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Ring Classification of Ideal-Based Zero Divisor Graph with Vertices 9

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Keywords: Zero Divisor Graph, Ideal-Based Zero Divisor Graph, Direct Product, Finite Ring, Local Ring.

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Abstract:

Let R be a finite commutative ring with a non-zero unit, and L be an ideal of R. focuses on expanding the notation of the Zero Divisor Graph to create what is known as the Ideal-Based Zero Divisor Graph. The main goal is to classify rings using the ideal-based Zero divisor graph that consists of 9 vertices and symbolizes ($\Gamma_L(R)$) by using the properties $|V(\Gamma_L(R))|=|L|.|V((\Gamma(R/L))|$, $|L|\geq 2$.

Keywords: Zero Divisor Graph, Ideal-Based Zero Divisor Graph, Direct Product, Finite Ring, Local Ring.

(Immediately after the abstract, provide 5-7 keywords and arrange them alphabetically, using American spelling and avoiding general and plural terms and multiple concepts (avoid, for example, ‘and’, ‘of’). Be sparing with abbreviations: only abbreviations firmly established in the field may be eligible. These keywords will be used for indexing purposes).

تصنيف الحلقات ببيان قاسم الصفر أساسه المثالي بتسعة رؤوس

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الخلاصة:

لتكن G حلقة ابدالية منتهية بعنصر محايد ليس صفريا، وليكن L مثالي في الحلقة R، تمركز هذا البحث على تعميم وتوسيع مصطلح بيان قاسم الصفر الى بيان قاسم الصفر الذي أساسه المثالي والذي يرمز له $(\Gamma_L(R))$. في هذا البحث قمنا بتصنيف جميع الحلقات التي لها بيان قاسم الصفر الذي أساسه المثالي وعدد رؤوسه 9. بالاعتماد على الخاصية $|V(\Gamma_L(R))|=|L|.|V((\Gamma (R/L))|$, $|L|\geq 2$.

الكلمات المفتاحية: بيان قاسم الصفر، بيان قاسم الصفر أساسه المثالي، الضرب المباشر، الحلقة المنتهية، الحلقة المحلية.

1. Introduction:

Let R be a commutative ring with identity and Z(R) the collection of all zero divisor elements in R. The polynomial ring R[X] is defined to be the set of all formal sums $a_0+ a_1 X+\ldots+a_nX_n+ a_{n+1}X_{n+1} +\ldots= \Sigma a_iX_i$, where the coefficients $a_i \in R$ for i=0, 1, 2, …n, n+1, … When working with polynomial rings, say R[X]/L, we will let X denote the coset X+L. It is well-known that any finite ring R is a direct product of local rings R_i for i=1, 2, …, n.

In 2003, SP. Redmond [1]introduced a new definition called an ideal-based zero-divisor graph, denoted by $\Gamma_L(R)$, which is defined as a statement in which the vertices, $r_1, r_2 \in R-L$ are adjacent if, $r_1.r_2 \in L$, where L is an ideal of R. This definition generalizes the zero-divisor graph proposed by D. F. Anderson and P. S. Livingston [2], which has the vertices, $r_1, r_2 \in Z(R)^*$. adjacent if and only if $r_1.r_2 = 0$, which is denoted by Γ(R). If L = 0 then $\Gamma_L(R) = \Gamma(R)$ and $\Gamma_L(R) = \emptyset$ if and only if L is a prime ideal of R, additionally, $\Gamma_L(R)$ is a connected graph. [3] The study of algebraic graphs, which is considered a modern and important topic linking the theory of rings in abstract algebra and graph theory, has received wide attention from researchers, for example, [4,5,6].

Clearly, every ring has $\Gamma_L(R)$ graph, and if $R_1 \cong R_2$, then $\Gamma_L(R_1) \cong \Gamma_L(R_2)$, but the convers is not true in general as well as there are $\Gamma_L(R_1) \cong \Gamma_L(R_2)$,but $R_1 \ncong R_2$. Redmond proved the following relationship: $|V(\Gamma_L(R)| = |L| \cdot |V(\Gamma(R/L))|$ and $|L| \geq 2$. Using this mathematical expression, researchers in [7,8] were able to find all rings corresponding to the ideal L with the number of vertices n, where 1≤ n≤ 7 or n is a prime number. In this work, we used this relationship to classify a ring when an ideal-based zero divisor graph with vertices 9. In [2] it is provided that Γ(R) is connected, if R is an integral domain, then Γ(R) is empty, Γ(R) has - finitely many vertices if, either R is finite or an integral domain, diam (Γ (R)) ≤ 3 [9,10] Gan and Yang presented the Zero-divisor graphs of MV-algebras

2. Ring | $V(\Gamma_L(R))$| =9

In this section, we give all possible rings with $|V(\Gamma_L(R))| = 9$.

2.1 Remark: We consider when a non-trivial $\Gamma_L(R)$ is the graph on 9 vertices since $|V(\Gamma_L(R)|=|V(\Gamma(R/L)|.|L|$ and $|L| \geq 2$ we get two possibilities :-

1- $|L| = 9$ and $|V(\Gamma(R/L)| = 1$.

2- $|L| = 3$ and $|V(\Gamma(R/L)| = 3$.

2.2 Proposition: When R is a ring with $|\Gamma_L(R)| = 9$, then R is nonlocal ring

Proof: We impose R to be a local ring with $|\Gamma_L(R)| = 9$, therefore by remark 2.1 there are two cases:

Case 1: If $|L|=9$ and $|V(\Gamma(R/L)| =1$, then by [11,12] , $R/L \cong Z_4$ or $Z_2[A]/(A^2)$ and so $|(R/L)| = 4$ which implies that $|R| = 9.4=36$, but R local which is a contradiction.

Case 2: If $|L|=3$ and $|V(\Gamma(R/L)| =3$, then by [4,12] $R/L \cong Z_8$, $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, $Z_4[A]/(A^2-2,2A)$ or $F_4[A]/(A^2)$. If $R/L \cong Z_8$, $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$ or $Z_4[A]/(A^2-2,2A)$ then $|(R/L)| =8$ therefore $|R| =24$, but R local which is a contradiction. When $R/L \cong F_4[A]/(A^2)$, the order $|R/L| = 16$, therefore $|R| =48$, but R local which is a contradiction. So R is not local.

2.3 Theorem: let R be isomorphic with $R_1 \times R_2$, when R_1, R_2 will be local rings and $|V(\Gamma_L(R))| =9$, therefore R isomorphic to one of 12 rings with corresponding ideal L from **Table 1**.

Table 1: R isomorphic with $R_1 \times R_2$

Ring	Ideal	Figure
$Z_9 \times Z_4$	$Z_9 \times (0)$	K_9
$Z_9 \times Z_2[A]/(A^2)$	$Z_9 \times (0)$	K_9
$Z_3[A]/(A^2) \times Z_4$	$Z_3[A]/(A^2) \times (0)$	K_9
$Z_3[A]/(A^2) \times Z_2[A]/(A^2)$	$Z_3[A]/(A^2) \times (0)$	K_9
$Z_9 \times Z_2$	$(3) \times (0)$	1
$Z_3[A]/(A^2) \times Z_2$	$(A) \times (0)$	1
$Z_8 \times Z_3$	$(0) \times Z_3$	K_9
$Z_4[A]/(2A, A^2) \times Z_3$	$(0) \times Z_3$	K_9
$Z_2[A]/(A^3) \times Z_3$	$(0) \times Z_3$	2
$Z_2[A,B]/(A,B)^2 \times Z_3$	$(0) \times Z_3$	K_9
$Z_4[A]/(2A, A^2-2) \times Z_3$	$(0) \times Z_3$	K_9
$F_4[A]/(A^2) \times Z_3$	$(0) \times Z_3$	K_9

Proof: If |L| = 9 and $|V(\Gamma(R/L)| = 1$, then $(R/L) \cong Z_4$ or $Z_2[A]/(A^2)$ so that | R | = 36, then $|R_1| = 9$ and $|R_2| = 4$ so $R_1 \cong Z_9$ or $Z_3[A]/(A^2)$ and $R_2 \cong Z_4$, or $Z_2[A]/(A^2)$, then R isomorphic one of 4 rings with corresponding ideal L from **Table 2**.

Table 2: | R | = 36

Ring	Ideal
$Z_9 \times Z_4$	$Z_9 \times (0)$
$Z_9 \times Z_2[A]/(A^2)$	$Z_9 \times (0)$
$Z_3[A]/(A^2) \times Z_4$	$Z_3[A]/(A^2) \times (0)$
$Z_3[A]/(A^2) \times Z_2[A]/(A^2)$	$Z_3[A]/(A^2) \times (0)$

If |L| = 3 and $|V(\Gamma(R/L)| = 3$, then by **[4,5]** $R/L \cong Z_6, Z_8, Z_2[A]/(A^3), Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, $Z_4[A]/(A^2 - 2,2A)$ or $F_4[A]/(A^2)$.

If $(R/L) \cong Z_6$, then | R | = 18, $|R_1| = 9$, and $|R_2| = 2$ then $R_1 \cong Z_9$ or $Z_3[A]/(A^2)$ and $R_2 \cong Z_2$. Hence R isomorphic one of 2 rings with depending on the ideal L from **Table 3**.

Table 3: | R | = 18

Ring	Ideal
$Z_9 \times Z_2$	$(3) \times (0)$
$Z_3[A]/(A^2) \times Z_2$	$(A) \times (0)$

Now, when $R/L \cong Z_8$, $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, or $Z_4[A]/(A^2 - 2,2A)$, then | R | = 24, and we get $|R_1| = 8$, $|R_2| = 3$, then R will be one of 5 rings with corresponding ideal L from **Table 4**.

Table 4: | R | = 24

Ring	Ideal
$Z_2[A]/(A^3) \times Z_3$	$(A) \times (0)$
$Z_8 \times Z_3$	$(0) \times Z_3$
$Z_4[A]/(2A,A^2) \times Z_3$	$(0) \times Z_3$
$Z_2[A]/(A^3) \times Z_3$	$(0) \times Z_3$
$Z_2[A,B]/(A,B)^2 \times Z_3$	$(0) \times Z_3$

When $R/L \cong F_4[A]/(A^2)$, we have | R/L | =16, so | R | = 48 where $|R_1| = 16$, then R_1 isomorphism of one 18 rings from **Table 5** and $|R_2| = 3$, $R_2 \cong Z_3$, there is only one ring of order 16 it is $F_4[A]/(A^2)$, $R \cong F_4[A]/(A^2) \times Z_3$, with $I = (0) \times Z_3$.

Table 5: | R | = 16

Ring
$F_2[A]/(A^4)$
$Z_4[A]/(A^2-2)$
$Z_4[A]/(A^2–2A-2)$
$Z_4[A]/(A^2-2A)$
$Z_4[A]/(A^3–2,2A)$
$F_2[A,B]/(A^3,AB,B^2)$
$F_2[A,B]/(AB,A^2 -B^2)$
$F_2[A,B]/(A^2,B^2)$
$Z_{p^2}[A,B]/(A^2,AB- 2,B^2)$
$Z_4[A]/(A^2)$
$Z_4[A,B]/(A^2 – 2,AB,B^2,2A)$
$Z_4[A]/(A^3 ,2A)$
$Z_8[A]/(A^2 - 4,2A)$
$Z_8[A]/(A^2,2A)$
$Z_4[A,B]/(A,B,2)^2$
$F_2[A,B,C]/(A,B,C)^2$
$Z_4[A,B]/(A^2–2,AB,B^2 - 2,2A)$
Z_{16}

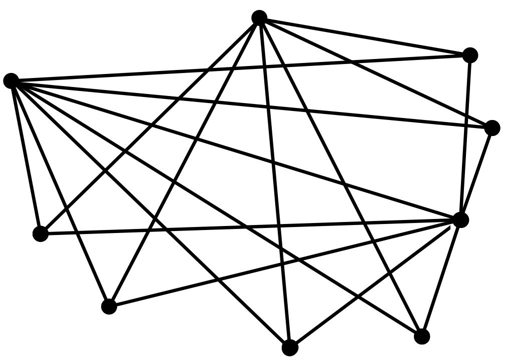

Figure 1

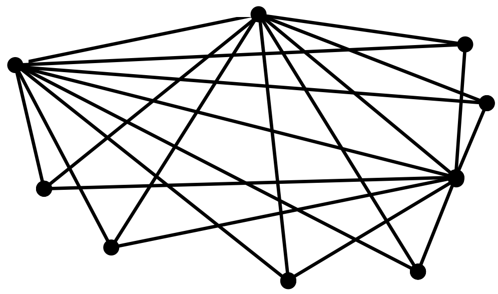

Figure 2

Figures 1 & 2: Ideal-Based Zero Divisor Graph with Vertices 9

2.4 Theorem:

When $R \cong R_1 \times R_2 \times R_3$, where R_i local ring $i \in \{1,2,3\}$ as well as $|V(\Gamma_L(R))|=9$; therefore, R isomorphic to one of 3 rings with depending on ideal L from **Table 6**.

Table 6: | R | = 36

Ring	Ideal	Figure
$Z_3 \times Z_3 \times Z_2$	$Z_3 \times Z_3 \times (0)$	K_9
$Z_3 \times Z_3 \times Z_2 [A]/(A^2)$	$Z_3 \times Z_3 \times (0)$	K_9
$Z_3 \times Z_3 \times Z_2$	$Z_3 \times (0) \times (0)$	1

Proof:

1- Suppose $|L| = 9$ and $|V(\Gamma(R/L)| = 1$, we note $(R/L) \cong Z_4$ or $Z_2[A]/(A^2)$ so that $|R| = 36$, which implies that $R \cong R_1 \times R_2 \times R_3$, where $|R_1| = |R_2| = 3$ and $|R_3| = 4$, so that R_1 and $R_2 \cong Z_3$,& $R_3 \cong Z_4$ or $Z_2[A]/(A^2)$, then $R \cong Z_3 \times Z_3 \times Z_4$ or $Z_3 \times Z_3 \times Z_2[A]/(A^2)$ with $L = Z_3 \times Z_3 \times (0)$.

2- When $|L| = 3$ and $|V(\Gamma(R/L)| = 3$, we see $R/L \cong Z_6$ or Z_8 or $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, $Z_4[A]/(A^2 - 2,2A)$ or $F_4[A]/(A^2)$.

If $R/L \cong Z_6$, then $|R| = 18$, we have $R \cong R_1 \times R_2 \times R_3$, we see $|R_1| = |R_2| = 3$ & $|R_3| = 2$. Then R is isomorphic $Z_3 \times Z_3 \times Z_2$, with $L = Z_3 \times (0) \times (0)$.

But if $R/L \cong Z_8$, $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$ or $Z_4[A]/(A^2 - 2,2A)$ then $|R| = 24$, we have $R \cong R_1 \times R_2 \times R_3$, such that $|R_1| = 2$, $|R_2| = 3$ and $|R_3| = 4$. Then $R \cong Z_2 \times Z_3 \times Z_4$ or $Z_3 \times Z_2 \times Z_2[A]/(A^2)$, with $L = (0) \times Z_3 \times (0)$. But $R/L \ncong Z_8, Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$ or $Z_4[A]/(A^2 - 2,2A)$ we get contradiction.

If $G/L \cong F_4[A]/(A^2)$, then $|(R/L)| = 16$, so $|R| = 48$ where $|R_1| = |R_2| = 4$ and $|R_3| = 3$. Then $R \cong Z_4 \times Z_4 \times Z_3$, $Z_2[A]/(A^2) \times Z_2[A]/(A^2) \times Z_3$ or $Z_4 \times Z_2[A]/(A^2) \times Z_3$, with $L = (0) \times (0) \times Z_3$. But $G/L \ncong F_4[A]/(A^2)$ we get a contradiction.

2.5 Theorem:

If R is isomorphic with $R_1 \times R_2 \times R_3 \times \ldots \times R_n$, where $n \geq 4$, and R_i local rings for all $i = 1,\ldots,n$ with $|V(\Gamma_L(R))| = 9$, we get G no isomorphic with any ring

Proof: If $n \geq 5$, then by same method in proof theorem 1.4, then we get contradiction. Let $n = 4$, then by remark 1.1, we have two cases:

Case 1: If $|L| = 9$ and $|V(\Gamma(R/L)| = 1$, then $(R/L) \cong Z_4$ or $Z_2[A]/(A^2)$ so that $R| = 36$, which implies that $R \cong R_1 \times R_2 \times R_3 \times R_4$, such that $|R_1| = |R_2| = 3$ & $|R_3| = |R_4| = 2$, so that R_1 & $R_2 \cong Z_3$ but $R_3 \cong Z_2$, then $R \cong Z_3 \times Z_3 \times Z_2 \times Z_2$ with $L = Z_3 \times Z_3 \times (0) \times (0)$. But $R/L \ncong Z_4$ or $Z_2[A]/(A^2)$, we get a contradiction.

Case 2: If $|L| = 3$ and $|V(\Gamma(R/L)| = 3$, then $R/L \cong Z_6$, Z_8, $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, $Z_4[A]/(A^2 - 2,2A)$ or $F_4[A]/(A^2)$. If $R/L \cong Z_6$, then $|R| = 18$ there is no existence R of order 18, ∴ R is isomorphic $R_1 \times R_2 \times R_3 \times R_4$, we get contradicts. If $R/L \cong Z_8$,

$Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$ or $Z_4[A]/(A^2 - 2,2A)$, then | R | = 24 and $R \cong Z_2 \times Z_2 \times Z_2 \times Z_3$ with $L = (0) \times (0) \times (0) \times Z_3$, but $R/L \ncong Z_8$ or $Z_2[A]/(A^3)$, $Z_2[A,B]/(A,B)^2$, $Z_4[A]/(A^2,2A)$, $Z_4[A]/(A^2 - 2,2A)$, we have a contradiction. If $R/L \cong F_4[A]/(A^2)$, then | R | =48. So that $R \cong Z_2 \times Z_2 \times Z_3 \times Z_4$ with $L = (0) \times (0) \times Z_3 \times (0)$. But $R/L \ncong F_4[A]/(A^2)$ contradiction.

3. Results

Ring| V(Γ$_L$(R))| =9 shown in **Tables [1-6]**

4. Discussion

If R commutative non local ring with identity, I an ideal of R and $\Gamma_I(R)$ zero divisor graph with ideal based I and $V(\Gamma_I(R)) = 9$, then there are three graphs (Fig 1, Fig 2, K_9) realized ring R with respect ideal I. Additionally, if R direct product n local rings, where $n \geq 4$, then no graph realized R.

5. Conclusions

Finally, we expand the notation of the Zero divisor graph to create what is known as the Ideal-Based Zero Divisor Graph, in our research we find and classify all rings using the ideal-based Zero divisor graph that consists of 9 vertices as shown in **Tables (1-6)** and find the graph of Ring | V(Γ$_L$(R))| =9 as shown in figures **(1,2)**.

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