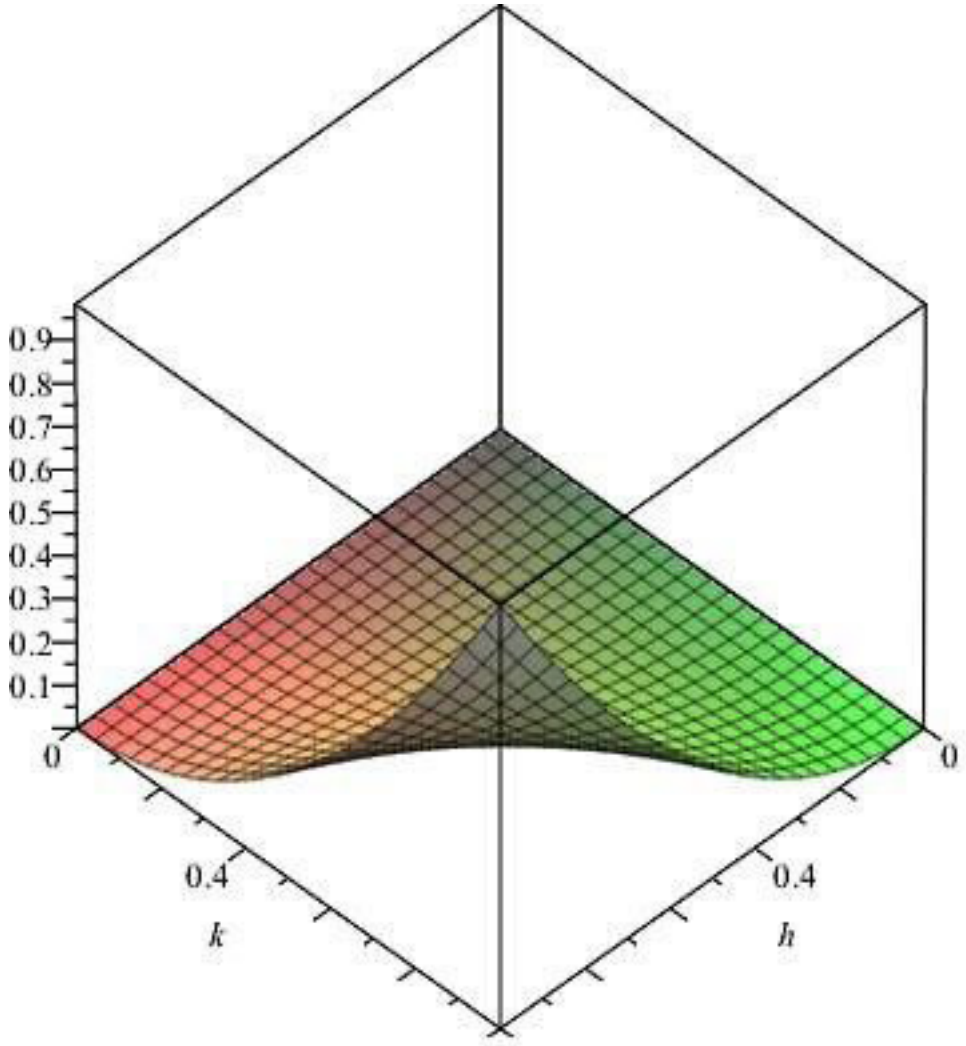

Fig. 6. The approximate solution of example 2 when $\beta = 1$.

one can referee to Agrawal etal and Deniz and Bildik [22–24].

4. Illustrative examples

Example 4.1: *Consider the PIDE of the form*

$$ {}^{c}D_{k}^{\beta}p(h,k) = -h + (h-h^2)\left(\frac{1}{2}k^2 - \frac{1}{3}k^3\right) + \int_0^k (kw - w)p(h,w)dw \tag{4.1} $$

such that

$$ p(h,0) = h \tag{4.2} $$

The exact solution of problem (4.1)–(4.2) is given by:

$$ p(h,k) = h(1-k) \tag{4.3} $$

Taking the Laplace transform on both sides of Eq. (4.1), we have:

$$ \mathcal{L}\{p(h,k)\} = \frac{1}{s}p\,(h,0) + \left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{-h + (h-h^2)\left(\frac{1}{2}k^2 - \frac{1}{3}k^3\right) + \int_0^k (kw - w)p(h,w)dw\right\} \tag{4.4} $$

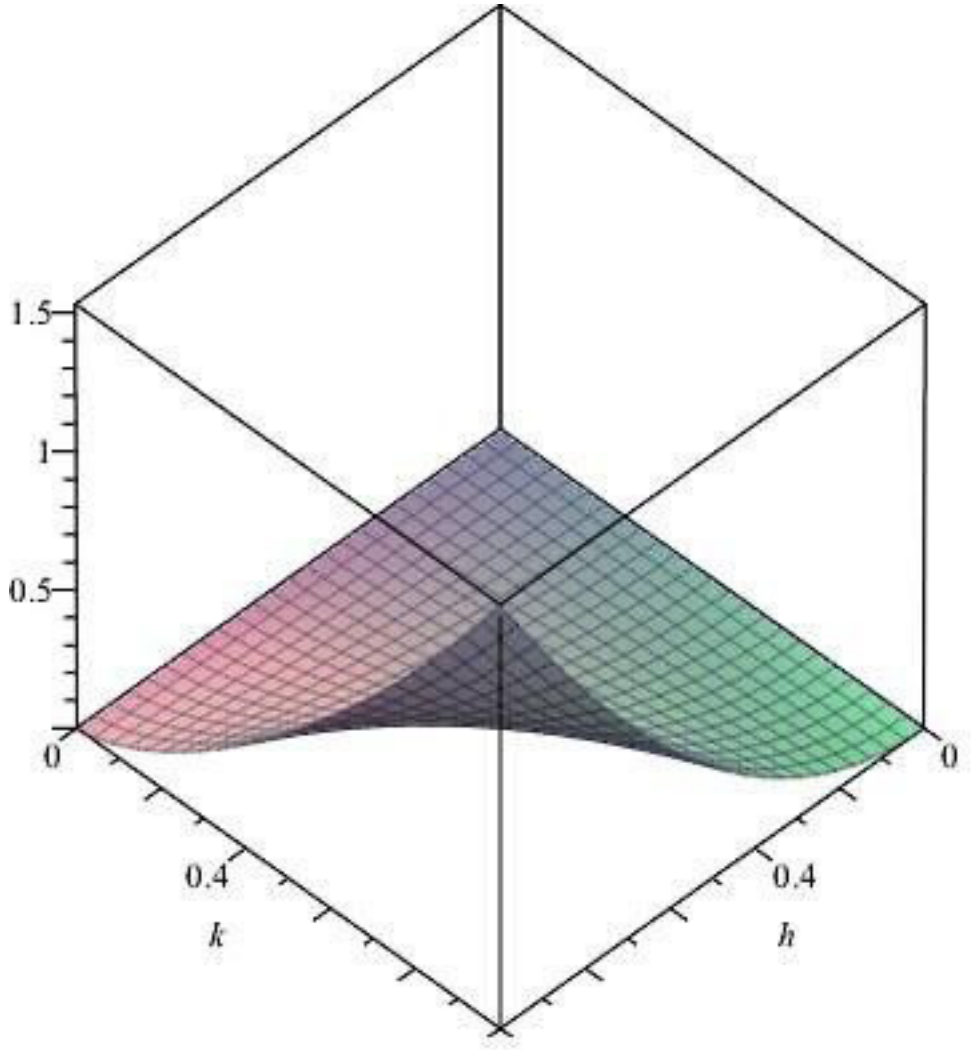

Fig. 7. The approximate solution of example 2 when $\beta = 0.5$.

Applying inverse Laplace transform on both sides of Eq. (4.4) with used the initial condition of Eq. (4.1), as:

$$p(h,k) = \mathcal{L}^{-1}\left\{\frac{1}{s}h\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{-h + (h-h^2)\left(\frac{1}{2}k^2 - \frac{1}{3}k^3\right) + \int_0^k (kw - w)p(h,w)dw\right\}\right\} \quad (4.5)$$

Now, applying the OPIM and setting $\varepsilon = 1$, yield:

$$p_n(h,k) + (p_c)_n(h,k) = h + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{-h(h-h^2)\left(\frac{1}{2}k^2 - \frac{1}{3}k^3\right) + \int_0^k (kw - w)p_n(h,w)dw\right\}\right\} \quad (4.6)$$

From the Eq. (4.6), we can be determine the correction term $(p_c)_n$. So, the approximation solutions are given as (with the initial function $p_0(h,k) = 0$),

$$\begin{aligned} p_1(h,k) &= p_0(h,k) + a_0(p_c)_0(h,k) \\ p_2(h,k) &= p_1(h,k) + a_1(p_c)_1(h,k) \\ &\cdots \end{aligned} \quad (4.7)$$

Following Table 1 represent the values of the control parameters a_1 and a_2 of problem (4.1)–(4.2) at $\beta = 1, 0.5$ and 0.75. While Table 2 represents a comparison between the mean absolute errors MAE that results from the proposed method OPITM and the Laplace Adomian decomposition method LADM [28], which shows that the OPITM is accurate and at the same time dose not require many calculations.

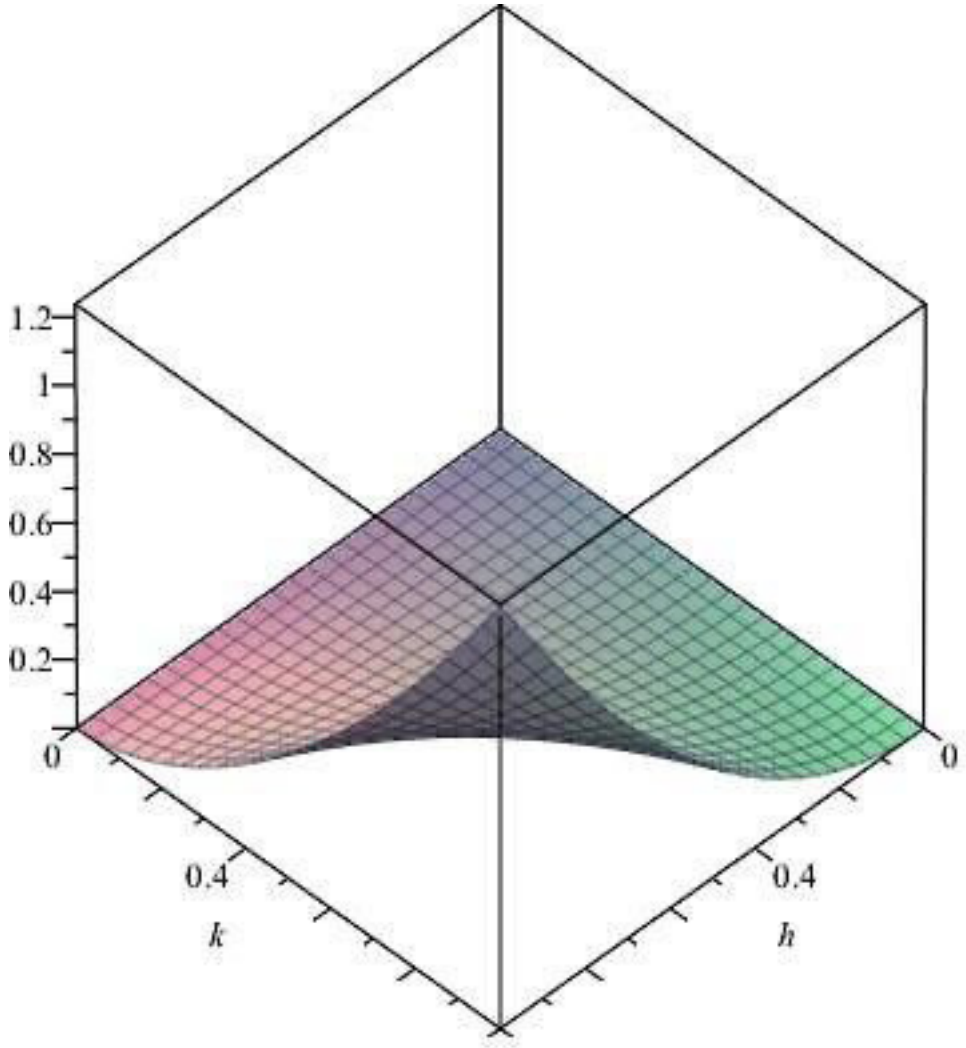

Fig. 8. The approximate solution of example 2 when $\beta = 0.75$.

Table 1. The control parameter of problem (4.1)–(4.2).

β	a_1	a_2
1	0.995	0.995
0.5	33.797	1.034
0.75	1.045	8.666

Table 2. Comparison between MAE of the proposed method and method [28] of problem (4.1)–(4.2).

k	MAE of OPITM	MAE of LADM [28]
0.01	1.243×10^{-5}	5.742×10^{-8}
0.05	1.851×10^{-5}	6.729×10^{-6}
0.09	4.755×10^{-5}	3.671×10^{-5}

Figs. 1 and 2 represent the exact and the approximate solutions of problem (4.1)–(4.2) at $\beta = 1$. Figs. 3 and 4 the approximate solutions of problem (4.1)–(4.2) at $\beta = 0.5$ and $\beta = 0.75$ respectively.

Example 4.2: *Consider the PIDE of the form*

$$ {}^{c}D_{k}^{\beta} p(h,k) = 2h^2k - \frac{1}{3}h^3k^3 + \frac{1}{4}h^2k^4 + \int_0^k (k-w)p(h,w)dw \tag{4.8} $$

Such that

$$ p(h,0) = 0 \tag{4.9} $$

The exact solution of problem (4.8)–(4.9) is given by:

$$ p(h,k) = h^2k^2 \tag{4.10} $$

Table 3. The control parameter of problem (4.8)–(4.9).

β	a_1	a_2
1	0.991	1.009
0.5	1.026	1.004
0.75	1.013	1.002

Table 4. Comparison between MAE of the proposed method and method [28] of problem (4.8)–(4.9).

k	MAE of OPITM	MAE of LADM [28]
0.01	2.252×10^{-10}	2.268×10^{-10}
0.05	1.377×10^{-7}	1.359×10^{-7}
0.09	1.4×10^{-6}	1.366×10^{-6}

Taking the Laplace transform on both sides of Eq. (4.8), we get:

$$\mathcal{L}\{p(h,k)\} = \frac{1}{s}p(h,0) + \left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{2h^2k - \frac{1}{3}h^3k^3 + \frac{1}{4}h^2k^4 + \int_0^k (k-w)p(h,w)dw\right\} \tag{4.11}$$

Applying inverse Laplace transform on both sides of Eq. (4.11) with used the initial condition of Eq. (4.8), as:

$$p(h,k) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{2h^2k - \frac{1}{3}h^3k^3 + \frac{1}{4}h^2k^4 + \int_0^k (k-w)p(h,w)dw\right\}\right\} \tag{4.12}$$

Now, applying the OPIM and setting $\varepsilon = 1$, yield:

$$p_n(h,k) + (p_c)_n(h,k) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^{\beta}}\right)\mathcal{L}\left\{2h^2k - \frac{1}{3}h^3k^3 + \frac{1}{4}h^2k^4 + \int_0^k (k-w)p(h,w)dw\right\}\right\} \tag{4.13}$$

From the Eq. (4.13), we can be determine the correction term $(p_c)_n$. So, the approximation solutions are given as (with the initial function $p_0(h,k) = 0$),

$$\begin{aligned} p_1(h,k) &= p_0(h,k) + a_0(p_c)_0(h,k) \\ p_2(h,k) &= p_1(h,k) + a_1(p_c)_1(h,k) \\ &\cdots \end{aligned} \tag{4.14}$$

Figs. 5 and 6 represent the exact and the approximate solutions of problem (4.8)–(4.9) at $\beta = 1$. Figs. 7 and 8 the approximate solutions of problem (4.8)–(4.9) at $\beta = 0.5$ and $\beta = 0.75$ respectively.

5. Conclusions

In this paper, we created a successful hybrid strategy that to build the approximate solution of the fractional order partial integro-differential equations FPIDEs by combined the least squares method, the Laplace transformation and optimal perturbation iteration method. In the proposed method, we overcome the problem that appeared when the Caputo derivative is considered by using the Laplace transform. We can be written the approximation solutions of the FPIDEs have infinite series with unknown converge parameters must be determined. We explained the method OPITM through two examples of the FPIDEs. It is noteworthy that the suggested approach has the potential to decrease the amount of time spent on computing labor in comparison to traditional methods and quickly converges to the precise answers of the provided equations at a reduced iteration value.

References

1. H. Rafeiro, and S. Samko, "Fractional integrals and derivatives: mapping properties," *Fractional Calculus and Applied Analysis*, vol. 19, no. 3, pp. 580–607, 2016.
2. A. A. Kilbas, H. M. Srivastava, and J. J.Trujillo, "Theory and applications of fractional differential equations," vol. 204. elsevier, 2006.
3. G. M. Zaslavsky, "Hamiltonian chaos and fractional dynamics," *Oxford University Press*, USA, 2005.
4. R. Magin, "Fractional calculus in bioengineering, part 1," *Critical Reviews in Biomedical Engineering*, vol. 32, pp. 1, 2004.
5. D. Baleanu, K. Dumitru, E. Scalas, and J. J. Trujillo, "Fractional calculus: models and numerical methods," *World Scientific*, vol. 3. 2012.
6. F. Mainardi, "Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models," *World Scientific*, 2022.
7. V. E. Tarasov, "Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media," *Springer Science and Business Media*, 2011.
8. M. D. Ortigueira, "Fractional calculus for scientists and engineers," *Springer Science and Business Media*, vol. 84, 2011.
9. J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in nonlinear science and numerical simulation*, 16.3, pp. 1140–1153, 2011.
10. V. V. Uchaikin, "Fractional derivatives for physicists and engineers," *Berlin: Springer*, vol. 2, 2013.
11. A. Carpinteri and F. Mainardi, eds. "Fractals and fractional calculus in continuum mechanics," *Springer*, vol. 378, 2014.
12. B. J. West, "Fractional calculus view of complexity: tomorrow's science," *CRC Press*, 2016.
13. A. S. El-Karamany and M. A. Ezzat, "On fractional thermoelectricity," *Mathematics and Mechanics of Solids*, 16, no. 3, pp. 334–346, 2011.
14. V. V. Uchaikin and R. T. Sibatov, "Fractional kinetics in solids: anomalous charge transport in semiconductors, dielectrics and nanosystems," *World Scientific*, 2012.
15. C. A. Monje. Y. Q. Chen, B. M. Vinagre, D. Xue, and V. Feliu-Batlle, "Fractional-order systems and controls:

fundamentals and applications," *Springer Science and Business Media*, 2010.
16. R. Hilfer, "Applications of fractional calculus in physics," *World scientific*, 2000.
17. R. Klages, G. Radons, and I. M. Sokolov, "Anomalous transport," *Wiley-VCH Verlag*, 2008.
18. M. Caputo. "Linear models of dissipation whose Q is almost frequency independent-II," *Geophysical journal international*, 13, no. 5, pp. 529–539, 1967.
19. N. A. Abdulhameed, O. H. Mohammed, and A. A. Yousif, "A hybrid technique for approximating the solution of fractional order integro differential equations," *Partial Differential Equations in Applied Mathematics*, vol. 8, pp. 100552, 2023.
20. H. M. Srivastava, S. Deniz, and K. M. Saad, "An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator," *Journal of King Saud University-Science*, vol. 33, no. 2, pp. 101345, 2021.
21. H. A. Salim, B. M. Abdali, F. A. Abdulkhaleq, and O. H. Mohammed, "Perturbation iteration transform method for solving fractional order integro-differential equation," *Partial Differential Equations in Applied Mathematics*, vol. 11, pp. 100874, 2024.
22. S. Deniz and N. Bildik, "Applications of optimal perturbation iteration method for solving nonlinear differential equations," *AIP Conference Proceedings, AIP Publishing*, vol. 1798. no. 1. 2017.
23. S. Deniz and N. Bildik, "A new analytical technique for solving Lane-Emden type equations arising in astrophysics," *Bulletin of the Belgian Mathematical Society-Simon Stevin*, vol. 24. no. 2, pp. 305–320, 2017.
24. S. Deniz and N. Bildik, "Optimal perturbation iteration method for Bratu-type problems," *Journal of King Saud University-Science*, vol. 30, no. 1, pp. 91–99, 2018.
25. M. Dalir and M. Bashour, "Applications of fractional calculus," *Applied Mathematical Sciences*, vol. 4. no. 21, pp. 1021–1032, 2010.
26. J. K. Mohammed and A. Khudair, "Numerical solution of fractional integro-differential equations via fourth-degree hat functions," *Iraqi Journal for Computer Science and Mathematics*, vol. 4, no. 2, pp. 10–30, 2023.
27. R. Hussein and M. Khenissi, "Homotopy analysis method for solving fuzzy Nonlinear Volterra integral equation of the second kind," *Iraqi Journal for Computer Science and Mathematics*, vol. 5, no. 3, pp. 24, 2024.
28. A. E. Salih and T. T. Ashezua, "Solving partial integro-differential equations using Laplace-adomian decomposition method," *Unilag Journal of Mathematics and Applications* 1, no. 1, pp. 143–153, 2021.
29. H. F. Mohammed and O. H. Mohammed, "A hybrid technique for solving fractional delay variational problems by the shifted Legendre polynomials," *Partial Differential Equations in Applied Mathematics*, vol. 9 pp. 100635, 2024.
30. A. I. Khlaif, O. H. Mohammed, and M. Feki, "Conformable variational iteration method for solving fuzzy variable-order fractional partial differential equations with proportional delay," *Partial Differential Equations in Applied Mathematics*, vol.13, pp. 101064, 2025.
31. A. A. Yousif, F. A. AbdulKhaleq, A. K. Mohsin, O. H. Mohammed, and A. M. Malik, "A developed technique of homotopy analysis method for solving nonlinear systems of Volterra integro-differential equations of fractional order," *Partial Differential Equations in Applied Mathematics*, vol. 8 pp. 100548, 2023.
32. A. I. Khlaif, O. H. Mohammed, and M. Feki, "A modified method for solving delay fuzzy variable-order fractional partial differential equations," *Commun. Math. Biol. Neurosci*, 2024.