Iterated bivariate rayleigh distribution

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الخلاصة

ان توزيع رالي هو احد توزيعات الحياة وهو حالة خاصة من توزيع ويبل وله كثير من الاستخدامات في مجالات الحياة المختلفة، المالية، معالجة الإشارات والاتصالات. ان الصلات هي دوال تجمع دوال التوزيع الحدية والتي تعتبر متغيرات عشوائية ذات نوزيع منتظم على الفترة (1,0). فالصلة هي وسيله لبناء عوائل للتوزيعات الثنائية وهي مقباس للأعتمادية بين متغيرين لأنها تسمح لنا فصل تأثير الأعتمادية من تأثيرات التوزيعات الحدية.

في هذا البحث تم اشتتقاق توزيع رالي الثنائي المكرر باستخدام مفهوم الصلات مع مناقشة بعض الخواص، على سبيل المثال دالة التوزيع، دالة الكثافة الأحتمالية، الدوال الشرطية، التوقع الشرطي، التباين المشترك ومعامل الأرتباط.

الكلمات المفتاحية

توزيع رالي، الصلات، توزيع ثنائي المتغيرين، دالة كثافة الإحتمالية الشرطية، المعامل المترابط، التوقع المشروط.

Abstract

The Rayleigh distribution is one of the lifetime distributions and a special case from Weibull distribution. It has widely used in many fields in real life, finance, signal processing, and communications. Copulas are functions that join their one-dimensional marginal distribution functions which are uniform on the interval (0,1). The copula is an important tool for constructing families of bivariate distributions and it is measure of dependence between two variables since it allows us to separate the effect of dependence from the effects of the marginal distributions.

In this paper, we derive iterated bivariate Rayleigh distribution using the concept of copula with discussion of some properties, like the cdf, pdf, conditional pdf's, conditional expectation, covariance and correlation coefficient.

Keywords

copulas, Rayleigh distribution, bivariate distribution, conditional probability density function, conditional expectation ,correlation coefficient.

1. Introduction

Rayleigh, [1] noted that the data about the wave heights, wave length, wave induce pitch, wave and heave motions of the ships follow the Rayleigh distribution which was derived from the bivariate normal distribution when the variables are independent with equal variances.

The concept of copula was established by Sklar A. [2] when he studied the relationship between a multidimensional probability function and its lower dimensional margins.

Quesada-Molina, J., J., Rodrguez-Lallena, J. A., and beda-Flores, M., [3] presented a theory of copulas with some of the results and various examples.

Abdel-Hady, D.,[4] has studied the generalized bivariate Rayleigh (GBR) distribution its the cumulative distribution function, the probability density function, the conditional distribution of the BGR distribution and the maximum likelihood estimator.

Zeng, X., Ren, J., Wang, Z., Marshall, S., Durrani, T., [5] derived new bivariate copulas for Exponential, Weibull and Rician distributions. They proved that the three copula functions of these distributions are equivalent, and also showed that the copula function of log-normal distribution is quivalent to the Gaussian copula.

Sarabia, J., M., Prieto F. and Jord V., [6] introduced three new classes of bivariate beta-generated distributions with main properties.

In this search we derive the iterated Bivariate Rayleigh distribution which can be used in the lifetime phenomena, like finance, signal processing, and communications

2. Some important concepts

2. 1. Rayleigh distribution [7]

The Rayleigh random variable has the distribution function as

$$F(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0, \sigma > 0 \quad .(1)$$

And its probability density function is

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0, \sigma > 0 \quad .(2)$$

Therefore the mean and the variance of X are as follows

$$E(X) = \sigma\sqrt{\frac{\pi}{2}}, \quad V(X) = \frac{4-\sqrt{\pi}}{2}\sigma^2$$

2. 2. Copula [3]

Definition 2. 2. 1: A concept of copula can be defined as a function C: $[0, 1]^2 \rightarrow [0, 1]$ that satisfies the following:

a. $C(u, 0) = 0 = C(0, v)$, and $C(u, 1) = u$ and $C(1, v) = v$, for every u, v in [0, 1],

b. $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$, for every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

2. 3. Theorem [8]

Let be a joint distribution function with marginals F_1 and F_2. Then, there exists a copula C such that, for all $x, y \in [-\infty, \infty]$,

$$F(x, y) = C(F_1(x), F_2(y)) \quad .(3)$$

And the p. d, f, is

$$f(x, y) = c(F_1(x), F_2(y)) f_1(x) f_2(y) \quad .(4)$$

where f, f_1, f_2, and c be the density functions of F, F_1, F_2 and *C*, respectively.

3. Iterated F. G. M. bivariate rayleigh distribution [2]

The joint pdf, and cdf of "Iterated Bivariate Rayleigh Distribution", I BRD, using, (3), (4) which are as follows:

$$C(u_1,v_1) = u_1v_1[1 + \alpha(1 - u_1)(1 - v_1) + \beta u_1v_1(1 - u_1)(1 - v_1)] \quad .(5)$$

$$c(u_1,v_1) = 1 + \alpha(1 - 2u_1)(1 - 2v_1) + \beta u_1v_1(2 - 3u_1)(2 - 3v_1)$$

$$\text{where } u_1 = F_1(x_1), v_1 = F_2(x_2) \text{ and } -1 \leq \alpha \leq 1, -1 - \alpha \leq \beta \leq (3 - \alpha + \sqrt{9 - 6\alpha - 3\alpha^2})/2 \quad .(6)$$

That is, the cdf is

$$F(x_1,x_2) = F_1(x_1)F_2(x_2)\Big(1 + \alpha(1 - F_1(x_1))(1 - F_2(x_2)) + \beta F_1(x_1)F_2(x_2)(1 - F_1(x_1))(1 - F_2(x_2))\Big)$$

$$= \left(1 - e^{-\frac{x_1^2}{2\sigma_1^2}}\right)\left(1 - e^{-\frac{x_2^2}{2\sigma_2^2}}\right)\left[\left(1 + \alpha e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}}\right) + \beta\left(1 - e^{-\frac{x_1^2}{2\sigma_1^2}}\right)\left(1 - e^{-\frac{x_2^2}{2\sigma_2^2}}\right) e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}}\right]$$

$$= 1 - e^{-\frac{x_2^2}{2\sigma_2^2}} - e^{-\frac{x_1^2}{2\sigma_1^2}} + e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} + \alpha\Big(e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} - e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{\sigma_2^2}} - e^{-\frac{x_1^2}{\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} + e^{-\frac{x_1^2}{\sigma_1^2}} e^{-\frac{x_2^2}{\sigma_2^2}}\Big) + \beta\Big(e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} - 2e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{\sigma_2^2}} - 2e^{-\frac{x_1^2}{\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} + 4e^{-\frac{x_1^2}{\sigma_1^2}} e^{-\frac{x_2^2}{\sigma_2^2}} - 2e^{-\frac{3x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{\sigma_2^2}} + e^{-\frac{3x_1^2}{2\sigma_1^2}} e^{-\frac{x_2^2}{2\sigma_2^2}} + e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{3x_2^2}{2\sigma_2^2}} - 2e^{-\frac{x_1^2}{\sigma_1^2}} e^{-\frac{3x_2^2}{2\sigma_2^2}} + e^{-\frac{3x_1^2}{2\sigma_1^2}} e^{-\frac{3x_2^2}{2\sigma_2^2}}\Big) \quad .(7)$$

Fig.. (1) shows example of bivariate distribution (7)

The pdf is

$$f(x_1,x_2) = f_1(x_1)f_2(x_2)\left[1+\alpha\left(1-2F_1(x_1)\right)\left(1-2F_2(x_2)\right) + \beta F_1(x_1)F_2(x_2)\left(2-3F_1(x_1)\right)\left(2-3F_2(x_2)\right)\right]$$

$$= \frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + \alpha\left(4\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} - 2\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} - 2\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} + \frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}}\right) + \beta\left(16\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} - 4\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} - 4\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} + 3\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{3x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + 3\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{3x_2^2}{2\sigma_2^2}} - 12\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{3x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} - 12\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{3x_2^2}{2\sigma_2^2}} + 9\frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{3x_1^2}{2\sigma_1^2}}e^{-\frac{3x_2^2}{2\sigma_2^2}} + \frac{x_1}{\sigma_1^2}\frac{x_2}{\sigma_2^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}}\right) \quad .(8)$$

Fig.(2): shows example of bivariate distribution And the conditional pdf is

$$f(x_1/x_2) = \frac{f(x_1,x_2)}{f(x_2)}$$

$$= \frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}} + \alpha\left(4\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} - 2\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{\sigma_1^2}} - 2\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + \frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}}\right) + \beta\left(16\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} - 4\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{\sigma_1^2}} - 4\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + 3\frac{x_1}{\sigma_1^2}e^{-\frac{3x_1^2}{2\sigma_1^2}} + 3\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} - 12\frac{x_1}{\sigma_1^2}e^{-\frac{3x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{2\sigma_2^2}} - 12\frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} + 9\frac{x_1}{\sigma_1^2}e^{-\frac{3x_1^2}{2\sigma_1^2}}e^{-\frac{x_2^2}{\sigma_2^2}} + \frac{x_1}{\sigma_1^2}e^{-\frac{x_1^2}{2\sigma_1^2}}\right) \quad .(9)$$

Proposition 3. 1

If (X_1, X_2) ~IBRD (σ_1, σ_2), then

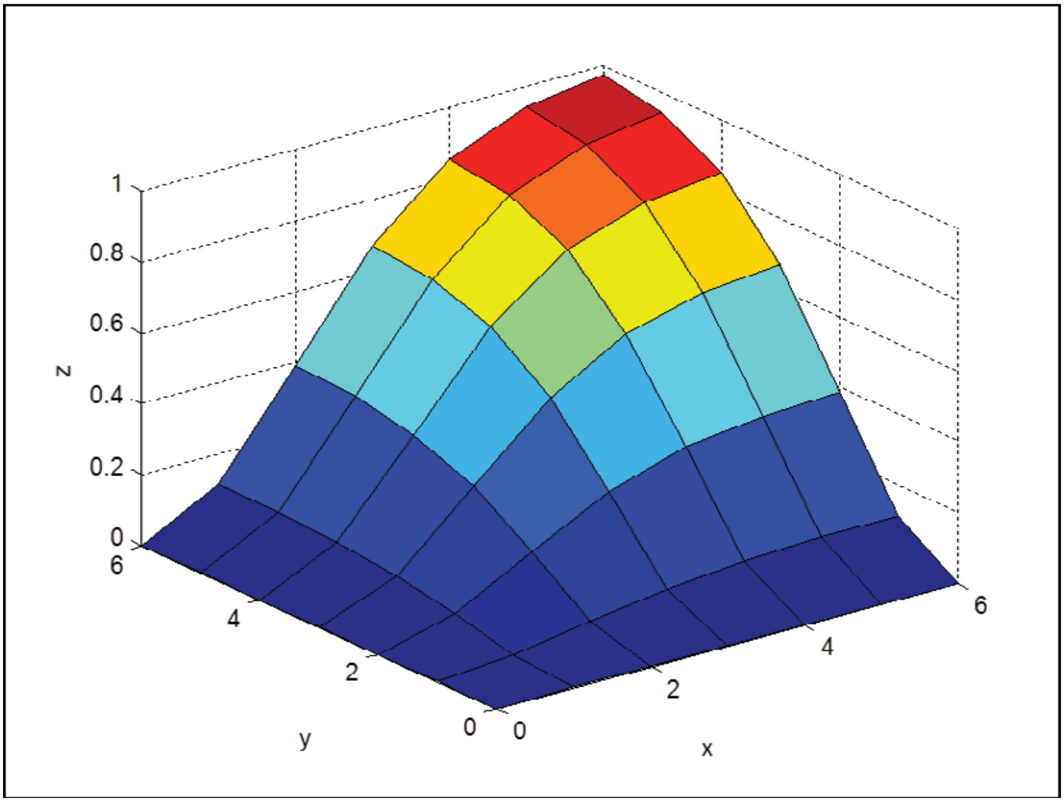

Fig.(1): Where

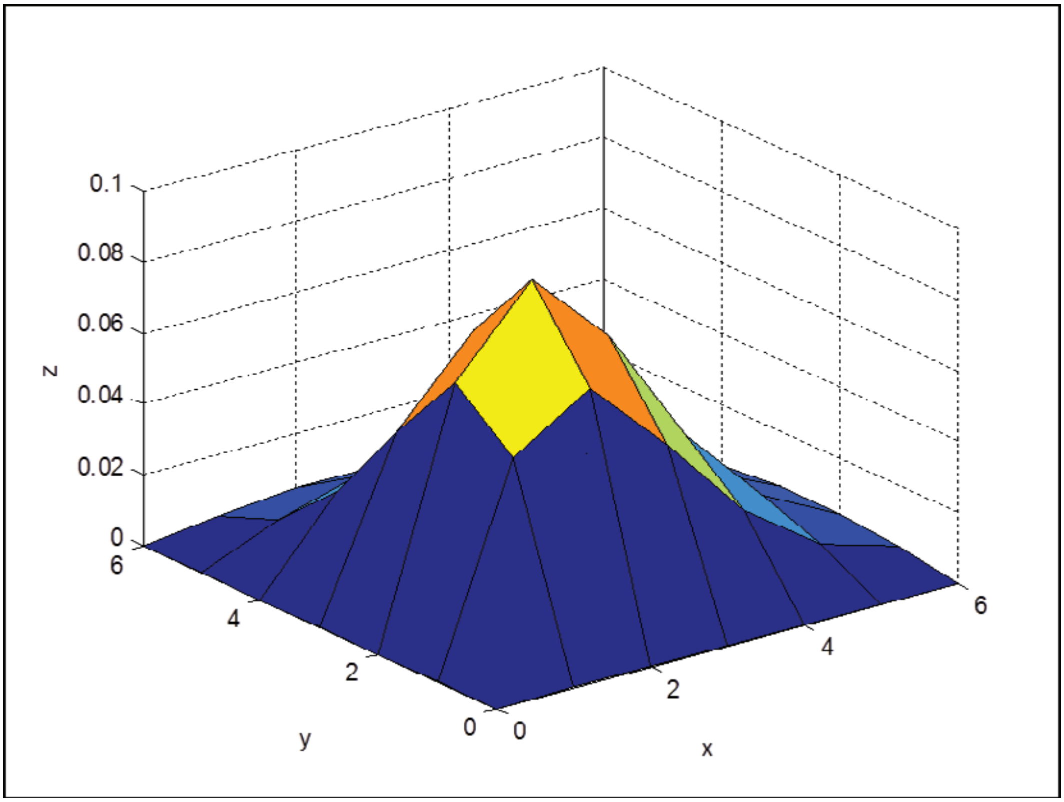

Fig.(2): Where

$$X_1 \sim R(\sigma_1) \text{and } X_2 \sim R(\sigma_2) \quad .1$$

$$E(X_1/X_2) = \gamma + \alpha\gamma\gamma_1 - 2\alpha\gamma\gamma_1 e^{-\frac{x_2^2}{2\sigma_2^2}} + \beta\gamma\gamma_2 - 4\beta\gamma\gamma_2 e^{-\frac{x_2^2}{2\sigma_2^2}} + 3\beta\gamma\gamma_2 e^{-\frac{x_2^2}{\sigma_2^2}} \quad .2$$

$$E(X_1X_2) = \gamma\lambda + \alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2 \quad .3$$

$$\operatorname{cov}(X_1, X_2) = \alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2 \quad .4$$

$$\operatorname{corr}(X_1, X_2) = \frac{\alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2}{\sqrt{4 - 2\lambda - 2\gamma + \lambda\gamma}} \quad .5$$

Proof of (1)

$$f(x_1) = \int_0^\infty f(x_1, x_2)\, dx_2$$

$$= \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}} + \alpha\left(2\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{\sigma_1^2}} - 2\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{\sigma_1^2}} - \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}} + \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}}\right)$$

$$+ \beta\left(8\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{\sigma_1^2}} - 4\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{\sigma_1^2}} - 2\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}} + 3\frac{x_1}{\sigma_1^2} e^{-\frac{3x_1^2}{2\sigma_1^2}} + \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}}\right.$$

$$\left. - 6\frac{x_1}{\sigma_1^2} e^{-\frac{3x_1^2}{2\sigma_1^2}} - 4\frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{\sigma_1^2}} + 3\frac{x_1}{\sigma_1^2} e^{-\frac{3x_1^2}{2\sigma_1^2}} + \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}}\right)$$

$$= \frac{x_1}{\sigma_1^2} e^{-\frac{x_1^2}{2\sigma_1^2}}$$

and f (x_2) is found similarly.

Proof of (2)

$$E(X_1/X_2) = \int_0^\infty x_1 f(x_1/x_2)dx_1$$

$$= \sigma_1\sqrt{\frac{\pi}{2}} + \alpha(\sigma_1\sqrt{\pi}e^{-\frac{x_2^2}{2\sigma_2^2}} - \frac{1}{2}\sigma_1\sqrt{\pi} - 2\sigma_1\sqrt{\frac{\pi}{2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + \sigma_1\sqrt{\frac{\pi}{2}})$$

$$+ \beta\left(4\sigma_1\sqrt{\pi}e^{-\frac{x_2^2}{2\sigma_2^2}} - \sigma_1\sqrt{\pi} - 4\sigma_1\sqrt{\frac{\pi}{2}}e^{-\frac{x_2^2}{2\sigma_2^2}} + \frac{1}{\sqrt{6}}\sigma_1\sqrt{\pi} + 3\sigma_1\sqrt{\frac{\pi}{2}}e^{-\frac{x_2^2}{\sigma_2^2}}\right.$$

$$\left. - \frac{4}{\sqrt{6}}\sigma_1\sqrt{\pi}e^{-\frac{x_2^2}{2\sigma_2^2}} - 3\sigma_1\sqrt{\pi}e^{-\frac{x_2^2}{\sigma_2^2}} + \frac{3}{\sqrt{6}}\sigma_1\sqrt{\pi}e^{-\frac{x_2^2}{\sigma_2^2}} + \sigma_1\sqrt{\frac{\pi}{2}}\right)$$

$$= \gamma + \alpha\gamma\gamma_1 - 2\alpha\gamma\gamma_1 e^{-\frac{x_2^2}{2\sigma_2^2}} + \beta\gamma\gamma_2 - 4\beta\gamma\gamma_2 e^{-\frac{x_2^2}{2\sigma_2^2}} + 3\beta\gamma\gamma_2 e^{-\frac{x_2^2}{\sigma_2^2}}$$

$$\text{where } \gamma = \sigma_1\sqrt{\frac{\pi}{2}}, \gamma_1 = \frac{2-\sqrt{2}}{2}, \gamma_2 = \frac{\sqrt{3}-\sqrt{6}+1}{\sqrt{3}} \qquad .(10)$$

Then by the Same way, we get

$$E(X_2/X_1) = \lambda + \alpha\lambda\gamma_1 - 2\alpha\lambda\gamma_1 e^{-\frac{x_1^2}{2\sigma_1^2}} + \beta\lambda\gamma_2 - 4\beta\lambda\gamma_2 e^{-\frac{x_1^2}{2\sigma_1^2}} + 3\beta\lambda\gamma_2 e^{-\frac{x_1^2}{\sigma_1^2}}$$

$$\text{where } \lambda = \sigma_2\sqrt{\frac{\pi}{2}} \qquad .(11)$$

Proof of (3)

$$E(X_1X_2) = \int_0^\infty x_2 E(X_1/X_2) f(x_2)dx_2 \qquad .(12)$$

$$= \gamma\sigma_2\sqrt{\frac{\pi}{2}} + \alpha\gamma\gamma_1\sigma_2\sqrt{\frac{\pi}{2}} - \frac{1}{2}\alpha\gamma\gamma_1\sigma_2\sqrt{\pi} + \beta\gamma\gamma_2\sigma_2\sqrt{\pi} + \frac{1}{\sqrt{6}}\beta\gamma\gamma_2\sigma_2\sqrt{\pi}$$

$$= \gamma\lambda + \alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2$$

Proof of (4)

$$cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2 . (13)$$

Proof (5)

$$corr(X_1, X_2) = \frac{cov(X_1, X_2)}{\sqrt{var(X_1)var(X_2)}} = \frac{\alpha\lambda\gamma\gamma_1^2 + \beta\lambda\gamma\gamma_2^2}{\sqrt{4 - 2\lambda - 2\gamma + \lambda\gamma}} . (14)$$

4. Conclusions

Copula function provides us with good tool to derive the extension of BRD (σ_1, σ_2), we denote it as IBRD (σ_1, σ_2), Therefore we present some of its properties, like the cdf, pdf, conditional pdf ′s, conditional expectations and correlation.

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