The Improvement of Galerkin-Conservation Finite Element Method for Coupled Burgers' Problem

Assi. Inst. Najat Jalil Noon

Department of Mathematics, College of Education for Pure Science
Basrah University.

The Improvement of Galerkin-Conservation Finite Element Method for Coupled Burgers' Problem

Assi. Inst. Najat Jalil Noon

Abstract: The Burgers' equation is an example of an equation that has unstable Galerkin-Conservation finite element method for very small viscosity coefficients, ϵ, in this paper, a stabilized finite element methods for solving 2-D coupled Burgers' problem is studied, the Galerkin-conservation partial artificial diffusion (G.-C.P.A.D.) and Galerkin-Conservation straight artificial diffusion (G.-C.S.A.D.) finite element methods are used to handle such problem. The Crank-Nicholson method for the time variable is considered. Two numerical examples have been carried out through implementation in MATLAB program to illustrate the efficiency of the proposed methods.

Mathematics Subject Classification: 65NXX, 65N30
keywords: Galerkin -Conservation, G.-C.P.A.D., G.-C.S.A.D., 2-D coupled Burgers' problem.

تحسين طريقة كالركن–المحافظة للعنصر المحدد لمسألة برغرز الثنائية

م.م نجاة جليل نون

المستخلص

معادلة برغرز هي مثال على المعادلة التي تكون فيها طريقة كالركن–المحافظة للعنصر المحدد غير مستقرة مع معاملات اللزوجة الصغيرة جداϵ، في هذا البحث، تم دراسة طرائق مستقرة للعنصر المحدد لحل مسألة برغرز الثنائية ذات البعدين ، للتعامل مع هذه المشكلة تم استخدام طريقة كالركن–المحافظة للانتشار الاصطناعي الجزئي وطريقة كالركن–المحافظة للانتشار الاصطناعي المباشر للعنصر المحدد . أُستُعملت طريقة كرنك–نيكولسن لمتغير الزمن، تم تنفيذ مثالين عدديين من خلال تطبيق برنامج ماتلاب لتوضيح كفاءة الطرائق المقترحة.

1 - Introduction

In recent years, Burgers' equation have received a considerable amount of attention due to the large number of physically important phenomena that can be modeled using this equation. Some attention has been given to the convection-dominated case. Several methods have been intensively studied to remove such a drawback for this problem, we can summarize some of this methods. Atwell and King [7] used the G. and G. L.-S. approximation for 1-D Burgers' equation with the linear feedback control law designed for the non-stabilized problem. Volkwein [15] considered upwind techniques and mixed finite elements for the steady-state Burgers' equation in 1-D to compute solutions for small viscosity parameters. Pugh [14] used G. and the G.−C. finite element methods for the 1-D Burgers' equation, he found that the G.− C. method was more accurate and computed more quickly than the G. method. Smith [8] applied G. and the G.−C. finite element methods for the 1-D Burgers' equation, he concluded that G.−C. computationally more efficient than G. finite element method. Kr$\ddot{a}$mer [1] studied the finite element method and the group finite element method to 1-D coupled Burgers' equation, he had seen similar numerical results that Smith and Pugh showed, then he applied the group proper orthogonal decomposition method for this equation. Kashkool and Noon [3] we presented the G. and G.-C. finite element methods for solving coupled Burgers' problem in 2-D, the fully discrete formulation was considered, the error estimate of these methods were $O(h,k)$ and the numerical results were compared with the exact solution, in [4] we used the classical artificial diffusion for G. and G.-C. finite element methods for the convection-dominated case, the error estimate of these methods were $O(h,k)$ and the numerical results were compared with the exact solution. Heitmann [9] applied subgrid scale eddy viscosity for convection dominated for convection-diffusion problem. The method consists of adding artificial viscosity term $\alpha(P_{L_H^{\perp}} \nabla u_h, P_{L_H^{\perp}} \nabla v_h)$ of orthogonal projection acting only on the fine scales, he give a comprehensive analysis of this method, in [10] he applied this method in a finite difference method by using an appropriate interpretation of the $\alpha(P_{L_H^{\perp}} \nabla u_h, \nabla v_h) \equiv \alpha(\nabla u_h, \nabla v_h) - \alpha(\nabla \bar{u}_h, \nabla v_h) \equiv \alpha\Delta u - \alpha\Delta\bar{u}$, where $\bar{u}$ is an average over itself and its five nearest discrete neighbors. Noon [11,12] we presented G. P. A. D. finite element method and consider semi and fully-discrete approximations, we proved stability and convergence for these approximations which was $O(h^{2r})$ and $O(h^{2r} + k^{2.5})$ respectively. The numerical solution of these approximations were compared with the exact solution . In this paper , we consider G.-C. P. A. D. and G.-C. S. A. D. finite element methods , a fully-discrete approximation with a Crank-Nicholson method for the time variable are present. The numerical solution of G.-C. P. A. D. is compared with the exact solution and[11,12] and the numerical solution of G.-C. S. A. D. is also compared with the exact solution. This paper is organized as follows. In section 2, we present the time-dependent modeling problem and a weak form of 2-D Burgers' problem. The discrete problem, fully-discrete approximation are presented in section 3. In section 4 the finite element approximation, test problems and numerical results are introduced. The conclusions is shown in section 5.

2. Time- Dependent Modeling Problem

Consider the nonlinear time-dependent for the two dimensional coupled Burgers' problem[13].

$$u_t - \epsilon\Delta u + u\,u_x + v\,u_y = f,$$
$$v_t - \epsilon\Delta v + u v_x + v v_y = g,$$

with boundary conditions

$$u(x,y,t) = 0, \quad v(x,y,t) = 0, \quad on \quad \partial\Omega \times (0,T],$$

and initial conditions

$$u(x,y,0) = u^0(x,y), \; v(x,y,0) = v^0(x,y),$$

where $\epsilon > 0$ is a viscosity constant, $\Omega \subset \mathrm{R}^2$ with boundary $\partial\Omega$, $u = u(x,y,t)$, $v = v(x,y,t)$, f and $g \in \mathrm{L}^2(\Omega)$.

The conservation form of Burger's equation was given by Fletcher[2]. Here the $(u\,u_x)$ and $(v\,v_y)$ terms are replaced by $\frac{1}{2}(u^2)_x$ and $\frac{1}{2}(v^2)_y$ respectively. Burgers' equation is then written as

$$u_t - \epsilon\,\Delta u + \frac{1}{2}(u^2)_x + v\,u_y = f, \qquad (2.1.a)$$
$$v_t - \epsilon\,\Delta v + u\,v_x + \frac{1}{2}(v^2)_y = g. \qquad (2.1.b)$$

The weak formulation of (2.1) is: find $u, v \in V = H_0^1(\Omega)$ such that:

$$(u_t,\varphi) + a(u,\varphi) + B(u,u,\varphi) + B(v,u,\varphi) = (f,\varphi), \quad (2.2.a)$$

$$(v_t,\varphi) + a(v,\varphi) + B(u,v,\varphi) + B(v,v,\varphi) = (g,\varphi),$$
$$\forall\, \varphi \in H_0^1(\Omega), \qquad (2.2.b)$$

$$(u(x,y,0),\varphi) = (u^0,\varphi), \qquad (v(x,y,0),\varphi) = (v^0,\varphi),$$

where $a(u,\varphi) = (\epsilon\,\nabla u, \nabla\varphi)$ and $a(v,\varphi) = (\epsilon\,\nabla v, \nabla\varphi)$, $B(u,u,\varphi) = (\frac{1}{2}(u^2)_x,\varphi)$, $B(v,u,\varphi) = (v u_y,\varphi)$, $B(u,v,\varphi) = (u\,v_x,\varphi)$, $B(v,v,\varphi) = (\frac{1}{2}(v^2)_y,\varphi)$,

3. The discrete problem

Given finite dimensional spaces $V_h \subset H_0^1(\Omega)$ then the approximate solution u_h, v_h to u, v respectively is the solution of the following problem:

$$(u_{h,t},\varphi_h) + a(u_h,\varphi_h) + B(u_h,u_h,\varphi_h) + B(v_h,u_h,\varphi_h) = (f,\varphi_h),$$
$$(v_{h,t},\varphi_h) + a(v_h,\varphi_h) + B(u_h,v_h,\varphi_h) + B(v_h,v_h,\varphi_h) = (g,\varphi_h),$$
$$\forall\varphi_h \in V_h .$$

3.1-Mathematical Formulation of an Artificial Viscosity Term

It is well known that when $\epsilon < h$, where h is mesh size, the convection term dominates over diffusion and the standard Galerkin finite element method produce an oscillating solution which is not close to exact solution. In the following we analyze an approach stabilizing the approximation through the introduction of an artificial viscosity terms which acts only on the fine scales of the finite element mesh. We add and subtract $\alpha(\nabla u, \nabla\varphi)$and $\alpha(\nabla v, \nabla\varphi)$ to (2.2.a) and (2.2.b) respectively where $\alpha = \alpha\,(h)$ is a positive constant, this gives,

$$(u_t,\varphi)+(\epsilon+\alpha)(\nabla u,\nabla\varphi)-\alpha(\nabla u,\nabla\varphi)+B(u,u,\varphi)+B(v,u,\varphi)=(f,\varphi),$$

$$(v_t,\varphi)+(\epsilon+\alpha)(\nabla v,\nabla\varphi)-\alpha(\nabla v,\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi)=(g,\varphi),\ \forall\,\varphi\in H_0^1(\Omega).$$

This suggests a mixed methods formulation wherein we define $q_1 \equiv \nabla u$ and $q_2 \equiv \nabla v \in (L^2(\Omega))^2$ [9]. We obtain the system, find $((u,v),\ (q_1,q_2)) \in (H_0^1, (L^2(\Omega))^2)$ satisfying ,

$$(u_t,\varphi)+(\epsilon+\alpha)(\nabla u,\nabla\varphi)-\alpha(q_1,\nabla\varphi)+B(u,u,\varphi)+B(v,u,\varphi)=(f,\varphi),$$

$$(v_t,\varphi)+(\epsilon+\alpha)(\nabla v,\nabla\varphi)-\alpha(q_2,\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi)=(g,\varphi),$$

$$(q_1-\nabla u, l)=0,\ (q_2-\nabla v, l)=0,\ \forall\,\varphi\in H_0^1(\Omega), l\in(L^2(\Omega))^2.$$

In the discretized problem, let h and H represent two mesh widths (with $h < H$). Let $L_H \subset (L^2(\Omega))^2$ and $V_h \subset H_0^1(\Omega)$ be finite element spaces. The problem then is to find $((u_h, v_h),(q_{1H}, q_{2H})) \in (V_h, L_H)$ satisfying

$$(u_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla u_h,\nabla\varphi_h)-\alpha(q_{1H},\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(f,\varphi_h)\ ,\ \text{(3.1.a)}$$

$$(v_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla v_h,\nabla\varphi_h)-\alpha(q_{2H},\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(g,\varphi_h)\ ,\ \text{(3.1.b)}$$

$$(q_{1H}-\nabla u_h, l_H)=0, (q_{2H}-\nabla v_h, l_H)=0, \forall\varphi_h\in V_h, l_H\in L_H.\ \text{(3.1.c)}$$

We note that,

- If $L_H = \{0\}$, L_H is small, then $q_{1H}, q_{2H} = 0$, and we have a straight artificial diffusion formulation (G.-C.S.A.D.),

$$(u_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla u_h,\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(f,\varphi_h)\ ,$$

$$(v_{h,t},\varphi_h)+(\epsilon+\alpha)(\nabla v_h,\nabla\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(g,\varphi_h)\ ,$$

- If L_H guided by numerical analysis so to obtain a beneficial balance, the key will be to select L_H, guided by precise and general error analysis, in such a way as to achieve a beneficial balance, we set $q_{1H} = P_{L_H}\,\nabla u_h$ and $q_{2H} = P_{L_H}\,\nabla v_h$ [9], where P_{L_H} is the orthogonal projection of L^2 onto L_H and $P_{L_H^{\perp}} = (I - P_{L_H})$ is orthogonal projection of L^2 on $L_H^{\perp}$, where,

$$P_{L_H}^{\perp} = \{v \in L^2(\Omega),\ (v,s) = 0,\ \forall s \in P_{L_H}\},$$

with these definitions we get the following lemma which represent a partial artificial diffusion formulation (G.-C.P.A.D.) .

***Lemma 3.1.1.*[11]** If $q_{1H} = P_{L_H}\ \nabla u_h$ and $q_{2H} = P_{L_H}\ \nabla v_h$ then the system (3.1) is equivalent to:

$(u_{h,t}\varphi_h) + a(u_h, \varphi_h) + \alpha\left(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h\right) + B(u_h, u_h, \varphi_h) + B(v_h, u_h, \varphi) = (f, \varphi_h)$ (3.2.a)

$(v_{h,t}\varphi_h) + a(v_h, \varphi_h) + \alpha\left(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla\varphi_h\right) + B(u_h, v_h, \varphi_h) + B(v_h, v_h, \varphi_h) = (g, \varphi_h),\ \ \forall\varphi_h \in V_h$ (3.2.b)

3.2 The Fully-Discrete Approximation

We consider a fully discrete formulation of (3.2), in particular, we will turn our attention to the Crank-Nicholson method. We use the subscript $n + \frac{1}{2}$ to represent the average of a quantity over the two discrete times for example $f^{n+\frac{1}{2}} = \frac{f^{n+1}+f^n}{2}$.

$\frac{1}{k}(u_h^{n+1} - u_h^n, \varphi_h) + A(u_h^{n+\frac{1}{2}}, \varphi_h) = (f^{n+\frac{1}{2}}, \varphi_h)$,(3.3.a)

$\frac{1}{k}(v_h^{n+1} - v_h^n, \varphi_h) + A(v_h^{n+\frac{1}{2}}, \varphi_h) = (g^{n+\frac{1}{2}}, \varphi_h)$, (3.3.b)

where,

$A(u_h, \varphi_h) = a(u_h, \varphi_h) + \alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h) + B(u_h, u_h, \varphi_h) + B(v_h, u_h, \varphi)$,

$A(v_h, \varphi_h) = a(v_h, \varphi_h) + \alpha\left(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla\varphi_h\right) + B(u_h, v_h, \varphi_h) + B(v_h, v_h, \varphi)$.

***Lemma 3.2.1.*[12]** The method described by (3.3) is stable over finite time. Specifically, for any $N > 0$,

$$\|u_h^N\| \le \|u_h^0\| + k\sum_{n=0}^{N-1} \|f^{n+\frac{1}{2}}\|,$$
$$\|v_h^N\| \le \|v_h^0\| + k\sum_{n=0}^{N-1} \|g^{n+\frac{1}{2}}\|.$$

For the error analysis we first need to establish the existence of the equilibrium projection pu_h, $pv_h \in V_h$ of u and v respectively which is given by,

$A(u - pu_h, \varphi_h) = a(u - pu_h, \varphi_h) + \alpha\left(P_{L_H}^{\perp}\nabla(u - pu_h), P_{L_H}^{\perp}\nabla\varphi_h\right) + B(u, u, \varphi_h) - B(pu_h, pu_h, \varphi_h) + B(u, u, \varphi_h) - B(pu_h, pu_h, \varphi_h) = 0$, (3.4.a)

$A(v - pv_h, \varphi_h) = a(v - pv_h, \varphi_h) + \alpha \quad (\quad P_{L_H^{\perp}} \nabla (v - pv_h), P_{L_H^{\perp}} \nabla\varphi_h) +$
$B(u, v, \varphi_h) - B(pu_h, pv_h, \varphi_h) + B(v, v, \varphi_h) - B(pv_h, pv_h, \varphi_h) = 0$,
$\forall\ \varphi_h \in\ V_h$. (3.4.b)

Lemma 3.2.2.[11] Let $u, v \in H_0^1(\Omega)$, the equilibrium projection pu_h, pv_h of u, v respectively, given by (3.4) exist uniquely.

Lemma 3.2.3.[11] Let $u, v \in H_0^1(\Omega)$, let pu_h, $pv_h \in V_h$ be the equilibrium projection given by (3.4) the assumptions of the finite element space there exists a constant C_3 and C_4 independent of ϵ, α, h and H such that

$$\| u - pu_h \|_{L^\infty(L^2)} \leq C_3(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}),$$
$$\| v - pv_h \|_{L^\infty(L^2)} \leq C_4(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}) .$$

***Lemma 3.2.4.*[12]** Let $u^{n+1}, v^{n+1} \in H_0^1(\Omega)$, let $pu_h^{n+1}, pv_h^{n+1} \in V_h \subset H_0^1$ be the equilibrium projection given by (3.4). Under the assumptions of lemma (3.2.3) there exists a constant C_1 and C_2 independent of ϵ, α, h and H such that

$$\max_{0 \leq n \leq N} \| u^{n+1} - pu_h^{n+1} \| \leq C_1\{h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}\},$$
$$\max_{0 \leq n \leq N} \| v^{n+1} - pv_h^{n+1} \| \leq C_2\{h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}\}.$$

***Theorem 3.2.1.*[12]** Let $u^{n+1}, v^{n+!}, u_h^{n+1}$ and v_h^{n+1}be the solutions of (2.2) and (3.3) respectively , then there exists constants C_1 ,C_2 independent of independent of ϵ, α, h and H such that,

$$\max_{0 \leq n \leq N} \| u^{n+1} - u_h^{n+1} \| \leq C_1 \left\{h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1} + k^{\frac{5}{2}} + \sqrt{\alpha}\right\},$$
$$\max_{0 \leq n \leq N} \| v^{n+1} - v_h^{n+1} \| \leq C_2 \left\{h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1} + k^{\frac{5}{2}} + \sqrt{\alpha}\right\}.$$

4. The Finite Element Approximation

To approximate the artificial viscosity terms in the equation (3.3), note that, from the definition of $P_{L_H^{\perp}}$,

$$\nabla\varphi_h = P_{L_H}\nabla\varphi_h + P_{L_H^{\perp}}\nabla\varphi_h ,$$
$$P_{L_H^{\perp}}\nabla\varphi_h = \nabla\varphi_h - P_{L_H}\nabla\varphi_h ,$$

then,

$$\alpha(P_{L_H^{\perp}} \nabla u_h , P_{L_H^{\perp}} \nabla\varphi_h) = \alpha(P_{L_H^{\perp}} \nabla u_h, \nabla\varphi_h) - \alpha(P_{L_H^{\perp}} \nabla u_h , P_{L_H}\nabla\varphi_h),$$

from the definition of $L_H^{\perp}$ the second term on the right hand side equal to zero, this implies,

$$\alpha(P_{L_H}^{\perp}\nabla u_h, P_{L_H}^{\perp}\nabla\varphi_h) \equiv \alpha(P_{L_H}^{\perp}\nabla u_h, \nabla\varphi_h),$$

similarly, $\alpha(P_{L_H}^{\perp}\nabla v_h, P_{L_H}^{\perp}\nabla\varphi_h) \equiv \alpha(P_{L_H}^{\perp}\nabla v_h, \nabla\varphi_h).$

The main point comes in finding an appropriate interpretation of the $\alpha(P_{L_H}^{\perp}\nabla u_h, \nabla\varphi_h)$ and $\alpha(P_{L_H}^{\perp}\nabla v_h, \nabla\varphi_h)$ terms, since ,

$$\nabla u_h = P_{L_H}\nabla u_h + P_{L_H}^{\perp}\nabla u_h,$$

$$\nabla v_h = P_{L_H}\nabla v_h + P_{L_H}^{\perp}\nabla v_h,$$

we rewrite,

$$\alpha\left(P_{L_H}^{\perp}\nabla u_h, \nabla\varphi_h\right) = \alpha(\nabla u_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla u_h, \nabla\varphi_h\right),$$

$$\alpha\left(P_{L_H}^{\perp}\nabla v_h, \nabla\varphi_h\right) = \alpha(\nabla v_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla v_h, \nabla\varphi_h\right).$$

As noted in section three , L_H is chosen such that $P_{L_H}^{\perp}$ is a projection onto fine scales and P_{L_H} is a projection onto the large scales, we can think of the large scale as representing average values, this implies[10],

$$\alpha(\nabla u_h, \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla u_h, \nabla\varphi_h\right) \approx \alpha(\nabla u_h, \nabla\varphi_h) - \alpha(\nabla\bar{u}_h , \nabla\varphi_h),$$

$$\alpha(\nabla v_h , \nabla\varphi_h) - \alpha\left(P_{L_H}\nabla v_h , \nabla\varphi_h\right) \approx \alpha(\nabla v_h, \nabla\varphi_h) - \alpha(\nabla\bar{v}_h, \nabla\varphi_h),$$

where $\bar{u}_h$ and $\bar{v}_h$ are an average over itself and its five nearest discrete neighbors.

The approximate solution is written as an expansion of the linear basis functions . In particular , we assume that,

$$(u_h^{n+1})^2 = \sum_{j=1}^{N}(d_j^{n+1})^2(t)\,\varphi_j(x,y),$$

$$(v_h^{n+1})^2 = \sum_{j=1}^{N}(\check{d}_j^{n+1})^2(t)\,\varphi_j(x,y).$$

where each $d_j(t)$, $\check{d}_j(t)$ are nodal unknowns and $\varphi_j(x,y)$ is the j^{th} linear basis function defined on Ω. Substitute the approximate solution in equation (3.4), we have,

$$2MD^{n+1} + kB_1D^{n+1} + kED^{n+1} = 2MD^n - kB_1D^n - kED^n + k(\breve{F}^{n+1} - \breve{F}^n), \quad (4.1.a)$$

$$2M\breve{D}^{n+1} + kB_2\breve{D}^{n+1} + kE\breve{D}^{n+1} = 2M\breve{D}^n - kB_2\breve{D}^n - kE\breve{D}^n + k(\breve{G}^{n+1} - \breve{G}^n), \quad (4.1.b)$$

where,

$M = (m_{ij}) = \int_{\Omega}\varphi_j\varphi_i dx\, dy,$

$$B_1 = \left(b_{1ijk}\right) = \int_{\Omega}\left[(\epsilon+\alpha)\nabla\varphi_j\nabla\varphi_i + \frac{1}{2}d_j\frac{\partial\varphi_j}{\partial x}\varphi_i + \check{d}_k\frac{\partial\varphi_j}{\partial y}\varphi_k\varphi_i\right]dx\, dy,$$

$E = (e_{ij}) = -\alpha \int_{\Omega} \nabla\bar{\varphi}_j \nabla\varphi_i \, dx\, dy$, $\breve{F} = (f_i) = \int_{\Omega} f \, \varphi_i dx\, dy$,

$$B_2 = (b_{2ijk}) = \int_{\Omega} \left[(\epsilon + \alpha)\nabla\varphi_j \nabla\varphi_i + d_k \frac{\partial\varphi_j}{\partial x}\varphi_k\varphi_i + \frac{1}{2}\breve{d}_j \frac{\partial\varphi_j}{\partial y}\varphi_i \right] dx\, dy,$$

$\breve{G} = (g_i) = \int_{\Omega} g \, \varphi_i dx\, dy$, $D = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$ and $\breve{D} = \begin{bmatrix} \breve{d}_1 \\ \vdots \\ \breve{d}_N \end{bmatrix}$,

for $i, j, k = 1,2, \ldots, N$.

- The system (4.1) represents (G.-C.P.A.D.) finite element method.
- The system (4.1) without the third term on the left and right hand sides represents (G.-C.S.A.D.) finite element method.

4.1 Numerical Results

In this subsection, we consider two test problems to illustrate G.-C.P.A.D. and G.-C.S.A.D. finite element method for 2-D coupled Burger problem as follow:

Problem 1: In this problem we illustrate G.-C.P.A.D. in system (4.1). The exact solutions of Burgers' equation(2.1) can be generated by using the Hopf–Cole transformation which are [6]:

$$u(x, y, t) = \frac{3}{4} - \frac{1}{4\left[1 + e^{\frac{(-4x + 4y - t)}{32\epsilon}}\right]}, \quad v(x, y, t) = \frac{3}{4} + \frac{1}{4\left[1 + e^{\frac{(-4x + 4y - t)}{32\epsilon}}\right]},$$

where $f = g = 0$. The domain Ω where the problem is to be solved is the unit square domain $\bar{\Omega} = [0,1] \times [0,1]$, we are discretized it using a uniform triangular mesh with mesh width parameter $h = \frac{1}{N-1}$ where $N = 18$. In this problem we take $\epsilon = \frac{1}{120}$ and $\frac{1}{240}$ respectively at $t = 0.5$ and $k = .01$ where $\epsilon < h$. In Figure (4.1.a) and (4.2.a) the problem run without P.A.D. (i.e. $\alpha = 0$), we see that the standard G.-C. finite element method produce an oscillating solution which is not close to the exact solution especially when ϵ decreasing with respect to h. In Figure (4.1.b)and (4.2.b) the problem run with G.-C.P.A.D., where $\alpha = .25*h$, the numerical solution became more convergent to the exact solution. In comparing with Case2[8,9], we see that the standard G.-C. finite element method produce slightly oscillated solution with respect to the exact solution than Case2 (Figure 5.2.2-a, 5.2.3-a, 2-a and 3-a) [11,12] respectively, also the G-C.P.A.D. finite element method is slightly more convergence to the exact solution than Case 2 (Figure 5.2.2-b, 5.2.3-b, 2-b and 3-b) [11,12] respectively.

Problem 2: In this problem we illustrate G.-C.S.A.D. in system (4.1) with the cancellation of the third term on the left and right hand sides . The exact solutions of Burgers' equation(2.1) are [5] :

$$u(x,y,t) = -\frac{4\pi\,\epsilon e^{-5\epsilon\pi^2 t}\cos(2\pi x)\sin(\pi y)}{2+e^{-5\epsilon\pi^2 t}\sin(2\pi x)\sin(\pi y)},$$

$$v(x,y,t) = -\frac{2\pi\,\epsilon e^{-5\epsilon\pi^2 t}\sin(2\pi x)\cos(\pi y)}{2+e^{-5\epsilon\pi^2 t}\sin(2\pi x)\sin(\pi y)},$$

where $f = g = 0$, $\bar{\Omega} = [0,1] \times [0,1]$ and $N = 18$. In this problem we take $\epsilon = \frac{1}{100}$ and $\frac{1}{500}$ respectively at $t = 1$ and $k = .01$ where $\epsilon < h$. In Figure (4.3.a) and (4.4.a) the problem run without S.A.D. (i.e. $\alpha = 0$), we see that the standard G.-C. finite element method produce an oscillating solution which is not close to the exact solution. In Figure (4.3.b)and (4.4.b) the problem run with G.-C.S.A.D., where $\alpha = 2$. As we see the numerical solution became more convergent to the exact solution.

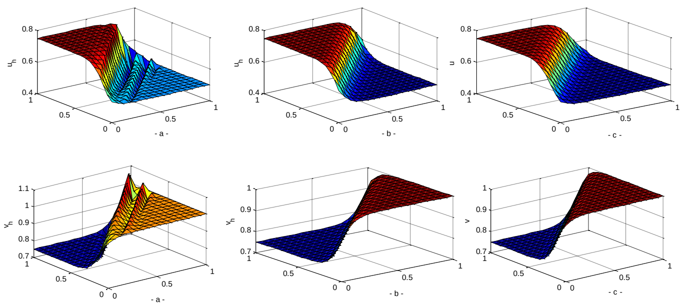

Figure 4. 1. a-Numerical solution of G.-C. method without P.A.D. of u and v , b-Numerical solution of G.-C.P.A.D. method of u and v, c-Exact solution of u and v , at $\epsilon=\frac{1}{120}$ and $t = .5$.

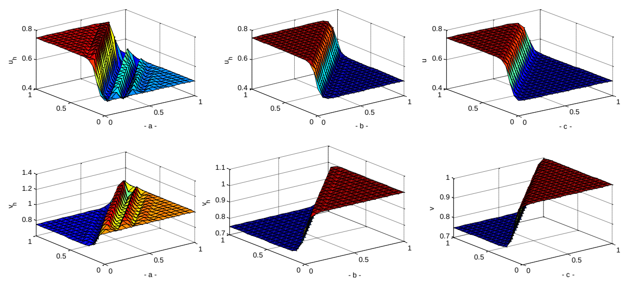

Figure 4. 2. a-Numerical solution of G.-C. method without P.A.D. of u and v , b-Numerical solution of G.-C.P.A.D. method of u and v, c-Exact solution of u and v , at $\epsilon=\frac{1}{240}$ and $t = .5$.

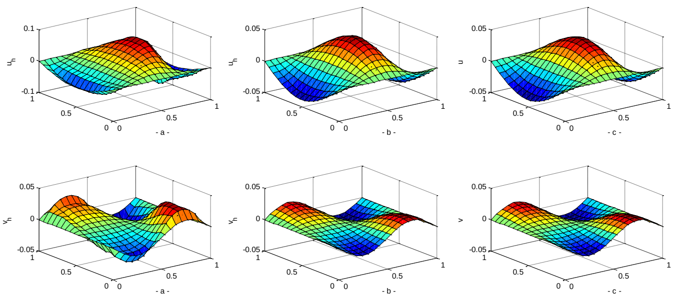

Figure 4. 3. a-Numerical solution of G.-C. method without S.A.D. of u and v , b-Numerical solution of G.-C.S.A.D. method of u and v, c-Exact solution of u and v, at $\epsilon=\frac{1}{100}$ and $t=1$.

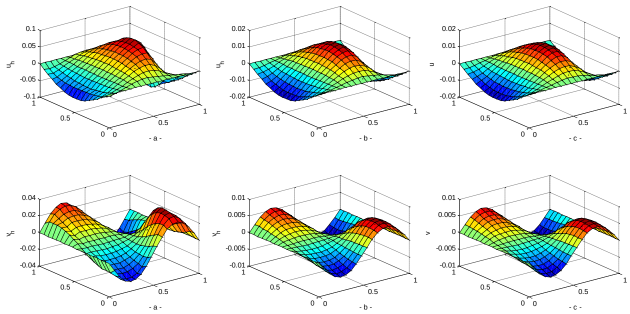

Figure 4. 4. a-Numerical solution of G.-C. method without S.A.D. of u and v , b-Numerical solution of G.-C.S.A.D. method of u and v, c-Exact solution of u and v, at $\epsilon=\frac{1}{500}$ and $t=1$.

5 – Conclusions

In this work, we considered the G.-C.P.A.D. and G.-C.S.A.D. finite element methods for 2-D coupled Burger problem in the fully discrete case using Cranck-Nicholson method for the time variable. For this studying, we can conclude the following:

1- The G.-C.P.A.D. and G.-C.S.A.D. finite element methods removed all oscillations occur when we use the standard G.-C. finite element method in the convection-dominated case, more over the numerical solutions obtained from these methods are consistent with the exact solutions.
2- We have concluded that the G.-C. without P.A.D. and G.-C.P.A.D. finite element method are more likely to capture the actual evolution of the solution than the G. without P.A.D. and G.P.A.D. finite element methods [11,12].

References

[1] B. Krämer, "Model reduction of the coupled Burgers equation in Conservation form", M.S. Thesis, Virginia Polytechnic Institute and State University, 2011.

[2] C. A. J. Fletcher, "*Computational Galerkin Methods*", Springer-Verlag, New York,1984.

[3] H. A. Kashkool, and N. J. Noon, " *The Numerical Simulation of Galerkin and Galerkin-Conservation Finite Element Methods for Coupled Burgers' Problem*", Scientific Journal of Education College - University of Thi-Qar, Vol.4, No.9, 2014.

[4] H. A. Kashkool, and N. J. Noon, "*The Modification of Galerkin and Galerkin-Conservation Finite Element Methods For Solving Coupled Burgers' Problem*", in Misan Journal of Academic Studies Vol. 9, No. 18, 2012.

[5] H. Aminikhah, " *A New Efficient Method for Solving Two-Dimensional Burgers' Equation*", International Scholarly Research Network, ID 603280, 2012.

[6] H. S. Shukla, Mohammad Tamsir, Vineet K. Srivastava, and Jai Kumar, " *Numerical solution of two dimensional coupled viscous Burger equation using modified cubic B-spline differential quadrature method* ", AIP advances 4, 117134, 2014.

[7] J. A. Atwell, and B.B. King, "*Stabilized finite element methods and feedback control for Burgers' equation*", Interdisciplinary center for applied Mathematics ,Virginia tech, Blacksburg, VA 24061-0531, Proceedings of the American Control Conference Chicago, Illinois, pp. 2745-2749 , 2000.

[8] L. C. Smith, " *Finite element approximations of Burgers' equation with Robin's boundary conditions*", M.S. Thesis, Virginia Polytechnic Institute and State University, 1997.

[9] N. F. Heitmann, "*Subgridscale eddy viscosity for convection dominated diffusive transport*", Department of Mathematics, Pittsburgh University, 2002.

[10] N. F. Heitmann, "*A stabilization scheme for convection dominated diffusive transport*", Department of Mathematics, Pittsburgh University, 2002.

[11] N. J. Noon, "*Semi discrete formulation of Galerkin-partial artificial diffusion finite element method for coupled Burgers' problem*", International Journal of Pure and Applied Research in Engineering and Technology, Vol. 2 (4), 2013.

[12] N. J. Noon, "*Fully discrete formulation of Galerkin -partial artificial diffusion finite element method for coupled Burgers' problem*", International Journal of Advances in Applied Mathematics and Mechanics, Vol. 1(3), 2014.

[13] Q. Yang, "*The upwind finite volume element method for two-dimensional Burgers equation*", Hindawi Publishing Corporation, Abstract and Applied Analysis, Article ID 351619, 11 pages, China, 2013.

[14] S. M. Pugh, "*Finite element approximations of Burgers' equation*", M.S. Thesis, Virginia Polytechnic Institute and State University, 1995.

[15] S. Volkwein, "*Upwind techniques and mixed finite elements for the steady-state Burgers' equation*", Institute of Mathematics, Karl -Franzens-University of Graz, Heinrichstrasse36, A-8010 Graz, Austria, 2003.