Graphs With Extensibility One Vertex

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Abstract.

In this paper, the concepts of extension of a graph (digraph) and the extensible class of graphs (digraphs) and the extensibility number of graphs (digraphs). The necessary and sufficient condition for regular, bipartite, tree, Hamiltonian to have extension one vertex.

الملخص :

في هذا البحث قدمنا توسيع البيان (والبيان الموجهه) وتوسيع الصف للبيان (والبيان الموجهه) وتوسيع البيانات والبيانات الموجهه بعدد من الرؤوس. قدمنا الشرط الضروري والكافي لبيان المجزء، والشجرة، هملتونين، وتوسيعهم برأس واحد.

Keywords: Extension graphs (digraphs), conncted graphs (digraphs), Hamiltonian of graphs, bipartition graphs (digraphs), reqular graphs

Introduction:

All graphs in this paper are finite and have no loop or multiple edges.

In 1980, plummer [6] studied the properties of n-extendable graphs and showed that every 2-extendable graphs is either bipartite of a brick. Motivated by this result he [7,8] further looked at the relationship between n-extend ability and other graphic parameters(e.g., degree,connectivity, genus, toughness). Recently, Schrage and Cammack[10] and Yn[11] classified the 2-extendable generalized Petersen graphs, and Chan, Chen and Yn[4] classified the 2-extendable cayley graphs one abelian groups. For more results and the motivations of n-extendable graphs, the interested reader is refereed to a recent survey paper by plummer[9]. Grant and Holton [5] gave good characterizations of 1-extendable graphs and 1-extendable bipartite graphs. Brualdi and perfect[3] in 1971 obtained a criterion of n-extendable bipartite graphs, but their result is described in terms of matrices and systems of distinct representatives. In 2009 Attar [1] introduc- eed the concept of extension graphs (digraphs), and he characterized the extensibility number for some graphs(digraphs) : Let G be a nontrivial graph. The extension of G is a simple graph denoted by $G + S$ obtained from G by adding a nonempty set of independent vertices S such that every vertex in S is adjacent to every vertex in G exactly one. In such a way S is called extension set of G.

2 Extension of Graphs

In this section, we introduced the concepts extension of graph, extensible class of graphs and the extensibility number of graphs.

Definition 2.1[2]

Let G_1 and G_2 be two graphs withs no vertex in common. We defined the join of G_1 and G_1 and G_1 denoted by $G_1 + G_2$ to be graph with vertex set and edge set given as follows:

V($G_1 + G_2$)=V(G_1)∪ $V(G_2)$,

E($G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$

Where J={ $x_1 \ x_{2:} \ x_1 \in V(G_1), x_{2:} \in V(G_2)$}.

Thus J consists of edges which join every of G_1 to every vertex of G_2.

Attar [1] defined the extension of graphs as follows:

Definition 2.2[1]

Let G a nontrivial graph. The extension of graphs of G is a simple graph denoted by G+S obtained from G by adding a nonempty set of independent vertices S such that every vertex in S is adjacent to every vertex in G.

In this work we define the extension of graphs as follows:

Definition 2.3

Let G be a nontrivial graph. The extension of G is a simple graph denoted by G*S obtained from G by adding a nonempty set S of independent vertices different from the vertices of G to the graph G such that every vertex in S is adjacent to at least one vertex in G. In such a way S is called extension set of G. In particular if S consists of a single element v, is called extension vertex of G.

Here, we define the extensible class of graphs.

Definition 2.4

Let ʒ be the class of graphs with certain property. Then ʒ is called extensible class of graphs, if for every graph G ∈ ʒ, there exists an extension vertex v different from the vertices of G such that G*v ∈ ʒ.

Now, we introduce the following two propositions.

Proposition 2.5

The class of connected graphs is extensible class of graphs.

Proof

It follows from Definition 3, the extension vertex preserve the connectedness of graph.

Proposition 2.6

1. The class of Hamiltonion graphs is not extensible class.
2. The class of trees is not extensible class.

3. The class of complete graphs is not extensible class.
4. The class of bipartite graphs is not extensible class.
5. The class of Eulerian graphs is not extensible class.
6. The class of regular graphs is not extensible class.

Proof

1. Let G be a Hamiltionian graph with n vertices. It is clear that the extension vertex v which is adjacent to exactly one vertex in G gives G*v is not Hamiltonian.
2. Let T be atree. If the extension vertex v is adjacent to more than one vertex in T, then T*v contains a cycle.
3. Let k_n be a complete graph. If the extension vertex v is adjacent to less than n vertices of k_n then $k_n * v$ is not complete.
4. Let G be X, Y-bipartite graph, and v be extension vertex of G such that v is adjacent to a vertices in X and a vertices in Y. It is clear that the resulting graph G*v is not bipartite.
5. Let ʒ be an Eulerian graph.nThen ʒ is conncted and every vertex of ʒ has an even degree. Suppose that v_0 is an extension vertex of ʒ. If v_0 is adjacent to odd number of vertices in ʒ. Then ʒ*v_0 is not Eulerian.
6. Let R be a regular graph with n vertices, and v_0 be an extension vertex of R. If v_0 is adjacent to h vertices in R such that h$< n$. Then R*v_0 is clearly not regular.

Now, the question is that, what is the samallest extension set of vertices which make the non extensible graph is extensible. In order to answer this question we introduced the following definition:

Definition 2.7

Let ʒ be the class of graphs writh certain property, and G ∈ ʒ be a nontrivial. The extensibility number of G with respect to ʒ is the samallest positive integer m, if exists such that there exists an extension set S of G with cardinality m in which the new graph G*S ∈ ʒ. We write m=$ext_ʒ$(G). If such a number dose not exist for G then we say that the corresponding extensibility number is ∞.
One can see immediate, the class of graphs ʒ is extensible class if and only if the extensibility number of every graph G ∈ ʒ is one.

3 Graphs with Number One Extensibility

In this section, we introduced the necessary and sufficient condition for som graphs to have extensibility number one.

Theorem 3.1

Let R be the class of regular graphs, R ∈ R . Then the ext_{R} =1 if and only if R is a trivial or complete graph and there exists a vertex v_0 different from the vertices of R such that v_0 is adjacent to every vertex in R exactly once.

Proof:

Let R be an r-regular graph with n vertices. Suppose that ext_{R} =1. Then by Definition 2.7, there exists an extension set of vertices with single element v_0, such that R*v_0 ∈ R. By Definition 2.3, we must have v_0 is adjacent to every vertex in R exactly once. Then d(v_0) $= n$ and the degree of every vertex of R in the graph R*v_0 is r+1, but R*v_0 is regular. Then we must have n=r+1. As R*v_0 is regular, then either r=0, then n=1 and R is a trival graph, or r=n-1 and R is complete graph.

Conversely, Let v_0 be a vertex defferent from the vertices of R such that v_0 is adjacent to every vertex in R exactly once, and R is trivial or complete. If R is trival graph then it is not difficult to see that ext_{R} =1.

Suppose that R is complete graph with n vertices. Then R is regular graph with regularity degree n-1, we prove that $ext_{\mathrm{R}}(R)$ =1.

If v_0 is adjacent to every vertex in R exactly once. Then d(v_0) $= n$ and the degree of every vertex of R is n-1+1. Then the degree of every vertex in R*$v_0$is n. Thus the new graph R*v_0 is n-regular graph. As such v_0 is extension vertex of R with respect to R. Hence $ext_{\mathrm{R}}(R)$ =1.

Theorem 3.2

Let ʒ be the class of bipartite graphs, B be X, Y-bipartite graph in ʒ. Then $ext_{\mathrm{B}}(B)$ =1, if and only if there exists a vertex v, different from the vertices of B such that v is adjacent to at least one vertex in B and v is not commn neighbor to a vertex X and a vertex in Y.

Proof:

Let ʒ be the class of bipartite graphs, B be X, Y-bipartite in ʒ. Suppose that $ext_{\mathrm{B}}(B)$ =1. Then by Definition 2.7, there exist extension vertex v different from the vertices of B, such that v is adjacent to at least one vertex in B and B*v ∈ ʒ. Suppose that v is a common neigbhour to a vertex in X and a vertex in Y. In this case, we get a controduction to our assumption that B*v ∈ ʒ.

Conversel, Let v be avertex diffent from the vertice of B such that v is adjacent to at least one vertex in B and v is not common neighbor to a vertex in X and a vertex in Y. Then v either in X or in Y, suppose that v is in X. Then v is adjacent to at least one vertex in Y and v is not adjacent to any vertex in X. Hence $ext_{\mathrm{B}}(B)$ =1.

Theorem 3.3

Let ʒ be the class of trees, T ∈ ʒ. Then $ext_{\mathrm{T}}(T)$ =1, if and only if there exists a vertex v, different from the vertices of T such that v, is adijacent to exactly one vertex of T.

Proof

Let ʒ be the class of trees, T ∈ ʒ. Suppose that $ext_{\mathrm{T}}(T)$ =1, then by Definition 2.7, there exist a vertex v different from the vertices of T, such that T*v ∈ ʒ. Let w,u be distinct vertices in T, then by Definition of tree there exist a part from w to u. Suppose that v is adjacent to w as well as u in T. Then vw,…,uv forms a cycle in T a controduction. Hence v must be adjacent to exactly one vertex of T.

Conversely, Suppose that there exists a vertex v different from the vertices of T, such that v is adjacent to exactly one vertex. Then v is extension vertex of T and T*v ∈ ʒ. Hence that $ext_{\mathrm{T}}(T)$ =1.

Theorem 3.4

Let ʒ be the class of Hamiltonian graphs, H ∈ ʒ such that d(v)=2, for all v∈ H. Then $ext_{\mathrm{H}}(H)$ =1, if and only if there exists a vertex v different from the vertices of H such that v is adjacent to at least two vertices of H, and there exist two vertices in N(v) which are adjacent.

Proof:

Let ʒ be the class of Hamiltonian graphs, H ∈ ʒ such that d(v)=2, for all v ∈ H. Then $ext_{ʒ}(H)$ =1, if and only if there exists a vertex v different from the vertices of H such that, v is adjacent to at least two vertices of H, and there exist two vertices in N(v) which are adjacent.

Proof:

Let ʒ be the class of Hamiltonian graphs H ∈ ʒ such that d(v)=2, for all v ∈ H. Then $ext_{ʒ}(H)$ =1, then by Definition 2.7, there exists extension vertex v, such that H*v ∈ ʒ. Suppose that v adjacent to one vertex from the vertex of H, then H*v ∈ ʒ and this a controduction to our assumption. If v is adjacent to at least two vertices of H and the there is not two vertices in N(v) which are adjacent, then H*v can not have a Hamiltonian cycle a controducetion.

Conversely, Suppos that ther exists a vertex v different from the vertex of H, such that v is adjacent to at least two vertices of H, and there exist two vertices in N(v) whch are adjacent. Suppose that $C_0 = u_1, u_2, \ldots, u_n$ is Hamilionian cycle in H, and v is adjacent to u_i , $u_{(i+1)}$ in C_0,then the cycle $C_1 = u_1, u_1, \ldots u_n$ is Hamiltonian cycle in H*v, as such v extension vertex of H and H*v ∈ ʒ. Hence $ext_{ʒ}(H)$ =1.

4 Extension of Digraphs

In this section, we introduced the concepts extension of digraph, extension class of digraphs and the extensibility number of digraphs.

Attar [1] defined the extension of digraphs as follows:

Definition 4.1 [1]

Let D be a nontrivial graph. The extension of D is a simple digraph denoted by D+S obtained from D by adding a nonempty set of independent vertices S vertex in D.

In this work we defined the extension of digraphs follows:

Definition 4.2

Let D be a nontrivial digraph. The extension of D is a simple digraph denoted by D*S obtained from D by adding a nonempty set S of independent vertices different from the vertices of D such that every vertex in S is adjacent or adjacent by, but not both at least one vertex in D. In such away S is called extension set of D. In particular if S consists of single element v, then v is called extension vertex of D.

Now, we defined the extension class of digraphs.

Definition 4.3

Let ʒ be the class of graphs with certain property. Then ʒ is called extensible class of digraphs, if for every digraphs D ∈ ʒ, there exists an extension vertex v different from the vertices of D such that D*v ∈ ʒ. Hence we introduce the following two propositions.

Proposition 4.4

The class of connected digraphs is extensible class of digraphs with respect to connectedness.

Proof

The proof follows from Definition 4.2.

Proposition 4.5

Each of the classes: Hamiltonion digraph complete digraph, bipartite digraphs. Eulerian digraphs, and regular digraphs is not extensible class.

Proof

The proof is similar to that in proposition 2.6.

The definition of extensibility number of digraph is analogous to that in Definition 2.7 only replace every graph G by digraph D as follows:

Definition 4.6

Let ʒ be the class of digraphs with certain property, and D ∈ ʒ, be a nontrivial. The extensibility number of D with respect to ʒ is the smallest positive integer m, if exists such that ther exists an extension set S of D with cardinality m, in which the new digraph D*S ∈ ʒ. Write m=$ext_{ʒ} = (D)$. If such a number does not exist for D then we say that the corresponding extensibility numbern ∞.

5 Digraphs with Extensibility Number One

In this section, we introduced the necessary and sufficient condition for som digraphs to have extensibility number one.

Theorem 5.1

Let ʒ be the class of trees digraphs, T ∈ ʒ. Then $ext_{ʒ}(T)$ =1, if and only if there exists a vertex v, different from the vertices of T such that v, is adjacent only for any vertex or adjacent by only for any vertex of a vertex T.

Proof

Let ʒ be the class of trees digraphs, T ∈ ʒ. Suppose that Then $ext_{ʒ}(T)$ =1. Then by definition 4.6, there exist a vertex v different from the vertices of T, such that T*v ∈ ʒ. If v is adjacent and adjacent by for any vertex of a vertices T then there is a cycle in T*v and this a controduction with defenation of trees digraphs. Hence v is adjacent for any vertex or adjacent by only for any vertex of a vertices T.

Conversely, Suppose that there exists a vertex v, different from the vertices of T such that v, is adjacent only for any vertex or adjacent by only for any vertex of vertices T. Then v is extension of T and T*v ∈ ʒ.

Hence that $ext_{ʒ}(T)$ =1.

Theorem 5.2

Let ʒ be the class of bipartite digraphs, B be X, Y- bipartite digraphs in ʒ. Then $ext_{ʒ}(B)$ =1, if and only if there exists a vertex v, different from the vertices of B such that, v is adjacent to at least one vertex in D and v is not comn neighbor to verex in X and Y.

Proof

Let ʒ be the class of bipartite digraphs, B be X, Y- bipartite digraphs in ʒ. Suppose that $ext_{ʒ}(B)$ =1. Then by Definition 4.6, there exists extension vertex v different from the vertices of B, such that v is adjacent to at least one vertex in B and B*v ∈ ʒ. Suppose that v is a common neigbhour to a vertex in X and a vertex in Y. In this case we get a controdution to our assumption that B*v ∈ ʒ.

Conversely, Suppose that the condition is holds. Then v either in X or in Y, let v in X. Then v is adjacent to at least one vertex in Y and v is not adjacent to any vertex in X. That is v is extension vertex of B and B*v is bipartite digraphs. Simlarly, if v is in Y. Hence $ext_{ʒ}(B)$ =1.

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