Approximation of functions on unit sphere in terms of K-functional

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الخلاصة

قدمنا في هذا البحث مؤثرات معرفة على فضاء الدوال المعرفة على كرات الوحدة والتي تنتمي الى الفضاء Lp عندما p<1. باستخدام تلك المؤثرات قدمنا بعض النظريات المباشرة ونظريات اخرى معاكسة لها بدلالة الدالي K الذي يكون مكافئاً لمقياس نعومة تلك الدوال.

الكلمات المفتاحية

معرف المشغلات للدوال، فضاء الوحدة، بدلالة الدالي K.

Abstract

In this paper we introduce operators defined for functions from Lp for p<1 defined on unit sphere and then we are using to prove direct inequalities in terms of K-functional. Also we are to prove some propped related to these operator.

Keywords

operators defined for functions, unit sphere, K-functional.

1. Introduction

For R^d, the unit sphere U^{d-1} is given by

$$U^{d-1}=\{x=(x_1,\dots,x_d):|x|=(x_1^2+\dots+x_d^2)^{1/2}=1\}$$

If $f \epsilon L_p(U^{d-1})$, p<1 and the mapping $f: U^{d-1} \to R$, then let us define:

$$\|f\|_{L_p(U^{d-1})}=\|f\|_p:=\left(\int_{U^{d-1}}|f|^p\right)^{1/p}$$

And

$$L_p^n:=\{f: f\in L_p, \grave{f},\dots,f^{(n)}\in L_p\} \;,\; p<1$$

For a function f(x) ($x\in U^{d-1}$), which is Lebesgue integrable on U^{d-1}, $d\geq 3$, the average on the cap of the sphere is given by [1]

$$B_t(f,y)=\frac{1}{\varphi(t)}\int_{\ell} f(x)d\sigma(x) \quad ,t>0 \quad (1.\ 1)$$

, where; $\ell=\{y: |y|=1, \cos t\leq x.y\leq 1, x, y\in U^{d-1}\}$ and x. y is the inner product in R^d is the measure on the sphere

$$\varphi(t)=\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\int_0^t \sin^{d-2}u\, du$$

For a function f(x) ($x\in U^{d-1}$) which is integrable on U^{d-1}, the average on the rim of the cap S_t (f, y) is given by [1]

$$S_t(f,y)=\frac{1}{\Psi(t)}\int_{x.y=\cos t} f(x)d\gamma(x), t>0 \quad (1.\ 2)$$

, where;

$d\gamma(\chi)$ is the measure (d-2 dimensional) of x on x. y = cost

$$\Psi(t)=\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\sin^{d-2}t$$

The Laplace – Beltrami operator on $x\in U^{d-1}$ is given by

, where; $\tilde{\Delta}f(x)=\Delta f(x/|x|)$ (1. 3)

$$\Delta f(x)=\frac{\partial^2}{\partial x_1^2}f(x)+\dots\dots\dots+\frac{\partial^2}{\partial x_d^2}f(x)$$

If $f\in L_p(U^{d-1})$, p<1, the K-functional can be defined as

$$K_r(f,\tilde{\Delta},t^{2r})_p^p = inf(\|f-g\|_p^p + t^{2r}\|\tilde{\Delta}^r g\|_{p'}^p;$$

$$\tilde{\Delta}^r g \in L_p(U^{d-1}))$$

$$K\,(f,\tilde{\Delta},t^2)_p^p \equiv K_1\,(f,\tilde{\Delta},t^2)_p^p. \quad (1.\,4)$$

Using the definition of $B_t(f, x)$, for $B_t(f, x)$ is bounded operator, we get that

$$\|B_t(f,x)\|_{L_p(U^{d-1})} = \|B_t(f,x)\|_p \; (1.\,5).$$

$$= \left\|\frac{1}{\varphi(t)}\int_{\ell} f(x)d\sigma(x)\right\|_p$$

$$\leq c(p)\|f\|_p$$

If $\tilde{\Delta}$ is the Laplace – Beltrami, for $\in L_p^2(U^{d-1})$, we get

$$\tilde{\Delta}B_t(f,x) = \Delta B_t(f(x)/|x|) \quad (1.\,6)$$

$$= \frac{\partial^2}{\partial x_1^2}B_t(f(x_1))/|x| + \cdots\cdots\cdots + \frac{\partial^2}{\partial x_d^2}B_t(f(x_d))/|x|$$

$$= \frac{\partial^2}{\partial x_1^2}(\frac{1}{\varphi(t)}\int_{\ell} f(x_1)d\sigma(x_1))/|x| + \cdots + \frac{\partial^2}{\partial x_d^2}(\frac{1}{\varphi(t)}\int_{\ell} f(x_d)d\sigma(x_d))/|x|$$

$$=(\frac{1}{\varphi(t)}\int_{\ell} \frac{\partial^2}{\partial x_1^2} f(x_1)d\sigma(x_1))/|x| + \cdots +$$

$$(\frac{1}{\varphi(t)}\int_{\ell} \frac{\partial^2}{\partial x_d^2} f(x_d)d\sigma(x_d))/|x|$$

$$=B_t(\Delta f(x)/|x|)$$

$$= B_t(\tilde{\Delta}f, x).$$

Then:

$$\tilde{\Delta}B_t(f,x) = B_t(\tilde{\Delta}f,x)$$

If the collection $v_1,\ldots\ldots, v_{d-1}$ is an orthonormal basis of the space orthogonal to x, the tangential gradient of f(x) is defined by [1]

$$grad_{tan} f(x) = \frac{\partial f(x)}{\partial \nu_1}, \ldots \ldots \ldots .., \frac{\partial f(x)}{\partial \nu_{d-1}}.$$

When $f \epsilon\, L_p^1(U^{d-1})$, p<1

$$|grad_{tan} f(x)| = \max_{\xi \perp x} \left|\frac{\partial f(x)}{\partial \xi}\right|$$

2. Auxiliary Result

2. 1. Lemma [3]

Suppose $f(x) \in L_p^2$, and

$$B_t(f,x) = \frac{1}{\varphi(t)} \int_{\ell} f(x) d\sigma(x) \ , t>0$$

$$S_t(f,x) = \frac{1}{\Psi(t)} \int_{x.y=cost} f(x) d\gamma(x) \ , t>0$$

$$\tilde{\Delta} f(x) = \Delta f(x/|x|) \quad for\ x \in U^{d-1}.$$

Then for $x \in U^{d-1}$ and $0 < t < \frac{\pi}{2}$, we have:

$$B_t(f,x) - f(x)$$

$$= \frac{1}{\varphi(t)} \int_0^t sin^{d-2}\theta \int_0^\theta sin^{2-d}\rho\, \varphi(\rho) B_\rho(\tilde{\Delta} f, x) d\rho\, d\theta$$

$$= \frac{1}{\varphi(t)} \int_0^t sin^{d-2}\theta \left\{ \int_0^\theta sin^{2-d}\rho \int_{\ell} \tilde{\Delta} f(y) d\,\sigma(y) d\rho \right\} d\theta \,.$$

And

$$S_t(f,x) - f(x) =$$

$$= \frac{1}{\Psi(t)} sin^{d-2} t \int_0^t sin^{2-d}\theta\, d\theta \int_{\ell} \tilde{\Delta} f(y) d\sigma(y)$$

$$= \frac{1}{\Psi(t)} \int_0^t sin^{2-d}\theta\, \varphi(\theta) B_\theta(\tilde{\Delta} f, x)\, d\theta$$

2. 2. Lemma [1]

for $\xi \perp x$, $B_t(f,x)$ is given by

$$B_t(f,x) = \frac{1}{\varphi(t)}\int_\Omega \int_{-\kappa}^{\kappa} f(\upsilon + (x\cos\theta + \xi\sin\theta)\sqrt{1-|\upsilon|^2})d\theta\, d\upsilon.$$

Where; $\varphi(t) = \frac{2\pi^{(d-1)/2}}{\Gamma_{(\frac{d-1}{2})}}\int_0^t \sin^{d-2} u\, du$

$\Omega = B_{x,\xi}\, sint = \{\upsilon : \upsilon.x = 0 , \upsilon.\xi = 0 , |\upsilon| \le sint\}$,

$\kappa = \arccos(cost/\sqrt{1-|\upsilon|^2})$, then

$$\frac{\partial}{\partial\xi}B_t(f,x) = \frac{1}{\varphi(t)}\int_\Omega \Big[f\Big(\upsilon + xcost + \xi\sqrt{1-|\upsilon|^2-cos^2 t}\Big)\alpha(t,\upsilon) - f\Big(x + xcost - \xi\sqrt{1-|\upsilon|^2-cos^2 t}\Big)\beta(t,\upsilon)\Big]\, d\upsilon$$

Where $\alpha(t, \upsilon)$ and $\beta(t, \upsilon)$ are close to 1 and arebounded by 1

2. 3. Lemma [4]

For $f \in L_\theta(U^{d-1})$, $1 \le \theta \le \infty$, there exsist $g \in L_{\grave\theta}(U^{d-1})$, such that $\frac{1}{\theta} + \frac{1}{\grave\theta} = 1$.

We have:

$$\begin{aligned}\|\tilde\Delta B_t B_\tau f\|_\theta - \varepsilon &\le |\langle g, \tilde\Delta B_t B_\tau f\rangle| \\ &\le |\langle g, B_t\tilde\Delta B_\tau f\rangle| \\ &\le |\langle B_t g, \tilde\Delta B_\tau f\rangle| \\ &\le |\langle grad_{tan}B_t g, grad_{tan}B_\tau f\rangle|\end{aligned}$$

Then

$$\|\tilde\Delta B_t B_\tau f\|_\theta - \varepsilon \le \|grad_{tan}B_t g\|_{\grave\theta} \cdot \|grad_{tan}B_\tau f\|_\theta.$$

3. The main results

In this section we shall introduce our main result

3. 1. Theorem

For $B_t(f,x), S_t(f,x), K(f,\tilde\Delta,t^2)$ are given by (1. 1), (1. 2), (1. 4) respectively, we have for p<1

$$\|f-B_t f\|_{L_{p(U^{d-1})}} \le c(p)\,K(f,\tilde{\Delta},t^2)_{L_p(U^{d-1})}$$

Proof: $\|f-S_t f\|_{L_{p(U^{d-1})}} \le c(p)\,K(f,\tilde{\Delta},t^2)_{L_p(U^{d-1})}$.

We choose $g\epsilon\, L_p^2$

$$\|f-g\|_p^p + t^2\left\|\tilde{\Delta}g\right\|_p^p \le 2\,K(f,\tilde{\Delta},t^2)_p^p$$

$$\|B_t(f-g)-(f-g)\|_p^p \le \|B(f-g)\|_p^p + \|f-g\|_p^p$$
$$\le c\left[\|B(f-g)\|_p^p + \|f-g\|_p^p\right] \quad ,c<1.$$

Then

$$\|B_t(f-g)-(f-g)\|_p^p \le 2\|f-g\|_p^p$$

$$\|S_t(f-g)-(f-g)\|_p^p \le \|S(f-g)\|_p^p + \|f-g\|_p^p$$
$$\le c\left[\|S(f-g)\|_p^p + \|f-g\|_p^p\right], \mathrm{c}<1.$$

Then

$$\|S_t(f-g)-(f-g)\|_p^p \le 2\|f-g\|_p^p.$$

Using Lemma2.1, we get

$$\|B_t g-g\|_p^p =$$
$$\left\|\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta\{\int_0^\theta \sin^{2-d}\rho\int_\ell \tilde{\Delta}g(y)d\sigma(y)d\rho\}d\theta\right\|_p^p$$
$$\le c(p)t^2\left\|\tilde{\Delta}g\right\|_p^p.$$

$$\|S_t g-g\|_p^p$$
$$=\left\|\frac{1}{\Psi(t)}\sin^{d-2}t\int_0^t \sin^{2-d}\theta d\theta\int_\ell \tilde{\Delta}g(y)d\sigma(y)\right\|_p^p$$
$$\le c(p)t^2\left\|\tilde{\Delta}g\right\|_p^p.$$

Then

$$\|f-B_t f\|_{L_p(U^{d-1})} \le 2\|f-g\|_p^p + c(p)t^2\left\|\tilde{\Delta}g\right\|_p^p$$
$$= c(p)K\,(f,\tilde{\Delta},t^2)_{L_p(U^{d-1})}$$
$$\|f-S_t f\|_{L_p(U^{d-1})} \le 2\|f-g\|_p^p + c(p)t^2\left\|\tilde{\Delta}g\right\|_p^p$$
$$= c(p)K\,(f,\tilde{\Delta},t^2)_{L_p(U^{d-1})}$$

3. 2. Theorem

If $L_p(U^{d-1})$, p<1, then $grad_{tan}\, B_{tf}$ is in $L_p(U^{d-1})$ and

$$\|grad_{tan}B_t f\|_{L_p} \le \frac{c(p)\Psi(t)}{\varphi(t)}\|f\|_{L_p} \le \frac{c(p)}{t}\|f\|_{L_p}$$

Proof:

By Lemma 2. 2 we get

$$\left|\frac{\partial}{\partial\xi}B_t(f,x)\right| =$$

$$\left|\frac{1}{\varphi(t)}\int_\Omega \left[f\left(v+xcost+\xi\sqrt{(1-|v|^2)-cos^2t}\right)\alpha(t,v) -\right.\right.$$

$$\left.\left. f\left(x+xcost-\xi\sqrt{(1-|v|^2)-cos^2t}\right)\beta(t,v)\right]dv\right|.$$

$$\leq \frac{2}{\varphi(t)}\{\int_\Omega \left|f(v+xcost+\xi\sqrt{(1-|v|^2)-cos^2t})\right|dv +$$

$$\int_\Omega \left|f(v+xcost+\xi\sqrt{(1-|v|^2)-cos^2t})\right|dv\}.$$

Since

$$\int_{U^{d-1}} f(x)dx \leq [\ measure\ of\ U^{d-1}][\max_{x\in U^{d-1}} f(x)]$$

, then

$$\left|\frac{\partial}{\partial\xi}B_t(f,x)\right| \leq \frac{2\psi(t)}{\varphi(t)}S_t(|f|,x).$$

$$|grad_{tan}B_t(f,x)| = \max_{\xi\perp x}\left|\frac{\partial}{\partial\xi}B_t(f,x)\right|.$$

$Then\ we\ get\ , for\ p<1\ and\ f\in L_p^1(U^{d-1})$, that

$$\|grad_{tan}B_t(f,x)\|_{L_p} = \int_{U^{d-1}} (|grad_{tan}B_t(f,x)|^p dx)^{1/p}$$

$$= \int_{U^{d-1}} \left(\left|\max_{\xi\perp x}\frac{\partial}{\partial\xi}B_t(f,x)\right|^p dx\right)^{1/p}$$

$$\leq \int_{U^{d-1}}\left(\left|\frac{2\psi(t)}{\varphi(t)}S_t(|f|,x)\right|^p dx\right)^{1/p}$$

$$\leq \frac{2\psi(t)}{\varphi(t)}\|S_t(|f|,x)\|_{L_p},$$

$$since\ \frac{2\psi(t)}{\varphi(t)} \leq \frac{c(p)}{t}\ ,then$$

$$\|grad_{tan}B_t(f,x)\|_{L_p} \leq \frac{c(p)}{t}\left\|\frac{1}{\psi(t)}\int_{x.y=cost} f(x)d\gamma(x)\right\|_{L_p}$$

$$\leq \frac{c(p)}{t}\|f\|_{L_p}$$

3. 3. Theorem

If $f \in L_p(U^{d-1})$, p<1. Then

$$\left\|\tilde{\Delta}^r B_{\tau_1} \cdots\cdots B_{\tau_{2r}} f\right\|_p \leq \frac{c_r(p)}{\tau_1 \cdots\cdots \tau_{2r}} \|f\|_p \qquad , \qquad p<1$$

Proof:

Since $\left\|\tilde{\Delta} B_t B_\tau f\right\|_p - \varepsilon \leq \left\|\tilde{\Delta} B_t B_\tau f\right\|_\theta - \varepsilon \quad , \theta \geq 1$

We choose g of in Lemma 2.3, then we get:

$\left\|\tilde{\Delta} B_t B_\tau f\right\|_p - \varepsilon \leq \|grad_{tan} B_t g\|_{\grave{\theta}} \cdot \|grad_{tan} B_\tau f\|_\theta$,

$\grave{\theta} \geq 1$, and $\dfrac{1}{\theta} + \dfrac{1}{\acute{\theta}} = 1$

$$\|grad_{tan} B_t g\|_{\grave{\theta}} = \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\acute{\theta}} \, dx \right)^{1/\acute{\theta}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\acute{\theta} + \frac{1}{\theta'} - \frac{1}{\theta'}} dx\right)^{\frac{1}{\theta'} + \acute{\theta} - \acute{\theta}}$$

$$\leq \left(\int_{U^{d-1}} |\mathrm{grad_{tan}} B_t g|^{\acute{\theta} - \frac{1}{\theta'}} |\mathrm{grad_{tan}} B_t g|^{\frac{1}{\theta'}} dx \right)^{\frac{1}{\theta'} - \acute{\theta}} \times$$

$$\left(\int_{U^{d-1}} |\mathrm{grad_{tan}} B_t g|^{\acute{\theta} - \frac{1}{\theta'}} |\mathrm{grad_{tan}} B_t g|^{\frac{1}{\theta'}} dx \right)^{\acute{\theta}}$$

Assume that $\frac{1}{\acute{\theta}} = q$, $so \quad \acute{\theta} = \frac{1}{q} \quad , \quad and \ \ q < 1$, then

$$\|\mathrm{grad_{tan}} B_t g\|_{\acute{\theta}} \leq \left(\int_{U^{d-1}} |\mathrm{grad_{tan}} B_t g|^{\frac{1}{q} - q} |\mathrm{grad_{tan}} B_t g|^q dx \right)^{q - \frac{1}{q}}$$

$$\times \left(\int_{U^{d-1}} |\mathrm{grad_{tan}} B_t g|^{\frac{1}{q} - q} |\mathrm{grad_{tan}} B_t g|^q dx \right)^{\frac{1}{q}}$$

$$\leq c(q) \times \left(\int_{U^{d-1}} c(q) |\mathrm{grad_{tan}} B_t g|^q dx \right)^{\frac{1}{q}}$$

$$\leq c(q) \|grad_{tan} B_t g\|_q , q<1 \qquad (1.\ 8).$$

And

$$\|grad_{tan}B_\tau f\|_\theta = \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^\theta dx\right)^{\frac{1}{\theta}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^{\theta+\frac{1}{\theta}-\frac{1}{\theta}} dx\right)^{\frac{1}{\theta}+\theta-\theta}$$

$$\leq \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^{\theta-\frac{1}{\theta}} |grad_{tan}B_\tau f|^{\frac{1}{\theta}} dx\right)^{\frac{1}{\theta}-\theta}$$

$$\times \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^{\theta-\frac{1}{\theta}} |grad_{tan}B_\tau f|^{\frac{1}{\theta}} dx\right)^{\theta}$$

Assume that $\frac{1}{\theta} = p$ so $\theta = \frac{1}{p}$ and $p < 1$, then

$$\|grad_{tan}B_\tau f\|_\theta$$

$$\leq \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^{\frac{1}{p}-p} |grad_{tan}B_\tau f|^{p}\, dx\right)^{p-\frac{1}{p}}$$

$$\times \left(\int_{U^{d-1}} |grad_{tan}B_\tau f|^{\frac{1}{p}-p} |grad_{tan}B_\tau f|^{p}\, dx\right)^{\frac{1}{p}}$$

$$\leq c(p) \times \left(\int_{U^{d-1}} c(p) |grad_{tan}B_\tau f|^{p}\, dx\right)^{\frac{1}{p}}$$

$\leq c(p)\|grad_{tan}B_\tau f\|_p$, $p < 1$ (1. 9).

From (1. 8), (1. 9), we get:

$\left\|\tilde{\Delta} B_t B_\tau f\right\|_p - \varepsilon \leq c(p,q)\, \|grad_{tan}B_t g\|_q \cdot \|grad_{tan}B_\tau f\|_p$

, where: p<1, q<1 and p + q=1

Let $\|g\|_q = c(q)$, and by Theorem 3.2 we get:

$$\left\|\tilde{\Delta} B_t B_\tau f\right\|_p - \varepsilon \leq \frac{c(q)}{t}\|g\|_q \cdot \frac{c(p)}{\tau}\|f\|_p$$

$$\leq \frac{c^2(q)\, c(p)}{t\tau}\|f\|_p.$$

Which, as is an arbitrary, implies our result for r=1.

Repetition of the above consideration implies.

$$\left\|\tilde{\Delta}^r B_{\tau_1} \cdots\cdots B_{\tau_{2r}} f\right\|_p \leq \frac{c_r(p)}{\tau_1 \cdots\cdots \tau_{2r}} \|f\|_p, p<1$$

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