Strong Converse Inequality

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الخلاصة

قدمنا في هذا البحث مؤثرات للدوال في الفضائات Lp عندما P<1 و المعرفة على كرة الوحده بعدها استخدمنا تلك المؤثرات لبرهان مبرهنه عكسية لمبرهنة مباشرة قدمناها مسبقا بدلالة الدالي K .

الكلمات المفتاحية

مؤثرات الدوال ، المبرهنة العكسية الاقوى ، الدالي K

Abstract

In this paper we introduce operators for functions from Lp for P<1, defined on unit sphere and then we use them to prove strong converse inequality for direct theorem that we introduce in terms of K-functional.

Keywords

Operators for functions, strong converse inequality, K-functional

1. Intoduction

For R^d, the unit sphere U^{d-1} is given by $U^{d-1} = \{ x = (x_1,\dots,x_d) : |x| = (x_1^2+\dots+x_d^2)^{1/2} = 1\}$. Let $L_p(U^{d-1})$, p<1 be the space of all mappings $U^{d-1} \to R$, with $\|f\|_{L_p(U^{d-1})} = \|f\|_p := \left(\int_{U^{d-1}} |f|^p\right)^{1/p} < \infty$.

And

$$L_p^n := \{f : f \in L_p, \acute{f}, \dots, f^{(n)} \in L_p(U^{d-1})\}, \quad p<1.$$

For a function ,$f \in L_p(U^{d-1})$, d≥3, the average on the cap of the sphere is given by[1]

$$B_t(f,y) = \frac{1}{\varphi(t)} \int_\ell f(x)d\sigma(x), t>0 \quad ....(1.1)$$

where ; $\ell = \{y : |y| = 1, cost \le x.y \le 1, x, y\ U^{d-1}\}$ and $x.y$ is the inner product defined on R^d. dσ(x) is the measure on the sphere

$$\varphi(t) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \int_0^t sin^{d-2}u\, du$$

For a function f(x) ($x \in U^{d-1}$) which is integrable on U^{d-1}, the average on the cap $S_t(f, y)$ is given by[1]

$$S_t(f,y) = \frac{1}{\Psi(t)} \int_{x.y=cost} f(x)d\gamma(x), t>0, x, y \in U^{d-1} ....(1.2)$$

where dγ(x) is the measure (d-2 dimensional) of x on x.y= cost,

$$\Psi(t) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} sin^{d-2}t.$$

The Laplace – Beltrami operator on $x \in U^{d-1}$ is given by

$$\tilde{\Delta}f(x) = \Delta f(x/(|x|)) \quad ....(1.3)$$

where ;

$$\Delta f(x) = \frac{\partial^2}{\partial x_1^2} f(x) + \dots\dots\dots + \frac{\partial^2}{\partial x_d^2} f(x)$$

If $f \in L_p(U^{d-1})$, p<1, the K-functional can be defined as

$$K_r(f,\tilde{\Delta},t^{2r})_p^p = \inf\left(\|f-g\|_p^p + t^{2r}\|\tilde{\Delta}^r g\|_p^p ; \tilde{\Delta}^r g \in L_p(U^{d-1})\right)$$

.......(1.4)

$$K(f,\tilde{\Delta},t^2)_p^p \equiv K_1(f,\tilde{\Delta},t^2)_p^p$$

Using the definition of $B_t(f,x)$, $B_t(f,x)$ is bounded operator. In fact

$$\|B_t(f,x)\|_{L_p(U^{d-1})} = \|B_t(f,x)\|_p = \left\|\frac{1}{\varphi(t)}\int_\ell f(x)d\sigma(x)\right\|_p$$

$$\le c(p)\|f\|\, p \quad ....(1.5)$$

If $\tilde{\Delta}$ is the Laplace – Beltrami , for $f \in L_p^2(U^{d-1})$, we get $\tilde{\Delta}B_T(f,x) = \Delta B_t(f(x)/|x|)$

$$= \frac{\partial^2}{\partial x_1^2}\left(B_t(f(x_1))/|x| + \dots\dots\dots + \frac{\partial^2}{\partial x_d^2} B_t(f(x_d))/|x|\right)$$

$$= \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{\varphi(t)}\int_\ell f(x_1)d\sigma(x_1)\right)/|x| + \dots + \frac{\partial^2}{\partial x_d^2}\left(\frac{1}{\varphi(t)}\int_\ell f(x_d)d\sigma(x_d)\right)/|x|$$

$$= \left(\frac{1}{\varphi(t)}\int_\ell \frac{\partial^2}{\partial x_1^2} f(x_1)d\sigma(x_1)\right)/|x| + \dots + \left(\frac{1}{\varphi(t)}\int_\ell \frac{\partial^2}{\partial x_d^2} f(x_d)d\sigma(x_d)\right)/|x|$$

$$= B_t(\Delta f(x)/|x|)$$

$$= B_t(\tilde{\Delta} f, x).$$

Then :

$$\tilde{\Delta}B_t(f,x) = B_t(\tilde{\Delta}f,x) \quad ....(1.6)$$

2. Auxiliary Results

In this section, let us introduce the results that we need in our work.

2.1. Lemma [3]

Suppose $f(x) \in L_p^2(U^{d-1})$, p <1 , and $B_t(f,x)$, $S_t(f,x)$, $\tilde{\Delta}f(x)$ are given by (1.1), (1.2) ,(1.3).Then for $x \in U^{d-1}$ and $0 < t < \frac{\pi}{2}$, we have :

$$B_t(f,x) - f(x) = \frac{1}{\varphi(t)} \int_0^{t} sin^{d-2}\theta \int_0^{\theta} sin^{2-d}\rho\, \varphi(\rho) B_\rho(\tilde{\Delta}f,x)d\rho\, d\theta$$

$$= \frac{1}{\varphi(t)} \int_0^t sin^{d-2}\theta \left\{\int_0^\theta sin^{2-d}\rho \int_\ell \tilde{\Delta}f(y)d\sigma(y)d\rho\right\} d\theta .$$

And

$$S_t(f,x) - f(x) = \frac{1}{\Psi(t)} sin^{d-2}t \int_0^t sin^{2-d}\theta\, d\theta \int_\ell \tilde{\Delta}f(y)d\sigma(y)$$

$$= \frac{1}{\Psi(t)} \int_0^t sin^{2-d}\theta\, \varphi(\theta) B_\theta(\tilde{\Delta}f,x)\, d\theta .$$

2.2. Lemma

For $f \in L_p^4(U^{d-1})$, p < 1 , and $B_t(g,x)$, $\tilde{\Delta}g(x)$ are given by (1.1) , (1.3). Then we have :

$$\|B_t g(x) - g(x) - \alpha(t)\tilde{\Delta}g(x)\|_p \le c(p)t^4 \|\tilde{\Delta}g(x)\|_p.$$

where ; $0<At^2\leq \alpha(t)\leq Bt^2$

Proof :

We can use Lemma 2.1

and write $B_t(g,x)-g(x)=$

$$\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)B_\rho(\tilde{\Delta}g,x)d\rho\,d\theta$$

$$=\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)(B_\rho(\tilde{\Delta}g,x)+\tilde{\Delta}g-\tilde{\Delta}g)d\rho\,d\theta$$

$$=\tilde{\Delta}g\left(\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)d\rho\,d\theta\right)+$$

$$\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)(B_\rho(\tilde{\Delta}g,x)-\tilde{\Delta}g)d\rho\,d\theta$$

$$=\alpha(t)\tilde{\Delta}g(x)+ct^2(B_\rho(\tilde{\Delta}g,x)-\tilde{\Delta}g(x))$$

$$B_t(g,x)-g(x)-\alpha(t)\tilde{\Delta}g(x)=ct^2(B_t(\tilde{\Delta}g,x)-\tilde{\Delta}g(x))$$

$$\|B_t(g,x)-g(x)-\alpha(t)\tilde{\Delta}g(x)\|_p\leq ct^2\|B_t(\tilde{\Delta}g,x)-\tilde{\Delta}g(x)\|_p$$

$$\leq ct^2\|\tilde{\Delta}B_t(g,x)-\tilde{\Delta}g(x)\|_p$$

$$\leq ct^2\|\tilde{\Delta}(B_t(g,x)-g(x))\|_p$$

$$\leq ct^2\left\|\tilde{\Delta}\left(\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)B_\rho(\tilde{\Delta}g,x)d\rho\,d\theta\right)\right\|_p$$

$$\leq ct^2\left\|\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)\tilde{\Delta}B_\rho(\tilde{\Delta}g,x)d\rho\,d\theta\right\|_p$$

$$\leq ct^2\left\|\frac{1}{\varphi(t)}\int_0^t \sin^{d-2}\theta \int_0^\theta \sin^{2-d}\rho\,\varphi(\rho)B_\rho(\tilde{\Delta}^2g,x)d\rho\,d\theta\right\|_p$$

$$\leq ct^2\|\tilde{\Delta}(B_t(g,x)-g(x))\|_p$$

$$\leq ct^2\|\tilde{\Delta}(B_t(g,x)-g(x))\|_p$$

$$\leq ct^2\|\tilde{\Delta}(B_t(g,x)-g(x))\|_p \blacksquare$$

2.3. Lemma

For $f\in L_p(U^{d-1})$, $p<1$, and $B_t(f,x)$ given by (1.1), $\tilde{\Delta}$ is the Laplace – Beltrami , then we get

$$\|\tilde{\Delta}B_t^2f(x)\|_p=\|\tilde{\Delta}B_t^2f(x)+\tilde{\Delta}B_t^4f(x)-\tilde{\Delta}B_t^4f(x)\|_p$$

$$\leq\|\tilde{\Delta}B_t^4f(x)\|_p+\|\tilde{\Delta}B_t^2f(x)-\tilde{\Delta}B_t^4f(x)\|_p$$

$$\leq\|\tilde{\Delta}B_t^4f(x)\|_p+\left\|\tilde{\Delta}B_t^2\left(f(x)-\tilde{\Delta}B_t^2f(x)\right)\right\|_p.$$

2.4. Lemma [1]

If ξx , $B_t(f,x)$ is given by

$$B_t(f,x)=\frac{1}{\varphi(t)}\int_\Omega\int_{-\kappa}^{\kappa}f(\upsilon+(x\cos\theta+\xi\sin\theta)\sqrt{1-|\upsilon|^2})d\theta\,d\upsilon$$

where;

$$\varphi(t)=\frac{2\pi^{(d-1)/2}}{\Gamma_{\left(\frac{d-1}{2}\right)}}\int_0^t \sin^{d-2}u\,du$$

$\Omega=B_{x,\xi}\ \sin t=\{\upsilon:\upsilon.x=0,\upsilon.\xi=0,|\upsilon|\leq\sin t\}$,

$$\kappa=\arccos(\cos t/\sqrt{1-|\upsilon|^2})$$

$$\frac{\partial}{\partial\xi}B_t(f,x)=\frac{1}{\varphi(t)}\int_\Omega\begin{bmatrix}f\left(\upsilon+x\cos t+\xi\sqrt{1-|\upsilon|^2-\cos^2t}\right)\alpha(t,\upsilon)-\\ f\left(x+x\cos t-\xi\sqrt{1-|\upsilon|^2-\cos^2t}\right)\beta(t,\upsilon)\end{bmatrix}d\upsilon,$$

where $\alpha(t,\upsilon)$ and $\beta(t,\upsilon)$ are close to 1 and are bounded by 1

2.5. Remark [4]

For $f\in L_\theta(U^{d-1})$, $1\leq\theta\leq\infty$, there exsists ,$g\in L_{\theta'}(U^{d-1})$,$1\leq\theta'\leq\infty$ such that for $\frac{1}{\theta}+\frac{1}{\grave{\theta}}=1$, We have :

$$\|\tilde{\Delta}B_tB_\tau f\|_\theta-\varepsilon\leq|\langle g,\tilde{\Delta}B_tB_\tau f\rangle|$$
$$\leq|\langle g,B_t\tilde{\Delta}B_\tau f\rangle|$$
$$\leq|\langle B_tg,\tilde{\Delta}B_\tau f\rangle|$$
$$\leq|\langle grad_{tan}B_tg,grad_{tan}B_\tau f\rangle|$$

Then : $\|\tilde{\Delta}B_tB_\tau f\|_\theta-\varepsilon\leq\|grad_{tan}B_tg\|_{\grave{\theta}}\cdot\|grad_{tan}B_\tau f\|_\theta$

3. The Main Results

In this section we shall introduce our main results.

3.1. Theorem

If f in $L_p(U^{d-1})$, $p<1$, then $grad_{tan}B_t f$ is in $L_p(U^{d-1})$ and

$$\|grad_{tan}B_tf\|_{L_p}\leq\frac{c(p)\Psi(t)}{\varphi(t)}\|f\|_{L_p}\leq\frac{c(p)}{t}\|f\|_{L_p}.$$

Proof :

By Lemma 2.4 we get

$$\left|\frac{\partial}{\partial\xi}B_t(f,x)\right|$$

$$= \left| \frac{1}{\varphi(t)} \int_{\Omega} \left[f\left(v + x\cos t + \xi\sqrt{(1-|v|^2) - \cos^2 t}\right)\alpha(t,v) \right.\right.$$

$$\left.\left. - f\left(x + x\cos t - \xi\sqrt{(1-|v|^2) - \cos^2 t}\right)\beta(t,v)\right] dv \right|$$

$$\le \frac{2}{\varphi(t)}\{\int_{\Omega} \left|f(v + x\cos t + \xi\sqrt{(1-|v|^2) - \cos^2 t})\right| dv$$

$$+ \int_{\Omega} \left|f(v + x\cos t + \xi\sqrt{(1-|v|^2) - \cos^2 t})\right| dv\}$$

Since $\int_{U^{d-1}} f(x)dx \le [\text{ measure of } U^{d-1}][\max_{x\in U^{d-1}} f(x)]$.

Then : $\left|\frac{\partial}{\partial\xi} B_t(f,x)\right| \le \frac{2\psi(t)}{\varphi(t)} S_t(|f|,x)$

$|grad_{tan} B_t(f,x)| = \max_{\xi\perp x} \left|\frac{\partial}{\partial\xi} B_t(f,x)\right|$.

Then we get, for $p < 1$ and $f\in L_p^1(U^{d-1})$,that

$$\|grad_{tan} B_t(f,x)\|_p = \int_{U^{d-1}} (|grad_{tan} B_t(f,x)|^p dx)^{1/p}$$

$$= \int_{U^{d-1}} (\left|\max_{\xi\perp x} \frac{\partial}{\partial\xi} B_t(f,x)\right|^p dx)^{1/p}$$

$$\le \int_{U^{d-1}} \left(\left|\frac{2\psi(t)}{\varphi(t)} S_t(|f|,x)\right|^p dx\right)^{1/p}$$

$$\le \frac{2\psi(t)}{\varphi(t)} \|S_t(|f|,x)\|_p$$

$since\ \frac{2\psi(t)}{\varphi(t)} \le \frac{c(p)}{t}$, then:

$$\|grad_{tan} B_t(f,x)\|_p \le \frac{c(p)}{t} \left\| \frac{1}{\psi(t)} \int_{x.y=cost} f(x)d\gamma(x) \right\|_p$$

$$\le \frac{c(p)}{t} \|f\|_p \ \blacksquare$$

3.2. Theorem

For $f \in L_p(U^{d-1})$, $p<1$. Then

$$\left\|\tilde{\Delta}^r B_{\tau_1} \cdots\cdots B_{\tau_{2r}} f\right\|_p \le \frac{c_r(p)}{\tau_1 \cdots\cdots \tau_{2r}} \|f\|_p.$$

Proof :

Since $\|\tilde{\Delta}B_t B_\tau f\|_p - \varepsilon \le \|\tilde{\Delta}B_t B_\tau f\|_\theta - \varepsilon$, $\theta\ge 1$

We choose g as in Remark2.5 , then we get :

$\|\tilde{\Delta}B_t B_\tau f\|_p - \varepsilon \le \|grad_{tan} B_t g\|_{\theta`} \cdot \|grad_{tan} B_\tau f\|_\theta$ , $\theta`\ge 1$,and $\frac{1}{\theta} + \frac{1}{\acute{\theta}} = 1$

$$\|grad_{tan} B_t g\|_{\grave{\theta}} = \left(\int_{U^{d-1}} |grad_{\tan B_t g}|^{\acute{\theta}} dx\right)^{1/\acute{\theta}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\acute{\theta} + \frac{1}{\theta'} - \frac{1}{\theta'}} dx\right)^{\frac{1}{\theta'} + \acute{\theta} - \acute{\theta}}$$

$$\le$$

$$\left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\acute{\theta} - \frac{1}{\theta'}} |grad_{tan} B_t g|^{\frac{1}{\theta'}} dx\right)^{\frac{1}{\theta'} - \acute{\theta}} \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\acute{\theta} - \frac{1}{\theta'}} |grad_{tan} B_t g|^{\frac{1}{\theta'}} dx\right)^{\acute{\theta}}$$

Assume that: $\frac{1}{\acute{\theta}} = q$, so $\acute{\theta} = \frac{1}{q}$, and $q < 1$, then

$$\|grad_{tan} B_t g\|_{\acute{\theta}} \le \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\frac{1}{q} - q} |grad_{tan} B_t g|^q dx\right)^{q - \frac{1}{q}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\frac{1}{q} - q + q} dx\right)^{q - \frac{1}{q} + \frac{1}{q}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_t g|^{\frac{1}{q} \times q^2} dx\right)^{q \times \frac{1}{q^2}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_t g|^q dx\right)^{\frac{1}{q}}$$

$$= \|grad_{tan} B_t g\|_q \ , \ q<1 \qquad \text{.......(3.1)}$$

And

$$grad_{tan} B_\tau f\|_\theta = \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^\theta dx\right)^{\frac{1}{\theta}} = \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\theta + \frac{1}{\theta} - \frac{1}{\theta}} dx\right)^{\frac{1}{\theta} + \theta - \theta}$$

$$\le \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\theta - \frac{1}{\theta}} |grad_{tan} B_\tau f|^{\frac{1}{\theta}} dx\right)^{\frac{1}{\theta} - \theta} \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\theta - \frac{1}{\theta}} |grad_{tan} B_\tau f|^{\frac{1}{\theta}} dx\right)^{\theta}$$

Assume that: $\frac{1}{\theta} = p$ so $\theta = \frac{1}{p}$ and $p < 1$, then

$$\|grad_{tan} B_\tau f\|_\theta$$

$$\le \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\frac{1}{p} - p} |grad_{tan} B_\tau f|^p dx\right)^{p - \frac{1}{p}} \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\frac{1}{p} - p} |grad_{tan} B_\tau f|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\frac{1}{p} - p + p} dx\right)^{p - \frac{1}{p} + \frac{1}{p}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^{\frac{1}{p} \times p^2} dx\right)^{p \times \frac{1}{p^2}}$$

$$= \left(\int_{U^{d-1}} |grad_{tan} B_\tau f|^p dx\right)^{\frac{1}{p}}$$

$$= \|grad_{tan} B_\tau f\|_p , p<1 \qquad \text{.......(3.2)}$$

From (3.1) and (3.2) ,we get:

$\|\tilde{\Delta}B_t B_\tau f\|_p - \varepsilon \le \|grad_{tan} B_t g\|_q \cdot \|grad_{tan} B_\tau f\|_p$

, where: $p < 1$, $q < 1$ and $p + q = 1$

Let $\|g\|_q = c(q)$, and by Theorem 3.2 we get:

$$\|\tilde{\Delta} B_t B_\tau f\|_p - \varepsilon \le \frac{c(q)}{t}\|g\|_q \cdot \frac{c(p)}{\tau}\|f\|_p$$

$$\le \frac{c^2(q)\,c(p)}{t\tau}\|f\|_p.$$

εis arbitrary , implies our result for r = 1

Repeting the above we get

$\|\tilde{\Delta}^r B_{\tau_1} \cdots\cdots B_{\tau_{2r}} f\|_p \le \frac{c_r(p)}{\tau_1 \cdots\cdots \tau_{2r}}\|f\|_p$, p<1■

3.3.Corollary

For $f \in L_p(U^{d-1})$,p < 1, and $B_t(f,x)$ given by (1.1) , we have

$$\|\tilde{\Delta} B_t^m(f,x)\|_p \le \frac{c(p)}{t^2}\|f(x)\|_p$$

3.4. Theorem

For $f \in L_p(U^{d-1})$, p<1, and $B_t(f,x)$,$K(f,\tilde{\Delta},t^2)$ are given by (1.1) , (1.4) we have, for some M independent of f , p or t

$$K(f,\tilde{\Delta},t^2)_p \approx \|f - B_t f\|_p + \|f - B_{t/M} f\|_p$$

Proof :

We note that we need to show only that

$$K(f,\tilde{\Delta},t^2)_p \approx \|f - B_t f\|_p + \|f - B_{t/M} f\|_p$$

We use Lemma 2.2 to write

$$\|B_{t/M} B_t^4 f - B_t^4 f - \alpha(t)\tilde{\Delta} B_t^4 f\|_p \le c(p)\frac{t^4}{M^4}\|\tilde{\Delta}^2 B_t^4 f\|_p$$

$$\le c(p)\frac{t^4}{M^4}\|\tilde{\Delta} B_t^2 f\, \tilde{\Delta} B_t^2 f\|_p$$

$$\le c(p)\frac{t^4}{M^4} \cdot \frac{c(p)}{t^2}\|\tilde{\Delta} B_t^2 f\|_p$$

$$\le c(p)\frac{t^2}{M^4}\|\tilde{\Delta} B_t^2 f\|_p$$

By Lemma 2.3

$$\|B_{t/M} B_t^4 f - B_t^4 f - \alpha(t)\tilde{\Delta} B_t^4 f\|_p$$

$$\le c(p)\frac{t^2}{M^4}\|\tilde{\Delta} B_t^4 f\|_p + c(p)\frac{t^2}{M^4}\|\tilde{\Delta} B_t^2(f - B_t^2 f)\|_p$$

$$\le c(p)\frac{t^2}{M^4}\|\tilde{\Delta} B_t^4 f\|_p + c(p)\frac{2}{M^4}\|f - B_t^2 f\|_p$$

Choosing *M* independent of *f* and t to have

$$c(p)\frac{t^2}{M^4}\|\tilde{\Delta} B_t^4 f\|_p \le \frac{1}{2}\,\alpha\left(\frac{t}{M}\right)\|\tilde{\Delta} B_t^4 f\|_p$$

Then we get :

$$K(f,\tilde{\Delta},t^2)_p \le c(p)\|f - B_t f\|_p + \|f - B_{t/M} f\|_p$$ ■

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